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Perron Frobenius Theorem for Positive Polynomial Matrices

Pham Huu Anh Ngoc[1*], Byung Soo Lee[1], and Nguyen Khoa Son[2]

[1] *Dept. of Mathematics, KyungSung University Busan, 608-736 Korea*
[2] *Institute of Mathematics, 18 Hoang Quoc Viet Road, 10307 Hanoi, Vietnam*

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Abstract. In this paper, we give an extension of the classical Perron–Frobenius theorem to positive polynomial matrices. Then the obtained result is applied to derive necessary and sufficient conditions for the exponential stability of positive linear discrete-time systems.

1. Introduction

It is well-known that the principal tool for the analysis of the stability and robust stability of a positive system is the Perron–Frobenius theorem, see, e.g. [1, 4 - 6, 10]. It is natural to ask whether it is possible to generalize this to classes of general (positive) linear systems and to positive non-linear systems. To our knowledge, there is a large number of extensions of the classical Perron-Frobenius theorem in the literature. We could find the latest extensions of Perron-Frobenius theorem to real, complex matrices in [12 - 14] and to non-linear systems in [1, 8]. However, the investigation of extension of Perron–Frobenius theorem to polynomial matrices and applying the obtained results to study the stability, robust stability of discrete time linear systems described by higher order difference equations of the form

$$A_\nu y(t+\nu) = A_{\nu-1} y(t+\nu-1) + \cdots + A_1 y(t+1) + A_0 y(t), \quad t \in \mathbb{N}, \qquad (1)$$

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where we suppose $\det A_\nu \neq 0$, has not been studied yet. In this paper, we give an extension of the classical Perron–Frobenius theorem to positive polynomial matrices. Then the obtained result is applied to derive necessary and sufficient conditions for the asymptotic stability of positive linear discrete-time systems of the form (1).

The organization of this paper is as follows. In the next section, we summarize some notations and recall the classical Perron–Frobenius theorem which will be used in the sequel. The main results of the paper will be presented in Sec. 3 where we extend the classical Perron–Frobenius theorem to positive polynomial matrices. In Sec. 4, as an application of the Perron-Frobenius theorem of positive polynomial matrices, we give necessary and sufficient conditions for the asymptotic stability of the discrete time linear systems of the form (1).

2. Preliminaries

Let $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$ and n, l, q be positive integers. Inequalities between real matrices or vectors will be understood componentwise, i.e. for two real $l \times q$-matrices $A = (a_{ij})$ and $B = (b_{ij})$, the inequality $A \geq B$ means $a_{ij} \geq b_{ij}$ for $i = 1, \dots, l$, $j = 1, \dots, q$. Furthermore, if $a_{ij} > b_{ij}$ for $i = 1, \dots, l$, $j = 1, \dots, q$ then we write $A \gg B$ instead of $A \geq B$. The set of all nonnegative $l \times q$-matrices is denoted by $\mathbb{R}_+^{l\times q}$. If $x = (x_1, x_2, ..., x_n) \in \mathbb{K}^n$ and $P = (p_{ij}) \in \mathbb{K}^{l\times q}$ we define $|x| = (|x_i|)$ and $|P| = (|p_{ij}|)$. Then $|CD| \leq |C||D|$. For any matrix $A \in \mathbb{K}^{n\times n}$ the *spectral radius* of the matrix A is denoted by $\rho(A) = \max\{|\lambda| : \ \lambda \in \sigma(A)\}$, where $\sigma(A) := \{z \in \mathbb{C} : \ \det(zI_n - A) = 0\}$ is the set of all eigenvalues of A. The next theorem summarizes some basic properties of nonnegative matrices, see e.g. [5].

Theorem 2.1. *Let $A \in \mathbb{R}_+^{n\times n}$, $t \in \mathbb{R}$. Then*

(i) (*Perron-Frobenius*) *$\rho(A)$ is an eigenvalue of A and there exists a nonnegative eigenvector $x \geq 0$, $x \neq 0$ such that $Ax = \rho(A)x$.*

(ii) *Given $\alpha \in \mathbb{R}, \alpha > 0$, there exists a nonzero vector $x \geq 0$ such that $Ax \geq \alpha x$ if and only if $\rho(A) \geq \alpha$.*

(iii) *$(tI_n - A)^{-1}$ exists and is nonnegative if and only if $t > \rho(A)$.*

(iv) *Given $B \in \mathbb{R}_+^{n\times n}, C \in \mathbb{C}^{n\times n}$. If $|C| \leq B$ then $\rho(A + C) \leq \rho(A + B)$.*

3. Main Results

Consider the ν-th order linear difference equation of the form (1) where $A_0, A_1, ...,$ $A_\nu \in \mathbb{R}^{n\times n}$ are given matrices. With this equation, we associate the polynomial matrix

$$P(z) := A_\nu z^\nu - A_{\nu-1} z^{\nu-1} - \cdots - A_0. \tag{2}$$

Denote by $\sigma(P(\cdot))$ the set of all roots of the characteristic equation of (2), that is $\sigma(P(\cdot)) = \{\lambda \in \mathbb{C} : \det P(\lambda) = 0\}$, and let $\rho(P(\cdot)) := \sup\{|\lambda| : \ \lambda \in \sigma(P(\cdot))\}$. Then, $\sigma(P(\cdot))$ and $\rho(P(\cdot))$ are called the *spectrum and spectral radius* of the polynomial matrix $P(\cdot)$, respectively. If $\det P(z) \equiv 0$ then $\sigma(P(\cdot)) = \mathbb{C}$, otherwise the spectrum of $P(\cdot)$ is a finite subset of $\mathbb{C}$ consisting of at most deg det $P(z)$ "eigenvalues". If $z \in \mathbb{C}$ is an eigenvalue of the polynomial matrix (2) then there exists a nonzero vector $x \in \mathbb{C}^n$ such that $P(z)x = 0$. Then the vector x is called an eigenvector of the polynomial matrix $P(\cdot)$ corresponding to the eigenvalue z.

Definition 3.1. *The difference equation* (1) *is called asymptotically stable if all its solutions $y(t)$ tend to zero as $t \to \infty$.*

It is well-known that the difference equation (1) is asymptotically stable if and only if the associated polynomial matrix $P(z) = A_\nu z^\nu - A_{\nu-1} z^{\nu-1} \cdots - A_0$ is *Schur stable* in the sense that all the roots of $p(z) = \det P(z)$ lie in the complex open unit disk.

Remark 1. If $\det P(z) \not\equiv 0$ then the difference equation (1) has a finite dimensional solution set (i.e. defines an autonomous behaviour, in terms of behavioural system theory [11]) and can be transformed equivalently into a first order difference equation $x(t) = Ax(t)$ (see e.g. [16]). It is known that $\lambda \in \sigma(A)$ if and only if $\det P(\lambda) = 0$ (see [15]). Hence the state space system described by $x(t) = Ax(t)$ is asymptotically stable if and only if the polynomial matrix $P(z)$ is Schur stable. If $\det A_\nu \neq 0$, an equivalent state space system is easily determined by introducing the state vector $x(t) = [y(t-\nu+1)^T, y(t-\nu+2)^T, ..., y(t)^T]^T$:

$$x(t+1) = Ax(t), \ t \in \mathbb{N}; \quad A = \begin{bmatrix} 0 & I_n & 0 & ... & 0 & 0 \\ 0 & 0 & I_n & ... & 0 & 0 \\ .. & .. & .. & ... & .. & .. \\ 0 & 0 & 0 & ... & 0 & I_n \\ A_\nu^{-1}A_{\nu-1} & A_\nu^{-1}A_{\nu-2} & .. & ... & .. & A_\nu^{-1}A_0 \end{bmatrix}. \tag{3}$$

If $\det A_\nu = 0$, the construction of an equivalent state space system is more complicate (see [16]).

In the rest of this paper, we always assume that A_ν is a *regular matrix.*

Definition 3.2. *The polynomial matrix* (2) *with* $\det A_\nu \neq 0$ *is called positive if the matrices* $A_\nu^{-1}A_{\nu-1}, ..., A_\nu^{-1}A_0$ *are nonnegative.*

Remark 2. From the above definition, it follows that $P(z)$ in (2) with $\det A_\nu \neq 0$ is a positive polynomial matrix if and only if the associated discrete state space system of the form (3) is positive.

We are now in a position to prove the main results of this paper.

Theorem 3.3. *Let the polynomial matrix* (2) *be positive and* $\rho_0 := \rho(P(\cdot))$. *Then,*

(i) (Perron-Frobenius theorem for positive polynomial matrices) ρ_0 *is an eigenvalue of the polynomial matrix* (2) *and there exists a nonnegative eigenvector* $x \in \mathbb{R}^n_+$ *such that*

$$P(\rho_0)x = \left(A_\nu {\rho_0}^\nu - A_{\nu-1}{\rho_0}^{\nu-1} - \cdots - A_0\right) x = 0.$$

(ii) *Given $\alpha \in \mathbb{R}_+$, there exists a nonzero vector $x \in \mathbb{R}^n_+$ such that*

$$\left(A_\nu^{-1}A_{\nu-1}\alpha^{\nu-1} + A_\nu^{-1}A_{\nu-2}\alpha^{\nu-2} + \cdots + A_\nu^{-1}A_0\right) x \geq \alpha^\nu x$$

if and only if $\rho_0 \geq \alpha$.

(iii) $t \in \mathbb{R}, t > \rho_0 \Leftrightarrow P(t)^{-1}A_\nu \geq 0$.

Proof.
(i) By definition of ρ_0, there exist $s \in \mathbb{C}, |s| = \rho_0$ and $y \in \mathbb{C}^n$, $y \neq 0$ such that

$$P(s)y = \left(A_\nu s^\nu - A_{\nu-1}s^{\nu-1} - \cdots - A_0\right) y = 0.$$

From $A_\nu^{-1}A_i \in \mathbb{R}^{n\times n}_+, i \in \underline{\nu} := \{0, 1, ..., \nu\}$, it follows that

$$\left(A_\nu^{-1}A_{\nu-1}{\rho_0}^{\nu-1} + A_\nu^{-1}A_{\nu-2}{\rho_0}^{\nu-2} + \cdots + A_\nu^{-1}A_0\right) |y| \geq {\rho_0}^\nu |y|.$$

By Theorem 2.1 (ii), we have

$$\rho\left(A_\nu^{-1}A_{\nu-1}{\rho_0}^{\nu-1} + A_\nu^{-1}A_{\nu-2}{\rho_0}^{\nu-2} + \cdots + A_\nu^{-1}A_0\right) \geq {\rho_0}^\nu. \tag{4}$$

Consider the continuous real function

$$f(\theta) := \theta^\nu - \rho\left(A_\nu^{-1}A_{\nu-1}\theta^{\nu-1} + A_\nu^{-1}A_{\nu-2}\theta^{\nu-2} + \cdots + A_\nu^{-1}A_0\right), \ \theta \in [0, +\infty). \tag{5}$$

By (4), $f(\rho_0) \leq 0$. Assume $f(\rho_0) < 0$. Since, clearly $\lim_{\theta\to+\infty} f(\theta) = +\infty, f(\theta_0) = 0$ for some $\theta_0 > \rho_0$. Therefore $\rho\big(A_\nu^{-1}A_{\nu-1}{\theta_0}^{\nu-1} + A_\nu^{-1}A_{\nu-2}{\theta_0}^{\nu-2} + \cdots + A_\nu^{-1}A_0\big) = {\theta_0}^\nu$. It follows from Theorem 2.1 (i) that ${\theta_0}^\nu$ is an eigenvalue of the nonnegative matrix $A_\nu^{-1}A_{\nu-1}{\theta_0}^{\nu-1} + A_\nu^{-1}A_{\nu-2}{\theta_0}^{\nu-2} + \cdots + A_\nu^{-1}A_0$, or equivalently, $\det\big(\theta_0^\nu I_n - (A_\nu^{-1}A_{\nu-1}{\theta_0}^{\nu-1} + A_\nu^{-1}A_{\nu-2}{\theta_0}^{\nu-2} + \cdots + A_\nu^{-1}A_0)\big) = 0$. By A_ν is a regular matrix, we get $\det\big(A_\nu\theta_0^\nu - A_{\nu-1}{\theta_0}^{\nu-1} - A_{\nu-2}{\theta_0}^{\nu-2} - \cdots - A_0\big) = 0$. This, however, conflicts with the definition of ρ_0. Thus, $f(\rho_0) = 0$. This gives

$$\rho_0^\nu = \rho\left(A_\nu^{-1}A_{\nu-1}\rho_0^{\nu-1} + A_\nu^{-1}A_{\nu-2}\rho_0^{\nu-2} + \cdots + A_\nu^{-1}A_0\right). \tag{6}$$

Applying Theorem 2.1 (i) to ρ_0^ν and taking into account that A_ν is a regular matrix, we get (i).
(ii) Let $\alpha \in \mathbb{R}$ and $\rho_0 \geq \alpha$. It follows from (i) that there exists a nonnegative vector $x \neq 0$ such that $\big(A_\nu{\rho_0}^\nu - A_{\nu-1}{\rho_0}^{\nu-1} - \cdots - A_0\big) x = 0$. Therefore

$$\left(A_\nu^{-1}A_{\nu-1}{\rho_0}^{\nu-1} + A_\nu^{-1}A_{\nu-2}{\rho_0}^{\nu-2} + \cdots + A_\nu^{-1}A_0\right) x = {\rho_0}^\nu x \geq \alpha^\nu x.$$

Conversely, suppose that there exists a nonzero vector $x_0 \geq 0$ such that

$$\left(A_\nu^{-1}A_{\nu-1}\alpha^{\nu-1} + A_\nu^{-1}A_{\nu-2}\alpha^{\nu-2} + \cdots + A_\nu^{-1}A_0\right) x_0 \geq \alpha^\nu x_0.$$

Since $A_\nu^{-1}A_{\nu-1}\alpha^{\nu-1} + A_\nu^{-1}A_{\nu-2}\alpha^{\nu-2} + \cdots + A_\nu^{-1}A_0$ is a nonnegative matrix, it follows from Theorem 2.1 (ii) that $\rho\big(A_\nu^{-1}A_{\nu-1}\alpha^{\nu-1} + A_\nu^{-1}A_{\nu-2}\alpha^{\nu-2} + \cdots + A_\nu^{-1}A_0\big) \geq \alpha^\nu$. Thus, $f(\alpha) \leq 0$, where the function f is defined by (5). It is important to note that from the arguments in the proof of (i), we have $f(\theta) > 0$ for every $\theta > \rho_0$. Therefore, we have $\rho_0 \geq \alpha$.

(iii) Since $t > \rho_0$, we have $f(t) > 0$, or equivalently,

$$t^\nu > \rho\left(A_\nu^{-1}A_{\nu-1}t^{\nu-1} + A_\nu^{-1}A_{\nu-2}t^{\nu-2} + \cdots + A_\nu^{-1}A_0\right).$$

By Theorem 2.1 (iii), the matrix

$$\left(t^\nu I_n - (A_\nu^{-1}A_{\nu-1}t^{\nu-1} + A_\nu^{-1}A_{\nu-2}t^{\nu-2} + \cdots + A_\nu^{-1}A_0)\right)^{-1}$$

exists and is nonnegative. Thus,

$$0 \leq \left(A_\nu^{-1}A_\nu t^\nu - A_\nu^{-1}A_{\nu-1}t^{\nu-1} - A_\nu^{-1}A_{\nu-2}t^{\nu-2} - \cdots - A_\nu^{-1}A_0\right)^{-1} = P(t)^{-1}A_\nu.$$

Using Theorem 2.1 (iii), it is easy to show the converse. This completes our proof. ∎

4. Application to Positive Linear Difference Equations

We now apply the above results to derive some necessary and sufficient conditions for the asymptotic stability of the positive linear difference equation of the form (1).

Theorem 4.1. *The positive linear difference equation of the form* (1) *is asymptotically stable if and only if the linear discrete-time system without delay*

$$x(t+1) = \Big(\sum_{i=0}^{\nu-1} A_\nu^{-1}A_i\Big)x(t), \quad t \in \mathbb{N} \tag{7}$$

is asymptotically stable.

Proof. Let the positive linear difference equation of the form (1) be asymptotically stable. That is $0 \leq \rho_0 := \rho\left(P(\cdot)\right) < 1$. Suppose that the system (7) is not asymptotically stable. Then, we have $\rho\Big(\sum\limits_{i=0}^{\nu-1} A_\nu^{-1}A_i\Big) \geq 1$. Thus, $f(1) \leq 0$, where the function f is defined as in the proof of Theorem 3.3 (ii) by

$$f(\theta) := \theta^\nu - \rho\left(A_\nu^{-1}A_{\nu-1}\theta^{\nu-1} + A_\nu^{-1}A_{\nu-2}\theta^{\nu-2} + \cdots + A_\nu^{-1}A_0\right), \quad \theta \in [0, +\infty).$$

On the other hand, by Theorem 3.3 (i), $f(\rho_0) = 0$. So, we derive $\rho_0 \geq 1$, a contradiction.

Conversely, assume the system without delay (7) is asymptotically stable. That is, $\rho\Big(\sum\limits_{i=0}^{\nu-1} A_\nu^{-1}A_i\Big) < 1$. Suppose that $\rho_0 > 0$, it follows from (6) that $\rho_0 = \rho\left(A_\nu^{-1}A_{\nu-1} + A_\nu^{-1}A_{\nu-2}{\rho_0}^{-1} + \cdots + A_\nu^{-1}A_0{\rho_0}^{-(\nu-1)}\right)$. By Theorem 2.1 (iv), $\rho_0 < 1$, as was to be shown.

Theorem 4.2. *Let the positive linear difference equation of the form* (1) *be asymptotically stable. Then, there exist a nonnegative vector* $p \in \mathbb{R}^n$, $p \neq 0$ *and* *a real number* $\gamma,\ 0 \le \gamma < 1$ *such that*

$$\Big(\sum_{i=0}^{\nu-1} A_\nu^{-1} A_i\Big)^T p = \gamma p. \tag{8}$$

Conversely, if the equality (8) *holds for some strictly positive vector* $p \in \mathbb{R}^n, p \gg 0$ *and some real number* $\gamma,\ 0 \le \gamma < 1$*, then the system* (1) *is asymptotically stable.*

Proof. The first assertion of this theorem follows from Theorem 4.1 and Theorem 2.1 (i).

Conversely, suppose that the equality (8) holds for some $p \gg 0, p \in \mathbb{R}^n$ and some real number $0 \le \gamma < 1$. Assume on the contrary that the positive system (1) is not asymptotically stable. By Theorem 4.1, it is equivalently to $\beta := \rho\Big(\sum_{i=0}^{\nu-1} A_\nu^{-1} A_i\Big) \ge 1$. From Theorem 2.1 (i), we get $\Big(\sum_{i=0}^{\nu-1} A_\nu^{-1} A_i\Big) p_0 = \beta p_0$ for some $p_0 \in \mathbb{R}^n_+, p_0 \ne 0$. This gives $p_0^T\Big(\sum_{i=0}^{\nu-1} A_\nu^{-1} A_i\Big)^T = \beta p_0^T$. Combining the last equality and the equality (7) gives $(\beta - \gamma) p_0^T p = 0$. However, this conflicts with $(\beta - \gamma) > 0$ and $p \gg 0, p_0 \ge 0, p_0 \ne 0$. This completes our proof. ■

We conclude the paper with a remark.

Remark 3. It is worth noticing that the vector $p \in \mathbb{R}^n_+$ in the converse statement in the last theorem must be strictly positive, i.e. $p \gg 0$. For clarity, we consider the following system

$$\begin{pmatrix} x_1(t+1) \\ x_2(t+1) \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}, \quad k \in \mathbb{N}. \tag{9}$$

Then, we have the equality (8) with $p := (1, 0)^T$ and $\gamma = 1/2$. However, the system (9) is unstable. Furthermore, the converse statement of Theorem 4.2 can be stated in the following form:

If there exist vectors $p, r \in \mathbb{R}^n;\ p, r \gg 0$ satisfying

$$\Big(\sum_{i=0}^{\nu-1} A_\nu^{-1} A_i\Big)^T p + r = p \tag{10}$$

then the system (1) is asymptotically stable.

In the particular case of $A_\nu = I_n$, we get back one of results in [4], where it was proved by using a Lyapunov–Krasovskii functional.

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