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Finite Rank Little Hankel Operators on Bergman-Type Spaces

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Abstract. In this paper we characterize the kernel of a little Hankel operator on the Bergman-type space $L^2_a(\Omega)$ in terms of inner divisors and obtain a characterization for finite rank little Hankel operators using the invariant subspace theory technique.

1. Introduction

Let Ω be a bounded symmetric domain in $\mathbb{C}^N$ for some integer $N \geq 1$. We assume that Ω is in its standard (Harish–Chandra) realization so that $0 \in \Omega$ and Ω is circular. The domain Ω is also star-like; i.e., $z \in \Omega$ implies that $tz \in \Omega$ for all $t \in [0, 1]$. Let $\text{Aut}(\Omega)$ be the Lie group of all automorphisms (biholomorphic mappings) of Ω, and G_0 the isotropy subgroup at 0; that is, $G_0 = \{\Psi \in \text{Aut}(\Omega) : \Psi(0) = 0\}$. Since Ω is bounded symmetric, we can canonically define [17] for each a in Ω an automorphism ϕ_a in $\text{Aut}(\Omega)$ such that
(1) $\phi_a \circ \phi_a(z) = z$;
(2) $\phi_a(0) = a, \phi_a(a) = 0$;
(3) ϕ_a has a unique fixed point in Ω.

Actually, the above three conditions completely characterize the ϕ_a's as the set of all (holomorphic) geodesic symmetries of Ω. As an example, we point out that if $\Omega = \mathbb{D}$, the open unit disk in $\mathbb{C}$, then

$$\phi_a(z) = \frac{a - z}{1 - \bar{a}z}$$

for all a and z in $\mathbb{D}$. They are involutive Mobius transformations on $\mathbb{D}$. Further assume that Ω is a simply connected Jordan domain bounded by an analytic

Jordan curve Γ and with finite area.

Let dA be the normalized Lebesgue measure on Ω. In particular, when $\Omega = \mathbb{D}$,

$$dA(z) = \frac{1}{\pi} dxdy = \frac{1}{\pi} r dr d\theta.$$

Let $L^2(\Omega, dA)$ denote the Hilbert space of complex-valued, absolutely square-integrable, Lebesgue measurable functions f on Ω with the inner product

$$\langle f, g \rangle = \int_\Omega f(z)\overline{g(z)} dA(z).$$

Let $L^\infty(\Omega, dA)$ denote the Banach space of Lebesgue measurable functions f on Ω with

$$\|f\|_\infty = \text{ess sup}\{|f(z)| : z \in \Omega\} < \infty.$$

Let $H^\infty(\Omega)$ be the space of bounded analytic functions on Ω. We consider the Bergman-type space $L^2_a(\Omega)$ of holomorphic functions in $L^2(\Omega, dA)$. The reproducing kernel $K(z, w)$ of $L^2_a(\Omega, dA)$ is holomorphic in z and anti-holomorphic in w, and

$$\int_\Omega |K(z, w)|^2 dA(w) = K(z, z) > 0$$

for all z in Ω. We shall write $K(z, w) = \overline{K_z(w)}$ for $z, w \in \Omega$. Thus we define for each $\lambda \in \Omega$, a unit vector k_λ in $L^2_a(\Omega)$ by $k_\lambda(z) = \frac{K(z,\lambda)}{\sqrt{K(\lambda,\lambda)}}$. For $\Omega = \mathbb{D}$, the reproducing kernel may be written down explicitly-namely $K(z, \lambda) = \frac{1}{(1-z\overline{\lambda})^2}$ and $k_\lambda(z) = \frac{K(z,\lambda)}{\sqrt{K(\lambda,\lambda)}} = \frac{(1-|\lambda|^2)}{(1-z\overline{\lambda})^2}$. If $\mathcal{B}_N$ is the unit ball in $\mathbb{C}^N$ and $K_{\mathcal{B}_N}$ is the Bergman reproducing kernel for $L^2_a(\mathcal{B}_N, dA)$ then for $z, \lambda \in \mathcal{B}_N$,

$$K_{\mathcal{B}_N}(z, \lambda) = \frac{N!}{(1 - z \cdot \overline{\lambda})^{N+1}},$$

where $z \cdot \overline{\lambda} = z_1\overline{\lambda_1} + \cdots + z_N\overline{\lambda_N}$. For details see [17].

Since $L^2_a(\Omega)$ is a closed subspace of $L^2(\Omega, dA)$(see [17]), there exists an orthogonal projection P from $L^2(\Omega, dA)$ onto $L^2_a(\Omega)$. For $\phi \in L^\infty(\Omega)$, we define the Toeplitz operator T_ϕ on $L^2_a(\Omega)$ by $T_\phi f = P(\phi f), f \in L^2_a(\Omega)$. The big Hankel operator H_ϕ is a mapping from $L^2_a(\Omega)$ into $(L^2_a(\Omega))^\perp$ defined by $H_\phi f = (I - P)(\phi f),\ f \in L^2_a(\Omega)$. Let $\overline{L^2_a(\Omega)} = \{\bar{f} : f \in L^2_a(\Omega)\}$. The little Hankel operator h_ϕ is a mapping from $L^2_a(\Omega)$ into $\overline{L^2_a(\Omega)}$ defined by $h_\phi f = \overline{P}(\phi f),\ f \in L^2_a(\Omega)$ where $\overline{P}$ is the projection from $L^2(\Omega, dA)$ onto $\overline{L^2_a(\Omega)}$. There are also many equivalent ways of defining little Hankel operators. For example define the map S_ϕ from $L^2_a(\Omega)$ into $L^2_a(\Omega)$ by $S_\phi f = P(J(\phi f))$ where J is the mapping from $L^2(\Omega)$ into $L^2(\Omega)$ defined by $J(h(z)) = h(\bar{z})$. It is easy to see that the operator S_ϕ is unitarily equivalent to the little Hankel operator h_ψ for some $\psi \in L^\infty(\Omega)$.

Let $\mathbb{T}$ denote the unit circle in $\mathbb{C}$. Let $L^2(\mathbb{T})$ be the space of complex-valued, absolutely square integrable, Lebesgue measurable functions on $\mathbb{T}$. Let $H^2(\mathbb{T})$ be the corresponding Hardy space of functions on $\mathbb{T}$ with vanishing negative

Fourier coefficients. For $\phi \in L^\infty(\mathbb{T})$, the space of essentially bounded measurable functions on $\mathbb{T}$, we define the Hankel operator L_ϕ from $H^2(\mathbb{T})$ into $H^2(\mathbb{T})$ as $L_\phi f = \tilde{P}(\tilde{J}(\phi f))$, where $\tilde{J}$ is the mapping from $L^2(\mathbb{T})$ into $L^2(\mathbb{T})$ defined by $\tilde{J}(h(z)) = h(\bar{z})$ and $\tilde{P}$ is the projection from $L^2(\mathbb{T})$ onto $H^2(\mathbb{T})$.

In the case of the Hardy space $H^2(\mathbb{T})$, where $\mathbb{T}$ is the unit circle, it is shown [14] that the following conditions are equivalent for a Hankel operator L:
1. The kernel of L is not trivial.
2. The range of L is not dense.
3. $L = L_{z\bar{u}h}$ for some inner function u and some h in H^∞.

Moreover, it was shown by Kronecker [14] that a Hankel operator L is of finite rank if and only if $L = L_{z\bar{u}h}$ with u a finite Blaschke product and $h \in H^\infty$. In this case, the rank of L is not greater than the number of zeroes of u (counted with multiplicity). We shall see whether some such results hold for Hankel operators on Bergman-type spaces $L^2_a(\Omega, dA)$ where $\Omega \subset \mathbb{C}$ (i.e when $N = 1$).

2. Decomposition of Functions and Little Hankel Operators

Let R be the ring of all polynomial functions on Ω. The Bergman space $L^2_a(\Omega)$ contains R as a dense subspace [17]. It is shown by Faour [7] that if T_z is the Toeplitz operator on the Bergman space $L^2_a(\mathbb{D})$ with symbol z then an operator S on $L^2_a(\mathbb{D})$ is a little Hankel operator if and only if

$$T_z^* S = S T_z.$$

Recall that for $\phi \in L^\infty(\Omega)$, the operator S_ϕ is the little Hankel operator on $L^2_a(\Omega)$. It can be easily established that $S_\phi T_z = T_{\bar{z}} S_\phi$, where T_z is the Bergman shift operator defined on $L^2_a(\Omega)$ by $T_z f = zf$ and $T_{\bar{z}} f = P(\bar{z} f)$. Thus, the little Hankel operatos S_ϕ are special instances of solutions of the operator equation

$$S T_z = T_{\bar{z}} S, \tag{$*$}$$

where S is a bounded operator on $L^2_a(\Omega)$. From $(*)$, it is established that $\langle Spq^+, 1\rangle = \langle Sp, q\rangle$, where p and q are polynomials in z, and $p^+(z) = \overline{p(\bar{z})}$. Hence, it follows that

$$\langle Sb_\zeta^{\frac{1}{2}}, (b_\zeta^{\frac{1}{2}})^+\rangle = \langle Sb_\zeta, 1\rangle,$$

where $b_\zeta(z) = \frac{(K_\zeta(z))^a}{\|K_\zeta\|^{a-1}}$, $\zeta, z \in \Omega$.

Definition 2.1. *A collection of points $\{\zeta_i\}_{i=1}^\infty$ in Ω is called an η-lattice if the pseudo-hyperbolic balls $\{z : d(z, \zeta_i) < \eta\}$ covers Ω, and the ζ_i are separated in the sense that $d(\zeta_i, \zeta_j) \geq c_0\eta, i \neq j$. Here c_0 is a constant associated with the η-lattice and $d(z, w) = |\phi_z(w)|$.*

Theorem 2.2. *Let $\{\zeta_i\}_{i=1}^\infty$ be an η-lattice with η sufficiently small then for $g \in L^1_a(\Omega)$ (that is, g is analytic on Ω and $\int_\Omega |g(z)| dA(z) < \infty$), there exists $\{\lambda_i\}$ in l^1 such that*

(i) $g = \sum_i \lambda_i b_{\zeta_i}$

(ii) $\sum_i |\lambda_i| \leq \alpha \|g\|_{L^1_a(\Omega)}$ *where α is a constant which depends only on η.*

The proof follows from [3, 8, 15, 16]. ■

In [3], a theorem similar to Theorem 2.2 is proved for the case $g \in L_a^2(\Omega)$, and thus it can be concluded that $\text{span}\{b_{\zeta_i}\}$ is dense in $L_a^1(\Omega)$ and $L_a^2(\Omega)$. Also, it is known that $\|b_\zeta\|_1 \leq c, \zeta \in \Omega$ and c is a constant. For a proof of the last statement see [8].

Theorem 2.3. *Let S be a bounded operator defined on the Bergman space $L_a^2(\Omega)$ such that $ST_z = T_{\bar{z}}S$. Then there exists $\phi \in L^\infty(\Omega)$ such that $S = S_\phi$.*

Proof. Let $M = \text{span}\{b_{\zeta_i}\}$ where $\{\zeta_i\}$ is an η-lattice. Define the linear functional G on M by $G(f) = \langle Sf, 1\rangle$. Note that $M \subset L_a^2(\Omega)$ and hence is contained in $L_a^1(\Omega)$. From Theorem 2.2, given $f \in M$ there exist $\{\lambda_i\}$ in l^1 such that $f = \sum_i \lambda_i b_{\zeta_i}$ with $\sum_i |\lambda_i| \leq \alpha\|f\|_1$. Given $0 < r < 1$, note that $\|b_{\zeta_i}(rz)\|_2 \leq l(r)$. Thus, with $f_r(z) = f(rz)$ we see that

$$\begin{aligned}\langle Sf_r, 1\rangle &= \Big\langle S\big(\sum_i \lambda_i b_{\zeta_i}(rz)\big), 1\Big\rangle = \sum_i \lambda_i \langle S\big(b_{\zeta_i}(rz)\big), 1\rangle \\ &= \sum_i \lambda_i \big\langle S\big(b_{\zeta_i}^{\frac{1}{2}}(rz)\big), \big(b_{\zeta_i}^{\frac{1}{2}}(rz)\big)^+\big\rangle.\end{aligned}$$

Therefore,

$$|\langle Sf_r, 1\rangle| \leq \sum_i |\lambda_i| \|S\| \sup_{\zeta_i} \|b_{\zeta_i}(rz)\|_1.$$

Consequently, it follows that

$$|\langle Sf_r, 1\rangle| \leq \alpha c\|S\|\|f\|_1.$$

But $f_r \to f$ in $L_a^2(\Omega)$. Thus, by the continuity of G it follows that $|G(f)| \leq \beta\|f\|_1$ for some constant β. Since $\text{span}\{b_{\zeta_i}\}$ is dense in $L_a^1(\Omega)$ it follows that G extends by continuity to an element of $(L_a^1(\Omega))^*$, and consequently, by the Hahn Banach theorem to an element of $(L^1(\Omega))^* = L^\infty(\Omega)$. Therefore, there exists $\phi \in L^\infty(\Omega)$, such that

$$\langle Sf, 1\rangle = \langle \phi f, 1\rangle = \langle J(\phi f), 1\rangle = \langle P(J(\phi f)), 1\rangle = \langle S_\phi f, 1\rangle.$$

Moreover, by [3], $\text{span}\{b_{\zeta_i}\}$ is dense in $L_a^2(\Omega)$. Thus, it follows that $\langle S_\phi h, 1\rangle = \langle Sh, 1\rangle, h \in L_a^2(\Omega)$. Using the fact that $\langle Spq^+, 1\rangle = \langle Sp, q\rangle$ where p, q are polynomials in z, it follows that $\langle Sp, q\rangle = \langle S_\phi p, q\rangle$, and hence, $S = S_\phi$, and this ends the proof of the theorem. ■

3. Finite Rank Hankel Operators

Theorem 3.1. *If ϕ is analytic in Ω then $H_{\bar{\phi}}$ is a finite rank big Hankel operator if and only if ϕ is a constant.*

Proof. Sufficiency is obvious. For the necessity, suppose $H_{\bar{\phi}}$ is a finite rank operator where ϕ is analytic on Ω. Then

$$\ker H_{\bar{\phi}} = \{f \in L_a^2(\Omega) : (I - P)(\bar{\phi} f) = 0\} = \{f \in L_a^2(\Omega) : \bar{\phi} f \in L_a^2(\Omega)\}$$

has finite codimension and is invariant under multiplication by z. By the results in [2, 5, 6, 9, 10], there exists a function $G \in L_a^2(\Omega)$ whose zeroes lie in Ω such that $\ker H_{\bar{\phi}} = GL_a^2(\Omega)$. Let $\phi(z) = \sum c_k z^k$; then $\bar{\phi}(z)G(z) \in L_a^2(\Omega)$ implies either that ϕ is a constant or $G = 0$. But $G \neq 0$ since $H_{\bar{\phi}}$ has finite rank, so the result follows. ∎

Arazy, Fisher and Peetre [1] also proved the analogous result for Hankel forms. Theorem 3.3 below describes the kernel of a big Hankel operator. The following theorem establishes that on the Bergman space $L_a^2(\Omega)$ the Toeplitz operator T_ϕ represents a multiplication operator on a closed subspace of $L_a^2(\Omega)$ if and only if the symbol ϕ is analytic.

Theorem 3.2. *Let $\mathcal{S}$ be a subspace of $L^\infty(\Omega)$ such that for $\phi \in S$ there exists a closed subspace $\mathcal{M}$ of $L_a^2(\Omega)$ for which $T_\phi f = \phi f$, for all $f \in \mathcal{M}$. Then $\mathcal{S} \subset H^\infty(\Omega)$.*

Proof. Let f be a fixed element of $\mathcal{M} \subset L_a^2$. Then $T_\phi f = \phi f$. Therefore $\phi(z) = \frac{T_\phi f(z)}{f(z)}$. Hence ϕ is analytic on $\Omega - \{\text{zeroes of } f\}$. Each isolated singularity of ϕ in Ω is removable since ϕ is assumed to be bounded. Thus ϕ is analytic on Ω. Since $\phi \in L^\infty$ hence $\phi \in H^\infty(\Omega)$. ∎

Theorem 3.3. *For $\phi \in L^\infty(\Omega)$ the kernel of a big Hankel operator H_ϕ is trivial.*

Proof. The kernel of the big Hankel operator is given by

$$\ker H_\phi = \{f \in L_a^2 : (I - P)(\phi f) = 0\} = \{f \in L_a^2 : \phi f \in L_a^2\}.$$

Now if $\ker H_\phi \neq \{0\}$ then $\phi \in H^\infty$ by Theorem 3.2. Therefore the operator H_ϕ is zero and $\ker H_\phi = L_a^2(\Omega)$. Thus if $H_\phi \neq 0$ then $\ker H_\phi = \{0\}$. ∎

Definition 3.4. *A function $G \in L_a^2(\Omega)$ ($G \in H^2$) is called an inner function in $L_a^2(\Omega)$ (respectively, H^2) if $|G|^2 - 1$ is orthogonal to $H^\infty(\Omega)$.*

This definition of an inner function in a Bergman space was given by Korenblum [13]. If N is a subspace of $L_a^2(\Omega)$, let $\mathcal{Z}(N) = \{z \in \Omega : f(z) = 0$ for all $f \in N\}$ which is called the zero set of functions in N. Here if z_1 is a zero of multiplicity at most n of all functions in N then z_1 appears n times in the set $\mathcal{Z}(N)$, and each z_1 is treated as a distinct element of $\mathcal{Z}(N)$. With the following result we begin to link the ideas of subspaces and zero-sets.

Proposition 3.5. *If N is a subspace of $L_a^2(\Omega)$ of finite codimension in $L_a^2(\Omega)$ then*

$$\mathcal{Z}(N) = \{z \in \Omega : f(z) = 0 \quad \text{for all} \quad f \in N\}$$

is a finite set.

Proof. Suppose $\mathcal{Z}(N)$ is an infinite set. Let $\{z_j\}_{j=1}^\infty$ be distinct points of $\mathcal{Z}(N)$

and let $f_1, f_2, f_3, \cdots$ be functions in $L^2_a(\Omega)$ such that

$$f_i(z_1) = \cdots = f_i(z_{i-1}) = 0, f_i(z_i) = 1 \quad \text{for all} \quad i \geq 2.$$

For example, we could take the functions (f_i) to be polynomials. Then $f_1, f_2, \cdots$ are linearly independent modulo N, i.e., if

$$\alpha_1 f_1 + \alpha_2 f_2 + \cdots + \alpha_n f_n \in N,$$

where $\alpha_1, \alpha_2, \cdots, \alpha_n \in \mathbb{C}$ then $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$. This contradicts the assumption that N has finite codimension in $L^2_a(\Omega)$ and the result is proved. ∎

When $\mathcal{I}$ is an invariant subspace of finite codimension we can characterise it by its zero set, as the next result shows.

Proposition 3.6. *If $\mathcal{I}$ is an invariant subspace of $L^2_a(\Omega)$ of finite codimension and $\mathcal{Z}(\mathcal{I}) = \{z \in \Omega : f(z) = 0 \ \ for\ all\ f \in \mathcal{I}\}$ then*

$$\mathcal{I} = I(\mathcal{Z}(\mathcal{I})) = \{f \in L^2_a(\Omega) : f(z) = 0 \quad \text{for all} \ z \in \mathcal{Z}(\mathcal{I})\}.$$

Proof. If $g \in \mathcal{I}$ then $g(z) = 0$ for all $z \in \mathcal{Z}(\mathcal{I})$. This implies that $g \in I(\mathcal{Z}(\mathcal{I}))$. If $g \in I(\mathcal{Z}(\mathcal{I}))$ then $g(z) = 0$ for all $z \in \mathcal{Z}(\mathcal{I})$. From [5] and [6] it follows that there exists a rational function $G \in H^\infty(\Omega)$ such that $g = Gh$ for some $h \in L^2_a(\Omega)$ since the set $\mathcal{Z}(\mathcal{I})$ is finite by Proposition 3.5. Furthermore, the function G is in $\mathcal{I}$. Also $g = Gh = \lim_{n\to\infty} Gh_n$ where $\{h_n\}$ is a sequence of functions in R. Now $Gh_n \in \mathcal{I}$ since $\mathcal{I}$ is invariant under multiplication by z and $G \in \mathcal{I}$. Thus $\{Gh_n\}$ is a sequence in $\mathcal{I}$ and $\mathcal{I}$ is closed. Hence $g \in I$. ∎

We are now able to characterize the kernel of a finite rank little Hankel operator in terms of inner functions, in our next result.

Theorem 3.7. *If S_ϕ is a finite rank little Hankel operator on $L^2_a(\Omega)$ then $kerS_\phi = GL^2_a(\Omega)$ for some inner function $G \in L^2_a(\Omega)$ such that the following conditions hold:*

(i) *G vanishes on $\mathbf{a} = \{a_j\}_{j=1}^N$, a finite sequence of points in Ω.*
(ii) $\|G\|_{L^2} = 1$.
(iii) *G is equal to a constant plus a linear combination of the Bergman kernel functions $K(z, a_1), K(z, a_2), \cdots, K(z, a_n)$ and certain of their derivatives.*
(iv) *$|G|^2 - 1$ is orthogonal to $H^\infty(\Omega)$.*

Proof. Since S_ϕ is a little Hankel operator on $L^2_a(\Omega)$ hence

$$T_z^* S_\phi = S_\phi T_z.$$

So $\ker S_\phi$ is invariant under multiplication by z and $\ker S_\phi$ has finite codimension since S_ϕ is of finite rank. Let $\mathbf{a} = \{a_j\}_{j=1}^N$ be the common zeroes of functions in $\ker S_\phi$ i.e., $\mathcal{Z}(\ker S_\phi) = \{a_j\}_{j=1}^N$. Let G be the extremal function for the problem

$$\text{Sup}\{\text{Re} f^{(k)}(0) : f \in L^2_a, \|f\|_{L^2} \leq 1, f = 0 \text{ on } \mathbf{a}\},$$

where k is the multiplicity of the number of times zero appears in $\mathbf{a} = \{a_j\}_{j=1}^N$ ($k = 0$ if $o \notin \{a_j\}_{j=1}^N$). It is clear from [5, 6, 9] that G satisfies the conditions (i)–(iv) and G vanishes precisely on $\mathbf{a}$ in $\overline{\Omega}$ counting multiplicities. Moreover, for every function $f \in L_a^2(\Omega)$ that vanishes on $\mathbf{a} = \{a_j\}_{j=1}^N$ there exists $g \in L_a^2(\Omega)$ such that $f = Gg$. Hence $\ker S_\phi = GL_a^2(\Omega)$. ■

If S_ϕ is of finite rank then $\text{rank} S_\phi$ = number of zeroes of G counting multiplicities. Moreover it is easy to check that $S_\phi = 0$ if and only if $\phi \in (\overline{L_a^2})^\perp$. We can now make the link between inner functions and finite rank Hankel operators, as follows.

Theorem 3.8. *If ψ is in $L^\infty(\Omega)$ and S_ψ is a finite rank little Hankel operator on $L_a^2(\Omega)$ then $\psi = \phi + \chi$ where $\chi \in (\overline{L_a^2})^\perp \cap L^\infty(\Omega)$ and $\bar{\phi}$ is a linear combination of the Bergman kernels and some of their derivatives.*

Proof. Suppose $\psi \in L^\infty(\Omega)$ and S_ψ is a finite rank little Hankel operator. By Theorem 3.7, there exists an inner function $G \in L_a^2(\Omega)$ such that $\ker S_\psi = GL_a^2(\Omega)$. Thus $\psi G \in (\overline{L_a^2})^\perp$. So $\langle \psi G, \bar{h} \rangle = 0$ for all $h \in L_a^2$, that is, $\langle Gh, \bar{\psi} \rangle = 0$ for all $h \in L_a^2$ and so $\bar{\psi} = \bar{\phi} + \bar{\chi}$ where $\bar{\chi} \in (L_a^2)^\perp$, the orthogonal complement of L_a^2 with respect to L^2 and $\bar{\phi} \in (GL_a^2)^\perp$, the orthogonal complement of GL_a^2 with respect to L_a^2. By Theorem 3.7, G vanishes precisely at $\mathbf{a} = \{a_j\}_{j=1}^N$, a finite sequence of points in Ω counting multiplicities. Since $K_{a_1}, K_{a_2}, \cdots, K_{a_N}$ and their derivatives form a basis for $(GL_a^2)^\perp$ (compare [5, 6, 9]), the assertion of the theorem follows. ■

This is the natural analogue of the Hardy space result which was shown by Kronecker [14], that a Hankel operator L is of finite rank if and only if $L = L_{z\bar{u}h}$ with u a finite Blaschke product and $h \in H^\infty$. In this case, the rank of L is not greater than the number of zeroes of u (counted with multiplicity). A similar result also appears in [12] for Hankel forms using different methods and in [4] for little Hankel operators on the Bergman space over $\mathbb{D}$. We use the invariant subspace theory technique, which is a more natural way of looking at this problem. This method may not be applied to certain weighted Bergman spaces [11].

Corollary 3.9. *If S_ϕ is a little Hankel operator on $L_a^2(\Omega)$ and if $\ker S_\phi = \{f \in L_a^2(\Omega) : f = 0$ on $\mathbf{b}\}$, where $\mathbf{b} = \{b_j\}_{j=1}^\infty$ is an infinite sequence of points in Ω, then there exists an inner function $G \in L_a^2(\Omega)$ such that $\ker S_\phi = GL_a^2(\Omega) \cap L_a^2(\Omega)$.*

Proof. The proof follows from the result of [5] and [6] since $\ker S_\phi$ is an invariant subspace of the operator of multiplication by z. ■

It is not known for the Bergman space $L_a^2(\Omega)$ whether the invariant subspaces determined by infinite zero-sets are generated by the corresponding canonical divisors (see [5, 6]). Now let $\mathbf{b} = \{b_j\}_{j=1}^\infty$ be an infinite sequence of points in

Ω. Let $\mathcal{I} = I(\mathbf{b}) = \{f \in L_a^2(\Omega) : f = 0 \text{ on } \mathbf{b}\}$. Let $G_{\mathbf{b}}$ be the solution of the extremal problem

$$\sup\{\mathrm{Re} f^{(n)}(0) : f \in \mathcal{I},\ \|f\|_{L^2} \leq 1\}, \tag{1}$$

where n is the number of times zero appears in the sequence $\mathbf{b}$ (that is, the functions in $\mathcal{I}$ have a common zero of order n at the origin). The natural question that arises at this point is to see if it is possible to construct a Hankel operator S_ϕ whose kernel is $G_{\mathbf{b}} L_a^2 \cap L_a^2$. In the case that $\mathbf{b} = \{b_j\}_{j=1}^N$ is a finite set of points in Ω, it is possible to construct a Hankel operator S_ϕ such that $\ker S_\phi = G_{\mathbf{b}} L_a^2$, as follows.

Theorem 3.10. *If $\mathbf{b} = (b_j)_{j=1}^N$ is a finite set of points in Ω and $\mathcal{I} = I(b) = \{f \in L_a^2(\Omega) : f = 0$ on $\mathbf{b}\}$ and $G_{\mathbf{b}}$ is the solution of the extremal problem* (1)*, then for*

$$\bar{\phi} = \sum_{j=1}^{N} \sum_{\nu=0}^{m_j - 1} c_{j\nu} \frac{\partial^\nu}{\partial \bar{b_j}^{\nu}} K_{b_j}(z),$$

where $c_{j\nu} \neq 0$ for all j, ν and m_j is the number of times b_j appears in $\mathbf{b}$, we have $\ker S_\phi = G_{\mathbf{b}} L_a^2(\Omega)$.

Proof. $\{K_{b_1}, \ldots, \frac{\partial^{m_1-1}}{\partial \bar{b}_1^{m_1-1}} K_{b_1}, \ldots, K_{b_N}, \ldots, \frac{\partial^{m_N-1}}{\partial \bar{b}_N^{m_N-1} K_{b_N}}\}$ forms a basis for $(G_{\mathbf{b}} L_a^2)^\perp$. By the Gram-Schmidt orthogonalization process, we can obtain an orthonormal basis $\{\psi_j\}_{j=1}^l$ for $(G_{\mathbf{b}} L_a^2)^\perp$. If $\bar{\phi} \in (G_{\mathbf{b}} L_a^2)^\perp$, then $\langle \bar{\phi}, G_{\mathbf{b}} t\rangle = 0$ for all $t \in L_a^2$, that is, $\langle \bar{t}, \phi G_{\mathbf{b}}\rangle = 0$ for all $t \in L_a^2$ and so $G_{\mathbf{b}} \in \ker S_\phi$. Since $\ker S_\phi$ is invariant under the operator of multiplication by z, we have

$$G_{\mathbf{b}} L_a^2 \subset \ker S_\phi. \tag{2}$$

Suppose $f \in \ker S_\phi$; then $\langle \phi f, \bar{h}\rangle = 0$ for all $h \in L_a^2$, so in particular, $\langle \phi f, \overline{K_{b_j}}\rangle = 0$ for all $j = 1, 2, \ldots, N$. Therefore, $\overline{\langle \bar{\phi} \bar{f}, K_{b_j}\rangle = 0}$ for all $j = 1, 2, \ldots, N$. Thus $\overline{\phi(b_j) f(b_j)} = 0$ for all $j = 1, 2, \ldots, N$. Since $\overline{\phi(b_j)} \neq 0$ for all $j = 1, 2, \ldots, N$, hence $\overline{f(b_j)} = 0$ for all $j = 1, 2, \ldots, N$. Thus $f \in \mathcal{I}$. Since $G_{\mathbf{b}}$ is the solution of the extremal problem (1), $f \in G_{\mathbf{b}} L_a^2$. Hence

$$\ker S_\phi \subset G_{\mathbf{b}} L_a^2. \tag{3}$$

From (2) and (3), $\ker S_\phi = G_{\mathbf{b}} L_a^2 = \mathcal{I}$ as required. ∎

We have thus shown that there is a strong connection between the theory of symbols for Hankel forms [12] and the theory of inner functions in the Bergman space. That the Bergman kernel and its derivatives should come up in both contexts is now explained and should be of importance in future applications.

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