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# Approximation and Furi–Pera Type Theorems for the S-KKM Class

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**Abstract.** Leray=-Schauder alternatives, Furi–Pera type fixed point and Fan type approximation results are established for multimaps in the S-KKM class.

## 1. Introduction

This paper discusses new fixed point results of Furi–Pera type [7] for the S-KKM multimaps. In the process we establish new Leray–Schauder alternatives for S-KKM multimaps defined on closed, convex, normal subsets of locally convex Hausdorff topological vector spaces. We also obtain several aproximation results of Fan type. Furi–Pera type results were given for $\mathfrak{A}_c^k$ and other classes in [2, 9, 10]. On the other hand, Fan type approximation [6] results were proved for $\mathfrak{A}_c^k$ and other classes in [8, 13, 14]. As a generalization of the $\mathfrak{A}_c^k$ class, the class of S-KKM maps was introduced and studied by Chang et al. [4] and further investigated by Agarwal and O'Regan [1], Chang et al. [3], O'Regan, Shahzad and Agarwal [12], and Shahzad [17]. Our results extend and complement those in [1 - 4, 7 - 10, 13, 14].

## 2. Preliminaries

Let $E$ and $E_1$ be Hausdorff topological vector spaces. Recall a polytope $P$ in $E_1$

is any convex hull of a nonempty finite subset of $E_1$. Given a class $\mathcal{X}$ of maps, $\mathcal{X}(E_1, E)$ denotes the set of maps $F : E_1 \to 2^E$ (the nonempty subsets of $E$) belonging to $\mathcal{X}$, and $\mathcal{X}_c$ the set of finite compositions of maps in $\mathcal{X}$. A class $\mathfrak{A}$ of maps [15, 16] is defined by the following properties:

(i) $\mathfrak{A}$ contains the class $\mathcal{C}$ of single valued continuous functions;
(ii) each $F \in \mathfrak{A}_c$ is upper semicontinuous and compact valued; and
(iii) for any polytope $P$, $F \in \mathfrak{A}_c(P, P)$ has a fixed point, where the intermediate spaces of composites are suitably chosen for each $\mathfrak{A}$.

**Definition 2.1.** *$F \in \mathfrak{A}_c^k(E_1, E)$ (i.e. $F$ is $\mathfrak{A}_c^k$–admissible ) if for any compact subset $K$ of $E_1$, there is a $G \in \mathfrak{A}_c(K, E)$ with $G(x) \subseteq F(x)$ for each $x \in K$.*

**Definition 2.2.** *Let $X$ be a convex subset of a Hausdorff topological vector space and $Y$ a topological space. If $S, T : X \to 2^Y$ are two set-valued maps such that $T(co(A)) \subseteq S(A)$ for each finite subset $A$ of $X$, then we say that $S$ is a generalized KKM map w.r.t. $T$. The map $T : X \to 2^Y$ is said to have the KKM property if for any generalized KKM w.r.t. $T$ map $S$, the family*

$$\{\overline{S(x)} : x \in X\}$$

*has the finite intersection property. We let*

$$KKM(X, Y) = \{T : X \to 2^Y : T \text{ has the KKM property}\}.$$

*Remark 2.1.* If $X$ is a convex space, then $\mathfrak{A}_c^\kappa(X, Y) \subset \mathrm{KKM}(X, Y)$ (see [5]).

**Definition 2.3.** *Let $X$ be a nonempty set, $Y$ a nonempty convex subset of a Hausdorff topological vector space and $Z$ a topological space. If $S : X \to 2^Y$, $T : Y \to 2^Z$, $F : X \to 2^Z$ are three set-valued maps such that $T(co(S(A))) \subseteq F(A)$ for each nonempty finite subset $A$ of $X$, then $F$ is called a generalized $S$-KKM map w.r.t. $T$. If the map $T : X \to 2^Z$ is such that for any generalized $S$-KKM w.r.t. $T$ map $F$, the family*

$$\{\overline{F(x)} : x \in X\}$$

*has the finite intersection property, then $T$ is said to have the $S$-KKM property. The class*

$$]dis\ S\text{-}KKM(X, Y, Z) = \{T : Y \to 2^Z : T \text{ has the } S\text{-KKM property}\}.$$

*Remark 2.2.* Note that $S$-KKM$(X, Y, Z) =$ KKM$(X, Z)$ whenever $X = Y$ and $S$ is the identity mapping $\mathbf{1}_X$. Moreover, KKM$(Y, Z)$ is contained in $S$-KKM $(X, Y, Z)$ for any $S : X \to 2^Y$ and usually this inclusion is proper (see [4, p. 214-215]). $S$-KKM$(X, Y, Z)$ also includes other important classes of multimaps (see [3, 4] for examples).

*Remark 2.3.* Let $X$ be a convex space, $Y$ a convex subset of a Hausdorff locally convex space, and $Z$ a normal space. Suppose $s : Y \to Y$ is surjective, $F \in s$-KKM$(Y, Y, Z)$ is closed, and $f \in \mathcal{C}(X, Y)$. Then $Ff \in \mathbf{1}_X$-KKM$(X, X, Z)$ (see [4]).

*Remark 2.4.* Let $X$ be a convex subset of a Hausdorff topological space, $Y$ a convex space, and $Z$, $W$ topological spaces and $S : X \to 2^Y$. If $F \in S$-KKM$(X, Y, Z)$ and $f \in \mathcal{C}(Z, W)$, then $f \circ F \in$ S-KKM$(X, Y, W)$ (see [4]).

Let $(E, d)$ be a pseudometric space. For any $C \subseteq E$, let $B(C, \epsilon) = \{x \in E : d(x, C) \leq \epsilon\}$, here $\epsilon > 0$ The measure of noncompactness of the set $M \subseteq E$ is defined by $\alpha(M) = \inf Q(M)$, where

$$Q(M) = \{\epsilon > 0 : M \subseteq B(A, \epsilon) \text{ for some finite subset } A \text{ of } E\}.$$

Let $C$ be a subset of a locally convex Hausdorff topological vector space $E$, and let $\mathcal{P}$ be a defining system of seminorms on $E$. Suppose $F : C \to 2^E$. Then $F$ is called countably $\mathcal{P}$-concentrative mapping if $F(C)$ is bounded, and for $p \in \mathcal{P}$ and each countably bounded subset $S$ of $C$, we have $\alpha_p(F(S)) \leq \alpha_p(S)$, and for $p \in \mathcal{P}$ for each countably bounded non- $p$-precompact subset $S$ of $C$ (i.e., $S$ is not precompact in the pseudonormed space $(E, p)$), we have $\alpha_p(F(S)) < \alpha_p(S)$; here $\alpha_p(.)$ denotes the measure of noncompactness in the pseudonormed space $(E, p)$.

Let $Q$ be a subset of a Hausdorff topological space $X$. We let $\overline{Q}$ (respectively, $\partial Q$, $\text{int}(Q)$) denote the closure (respectively, boundary, interior) of $Q$.

Let $C$ be a subset of a Hausdorff topological vector space $E$ and $x \in X$. Then the inward set $I_C(x)$ is defined by

$$I_C(x) = \{x + r(y - x) : y \in C, r \geq 0\}.$$

If $C$ is convex and $x \in C$, then

$$I_C(x) = x + \{r(y - x) : y \in C, r \geq 1\}.$$

The following results [1, 10] will be needed in the sequel.

**Theorem 2.1.** *Let $\Omega$ be a closed, convex subset of a locally convex Hausdorff topological vector space $E$ with $x_0 \in \Omega$ and $s : \Omega \to \Omega$ is surjective. Suppose $F \in s$-KKM$(\Omega, \Omega, \Omega)$ is closed with the following property holding:*

$$A \subseteq \Omega, A = \overline{co}(\{x_0\} \cup F(A)) \text{ implies } A \text{ is compact.} \tag{2.1}$$

*Then $F$ has a fixed point in $\Omega$.*

**Theorem 2.2.** *Let $\Omega$ be a closed, convex subset of a locally convex Hausdorff topological vector space $E$ and $x_0 \in \Omega$ and $s : \Omega \to \Omega$ is surjective. Suppose $F \in s$-KKM$(\Omega, \Omega, \Omega)$ is closed and satisfies the following properties:*

$$A \subseteq \Omega, A = co(\{x_0\} \cup F(A)) \text{ implies } \overline{A} \text{ is compact} \tag{2.2}$$

*and*

$$F(\overline{A}) \subseteq \overline{F(A)} \text{ for any relatively compact subset } A \text{ of } \Omega. \tag{2.3}$$

*Then $F$ has a fixed point in $\Omega$.*

**Theorem 2.3.** *Let $\Omega$ be a closed, convex subset of a locally convex Hausdorff topological vector space $E$ and $x_0 \in \Omega$ and $s : \Omega \to \Omega$ is surjective. Suppose*

*$F \in$ s-KKM$(\Omega, \Omega, \Omega)$ is closed, maps compact sets into relatively compact sets, satisfies* (2.3) *and suppose the following properties hold:*

$$\begin{cases} A \subseteq \Omega, A = co(\{x_0\} \cup F(A)) \text{ with } \overline{A} = \overline{C} \\ \text{and } C \subseteq A \text{ countable, implies } \overline{A} \text{ is compact,} \end{cases} \tag{2.4}$$

$$\begin{cases} \text{for any relatively compact subset } A \text{ of } \Omega, \text{ there} \\ \text{exists a countable set } B \subseteq A \text{ with } \overline{B} = \overline{A}, \end{cases} \tag{2.5}$$

*and*

$$\text{if } A \text{ is a compact subset of } \Omega, \text{ then } \overline{co}(A) \text{ is compact.} \tag{2.6}$$

*Then $F$ has a fixed point in $\Omega$.*

*Remark 2.5.* It is worth noting that if $E$ is metrizable, then (2.5) holds, and if $E$ is quasicomplete then (2.6) holds.

*Remark 2.6.* Following arguments similar to those given in [11, Theorem 2.3], one can remove (2.3) in Theorem 2.2 and Theorem 2.3 for certain subclasses of $s$-KKM maps.

**Theorem 2.4.** *Let $\Omega$ be a nonempty, closed, convex subset of a Fréchet space $E$ ($\mathcal{P}$ is a defining system of seminorms) and suppose $F \in KKM(\Omega, \Omega)$ is a closed, countably $\mathcal{P}$-concentrative map. Then $F$ has a fixed point in $\Omega$.*

## 3. Main Results

Our first results are nonlinear alternatives of Leray–Schauder type for maps in the s-KKM class.

Let $C$ be a convex subset of a Banach space $E$ with $0 \in \text{int}(C)$. The Minkowski functional $\mu : E \to [0, \infty)$ of $C$ is defined by

$$\mu(x) = \inf\{r > 0 : x \in rC\}.$$

The following properties of the Minkowski functional are well known:
(i) $\mu$ is continuous on $E$;
(ii) $\mu(x + y) \leq \mu(x) + \mu(y), x, y \in E$;
(iii) $\mu(\lambda x) = \lambda\mu(x), \lambda \geq 0, x \in E$;
(iv) $0 \leq \mu(x) < 1$ if $x \in \text{int}(C)$;
(v) $\mu(x) > 1,$ if $x \notin \overline{C}$;
(vi) $\mu(x) = 1,$ if $x \in \partial C$.

**Theorem 3.1.** *Let $E$ be a locally convex Hausdorff topological vector space, $C$ a closed, convex, normal subset of $E$, $U \subseteq C$ an open convex subset of $E$, and $0 \in U$. Suppose $s : \overline{U} \to \overline{U}$ is surjective and $F \in$ s-KKM$(\overline{U}, \overline{U}, C)$ is closed and satisfies the following two properties:*

$$A \subseteq C, A \subseteq \overline{co}(\{0\} \cup F(co(\{0\} \cup A))) \text{ implies } \overline{A} \text{ is compact} \tag{3.1}$$

*and*

$$x \notin \lambda F(x) \text{ for } x \in \partial U \text{ and } \lambda \in (0,1). \tag{3.2}$$

*Then $F$ has a fixed point in $\overline{U}$.*

*Proof.* Let $\mu$ be the Minkowski functional on $\overline{U}$ and let $r : E \to \overline{U}$ be given by

$$r(x) = \frac{x}{\max\{1, \mu(x)\}} \text{ for } x \in E. \tag{3.3}$$

Let $G = Fr$. Then, by Remark 2.3, $G \in \mathbf{1}_C$-KKM$(C, C, C)$. Furthermore $G$ is closed. Next we claim

$$\text{if } A \subseteq C \text{ and } A \subseteq \overline{co}(\{0\} \cup G(A)), \text{ then } \overline{A} \text{ is compact.} \tag{3.4}$$

To see this notice if $A \subseteq C$ and $A \subseteq \overline{co}(\{0\} \cup Fr(A))$, then since $r(A) \subseteq co(\{0\} \cup A)$ we have

$$A \subseteq \overline{co}(\{0\} \cup F(co(\{0\} \cup A))).$$

Now (3.1) implies $\overline{A}$ is compact, so (3.4) holds. Now Theorem 2.1 guarantees that there exists $x \in C$ with $x \in Fr(x)$. If we let $z = r(x) \in \overline{U}$, then $z \in rF(z)$. Thus $z = r(y)$ for some $y \in F(z)$. Now either $y \in \overline{U}$ or $y \notin \overline{U}$. If $y \in \overline{U}$, then $r(y) = y$ so $z = r(y) = y \in F(z)$, and we are finished. If $y \notin \overline{U}$, then $r(y) = \lambda y$ with $\lambda = \frac{1}{\mu(y)} \in (0,1)$. Thus $z = r(y) = \lambda y \in \lambda F(z)$ with $\lambda \in (0,1)$. This contradicts (3.2). ∎

**Theorem 3.2.** *Let $E$ be a locally convex Hausdorff topological vector space, $C$ a closed, convex, normal subset of $E$, $U \subseteq C$ an open convex subset of $E$, and $0 \in U$. Suppose $s : \overline{U} \to \overline{U}$ is surjective and $F \in s$-KKM$(\overline{U}, \overline{U}, C)$ is closed, satisfies* (3.2)*, and the following properties:*

$$A \subseteq C, A \subseteq co(\{0\} \cup F(co(\{0\} \cup A))) \text{ implies } \overline{A} \text{ is compact} \tag{3.5}$$

*and*

$$F(\overline{A}) \subseteq \overline{F(A)} \text{ for any relatively compact subset } A \text{ of } \overline{U}. \tag{3.6}$$

*Then $F$ has a fixed point in $\overline{U}$.*

*Proof.* Let $\mu$, $r$, $G$ be as in Theorem 3.1. Essentially the same reasoning as in Theorem 3.1 guarantees that $G \in \mathbf{1}_C$-KKM$(C, C, C)$ is closed with

$$\text{if } A \subseteq C \text{ and } A \subseteq co(\{0\} \cup G(A)), \text{ then } \overline{A} \text{ is compact.} \tag{3.7}$$

Next we claim

$$G(\overline{A}) \subseteq \overline{G(A)} \text{ for any relatively compact subset } A \text{ of } C. \tag{3.8}$$

This is immediate since $F$ satisfies (3.6) and $r$ is continuous i.e. if $A$ is a relatively compact subset of $C$, then

$$G(\overline{A}) = F(r(\overline{A})) \subseteq F(\overline{r(A)}) \subseteq \overline{F(r(A))} = \overline{G(A)}.$$

Now Theorem 2.2 guarantees that there exists $x \in C$ with $x \in G(x) = Fr(x)$. If we let $z = r(x) \in \overline{U}$, then $z \in rF(z)$. Thus $z = r(y)$ for some $y \in F(z)$ and as in Theorem 3.1 we have $y \in \overline{U}$. Consequently we have $z = r(y) = y \in F(z)$. ■

Essentially the same reasoning as in Theorem 3.2 (except here we use Theorem 2.3) establishes the following result.

**Theorem 3.3.** *Let $E$ be a locally convex Hausdorff topological vector space, $C$ a closed, convex, normal subset of $E$, $U \subseteq C$ an open convex subset of $E$, and $0 \in U$. Suppose $s : \overline{U} \to \overline{U}$ is surjective and $F \in s\text{-}KKM(\overline{U}, \overline{U}, C)$ is closed, maps compact sets into relatively compact sets. Assume* (3.2) *and* (3.6) *are satisfied and also suppose the following properties hold:*

$$\begin{cases} A \subseteq C, A \subseteq co(\{0\} \cup F(co(\{0\} \cup A))) \text{ with } \overline{A} = \overline{Q} \\ \text{and } Q \subseteq A \text{ countable, implies } \overline{A} \text{ is compact,} \end{cases} \tag{3.9}$$

$$\begin{cases} \text{for any relatively compact subset } A \text{ of } C, \text{ there} \\ \text{exists a countable set } B \subseteq A \text{ with } \overline{B} = \overline{A}, \end{cases} \tag{3.10}$$

*and*

$$\text{if } A \text{ is a compact subset of } C, \text{ then } \overline{co}(A) \text{ is compact.} \tag{3.11}$$

*Then $F$ has a fixed point in $\overline{U}$.*

**Theorem 3.4.** *Let $E$ be a Fréchet space, $C$ a closed, convex, bounded subset of $E$, $U \subseteq C$ an open convex subset of $E$, and $0 \in U$. Suppose $s : \overline{U} \to \overline{U}$ is surjective and $F \in s\text{-}KKM(\overline{U}, \overline{U}, C)$ is a closed countably $\mathcal{P}$-concentrative map and satisfies* (3.2)*. Then $F$ has a fixed point in $\overline{U}$.*

*Proof.* Let $\mu$, $r$, $G$ be as in Theorem 3.1. Essentially the same reasoning as in Theorem 3.1 guarantees that $G \in \mathbf{1}_C\text{-KKM}(C, C, C)$ is closed. Next we show that $G$ is countably $\mathcal{P}$-concentrative. Let $p \in \mathcal{P}$ and $\Omega$ be a countably bounded non-precompact subset of $C$. Then since

$$G(\Omega) \subseteq Fr(\Omega) \subseteq F(co(\Omega \cup \{0\}))$$

we have $\alpha_p(G(\Omega)) < \alpha_p(\Omega)$. Now Theorem 2.4 guarantees that there exists $x \in C$ with $x \in G(x) = Fr(x)$. If we let $z = r(x) \in \overline{U}$, then $z \in rF(z)$ and so $z = r(y)$ for some $y \in F(z)$. As in Theorem 3.1 we have $y \in \overline{U}$. Thus $z = r(y) = y \in F(z)$. ■

Our next result is a Furi–Pera type fixed point theorem for compact, closed $s$-KKM maps.

**Theorem 3.5.** *Let $Q$ be a closed convex subset of a metrizable locally convex topological vector space $E$ with $0 \in Q$. Suppose $s : Q \to Q$ is surjective and $F \in s\text{-}KKM(Q, Q, E)$ is a closed, compact (i.e., $F(Q)$ is a subset of a relatively compact subset of $Q$) map with the following condition satisfied:*

$$\begin{cases} \textit{if } \{(x_j, \lambda_j)\}_{j=1}^{\infty} \textit{ is a sequence in } \partial Q \times [0,1] \textit{ converging} \\ \textit{to } (x, \lambda) \textit{ with } x \in \lambda F(x) \textit{ and } 0 \le \lambda < 1, \textit{ then} \\ \{\lambda_j F(x_j)\} \subseteq Q \textit{ for } j \textit{ sufficiently large.} \end{cases} \tag{3.12}$$

*Then $F$ has a fixed point in $Q$.*

*Proof.* Let $r : E \to Q$ be a continuous retraction (the existence of $r$ follows from Dugundji's extension theorem).

*Remark 3.1.* If $0 \in \text{int}Q$, we may take

$$r(x) = \frac{x}{\max\{1, \mu(x)\}}, \quad x \in E,$$

where $\mu$ is the Minkowski functional on $Q$. Note if $\text{int}Q = \emptyset$, then $\partial Q = Q$.

From Remark 3.1, we may choose (and we do so) the retraction $r$ above so that $r(z) \in \partial Q$ if $z \in E \backslash Q$. Consider

$$B = \{x \in E : x \in Fr(x)\}.$$

Firstly $B \neq \emptyset$. To see this notice $Fr \in \mathbf{1}_E\text{-KKM}(E, E, E)$ is closed and also notice $Fr$ is a compact map ( $r$ is continuous and $F$ is compact). Now [4, Theorem 3.2] guarantees that $Fr$ has a fixed point and so $B \neq \emptyset$. In addition since $F$ is closed we have that $B$ is closed. In fact $B$ is compact since

$$B \subseteq Fr(B) \subseteq F(Q).$$

We claim that $B \cap Q \neq \emptyset$. Suppose not. Then since $B$ is compact and $Q$ is closed there exists a $\delta > 0$ with $\text{dist}(B, Q) > \delta$. Choose $m \in \{1, 2, ...\}$ with $1 < \delta m$. Define

$$U_i = \left\{ x \in E : d(x, Q) < \frac{1}{i} \right\} \text{ for } i \in \{m, m+1, ....\},$$

here $d$ is the metric associated with $E$ [18, p. 18, 29]. Fix $i \in \{m, m+1, ...\}$. Since $\text{dist}(B, Q) > \delta$ then we see that $B \cap \overline{U_i} = \emptyset$. In addition $U_i$ is open and convex (follows from the fact that open balls are convex), $0 \in U_i$ and $Fr \in \mathbf{1}_{\overline{U_i}}$-KKM$(\overline{U_i}, \overline{U_i}, E)$ is a closed, compact map. Now Theorem 3.1 guarantees, since $B \cap \overline{U_i} = \emptyset$, that there exists

$$(y_i, \lambda_i) \in \partial U_i \times (0, 1) \text{ with } y_i \in \lambda_i Fr(y_i).$$

We can do this for each $i \in \{m, m+1, ....\}$. Thus we have

$$\{\lambda_i Fr(y_i)\} \not\subseteq Q \text{ for each } i \in \{m, m+1, ...\}. \tag{3.13}$$

We now look at

$$D = \{x \in E : x \in \lambda Fr(x) \text{ for some } \lambda \in [0, 1]\}.$$

Notice $D \neq \emptyset$ is closed and in fact compact (so sequentially compact) since $F$ is a compact map. This together with

$$d(y_j, Q) = \frac{1}{j} \text{ and } |\lambda_j| \le 1 \text{ for } j \in \{m, m+1, ...\}$$

implies that we may assume without loss of generality that

$$\lambda_j \to \lambda^\star \in [0,1] \text{ and } y_j \to y^\star \in \partial Q.$$

In addition we have $y_j \in \lambda_j Fr(y_j)$ with $F$ closed, and so $y^\star \in \lambda^\star Fr(y^\star)$. Note that $\lambda^\star \neq 1$ since $B \cap Q = \emptyset$. Hence $0 \le \lambda^\star < 1$. However (3.12) with

$$x_j = r(y_j) \in \partial Q \text{ and } x = y^\star = r(y^\star)$$

implies $\{\lambda_j Fr(y_j)\} \subseteq Q$ for $j$ sufficiently large. This contradicts (3.13). Thus $B \cap Q \neq \emptyset$ so there exists $x \in Q$ with $x \in Fr(x) = F(x)$. ∎

Theorem 3.6 can be further improved if $E$ is a Hilbert space. In the following result, by a countably condensing map $F$ we mean $\alpha(F(B)) < \alpha(B)$ for all countably bounded sets $B$ of $C$ with $\alpha(B) \neq 0$ and $\alpha(F(D)) \le \alpha(D)$ for all countably bounded sets $D$ of $C$, where $\alpha(.)$ is the Kuratowski measure of noncompactness (see [19]).

**Theorem 3.6.** *Let $Q$ be a closed convex subset of a Hilbert space $E$ with $0 \in Q$. Suppose $s : Q \to Q$ is surjective and $F \in s$-KKM$(Q, Q, E)$ is a closed, countably condensing map with $F(Q)$ a bounded subset of $E$ and assume* (3.12) *holds. Then $F$ has a fixed point in $Q$.*

*Proof.* Define $r : E \to Q$ by $r(x) = P_Q(x)$ i.e. $r$ is the nearest point projection. Note $r$ is nonexpansive. Let $C_0 = \overline{co}(F(Q) \cup \{0\})$. Then $C_0$ is bounded. Now let

$$B = \{x \in E : x \in Fr(x)\}.$$

Then $Fr \in \mathbf{1}_{C_0}$-KKM$(C_0, C_0, C_0)$. Next we show $Fr$ is a countably condensing map. Let $\Omega$ be any countable subset of $C_0$ with $\alpha(\Omega) \neq 0$. Clearly $r(\Omega)$ is countable. Consequently, if $\alpha(r(\Omega)) \neq 0$, then

$$\alpha(Fr(\Omega)) < \alpha(r(\Omega)) \le \alpha(\Omega),$$

whereas if $\alpha(r(\Omega)) = 0$, then

$$\alpha(Fr(\Omega)) < \alpha(r(\Omega)) = 0 < \alpha(\Omega).$$

Now Theorem 2.4 guarantees that $Fr$ has a fixed point so $B \neq \emptyset$. Also $B$ is closed. Next we show $B$ is compact. Indeed, if $\{y_n\}_1^\infty$ is any sequence in $B$, then $y_n \in Fr(y_n)$, $n = 0, 1, ....$ Let $C = \{y_n\}_1^\infty$. Then $C$ is countable with $C \subseteq \overline{co}(Fr(C) \cup \{0\})$. Suppose $\alpha(C) \neq 0$. Then

$$\alpha(C) \le \alpha(Fr(C))) < \alpha(r(C)) \le \alpha(C), \text{ if } \alpha(r(C)) \neq 0,$$

whereas

$$\alpha(C) \le \alpha(Fr(C))) \le \alpha(r(C)) = 0 < \alpha(C), \text{ if } \alpha(r(C)) = 0.$$

We have a contradiction in both cases. Therefore, $\alpha(C) = 0$ and so $\overline{C}$ is compact. As a result, there exists a subsequence $N$ of $\{1, 2, ...\}$ and a $y \in \overline{C}$ such that $y_n \to y$ as $n \to \infty$ in $N$. The closedness of $F$ further implies that $y \in Fr(y)$ and so $y \in B = \overline{B}$. As a result $B$ is compact. Suppose now $B \cap Q = \emptyset$. Let $m$, $U_i$ be as in Theorem 3.5 and essentially the same reasoning as in Theorem 3.5 (since $Fr \in \mathbf{1}_{\overline{U_i}}$-KKM$(\overline{U_i}, \overline{U_i}, E)$ is a countably condensing map)(here we have used Theorem 3.4) guarantees that

$$\{\lambda_i Fr(y_i)\} \not\subseteq Q \text{ for each } i \in \{m, m+1, ....\}. \tag{3.14}$$

Let $D$ be as in Theorem 3.6 and notice $D$ is closed. We claim that $D$ is compact. Indeed, let $\{x_n\}_1^\infty$ be any sequence in $D$. Then there exists a sequence $\{\lambda_n\}_1^\infty$ of $[0, 1]$ with $x_n \in \lambda_n Fr(x_n)$, $n = 0, 1, ....$ We may assume, without loss of generality, that $\lambda_n \to \lambda \in [0, 1]$. Let $C = \{x_n\}_1^\infty$. Then $C$ is countable with $C \subseteq \overline{co}(Fr(C) \cup \{0\})$. Consequently, if $\alpha(C) \neq 0$ then

$$\alpha(C) \leq \alpha(Fr(C)) < \alpha(r(C)) \leq \alpha(C), \text{ if } \alpha(r(C)) \neq 0,$$

whereas

$$\alpha(C) \leq \alpha(Fr(C)) \leq \alpha(r(C)) = 0 < \alpha(C), \text{ if } \alpha(r(C)) = 0.$$

This gives that $\alpha(C) = 0$ and so $\overline{C}$ is compact. Consequently, there exists a subsequence $N$ of $\{1, 2, ...\}$ and a $x \in \overline{C}$ such that $x_n \to x$ as $n \to \infty$ in $N$. Since $x_n \in \lambda_n Fr(x_n)$, $n \in N$, by the closedness of $F$, we have $x \in \lambda Fr(x)$ and so $y \in D = \overline{D}$. Thus $D$ is compact.

As a result we may assume without loss of generality that

$$\lambda_j \to \lambda^\star \in [0, 1] \text{ and } y_j \to y^\star \in \partial Q.$$

This implies $y^\star \in \lambda^\star Fr(y^\star)$ with $0 \leq \lambda^\star < 1$. This together with (3.12) implies $\{\lambda_j Fr(y_j)\} \subseteq Q$ for $j$ sufficiently large. This contradicts (3.14) and so $B \cap Q \neq \emptyset$. Hence there exists $x \in Q$ with $x \in Fr(x) = F(x)$ ∎

**Theorem 3.7.** *Let $Q$ be a closed convex subset of a Hilbert space $E$ with $0 \in Q$. Suppose $s : Q \to Q$ is surjective and $F \in s$-KKM$(Q, Q, E)$ is a closed, countably 1-set-contractive map with $F(Q)$ a bounded subset of $E$ and assume* (3.12) *holds. In addition, suppose*

$$\begin{cases} \text{if } \{x_n\}_{j=1}^\infty \text{ is a sequence in } Q \text{ with } y_n \in F(x_n) \text{ for all } n \text{ and} \\ x_n - y_n \to 0 \text{ as } n \to \infty, \text{ then there exists } x \in Q \text{ with } x \in F(x). \end{cases} \tag{3.15}$$

*Then $F$ has a fixed point in $Q$.*

*Proof.* For each $n \in \{2, 3, ...\}$, let $T_n$ be defined by

$$T_n(x) = \left(1 - \frac{1}{n}\right) F(x).$$

Consider the mapping $g_n(y) = \left(1 - \frac{1}{n}\right) y$. Then each $g_n$ is continuous. Since $F \in s$-KKM$(Q, Q, E)$, we have $T_n = g_n F \in s$-KKM$(Q, Q, E)$ (by Remark 2.4). Since $F$ is closed, each $T_n$ is closed. Moreover, each $T_n$ is countably condensing.

Let $\{(x_j, \lambda_j)\}_{j=1}^{\infty}$ be a sequence in $\partial C \times [0,1]$ converging to $(x, \lambda)$ with $x \in \lambda T_n(x)$ and $0 \leq \lambda < 1$. Then

$$\lambda_j T_n(x_j) = \lambda_j \Big(1 - \frac{1}{n}\Big) F(x_j) = \mu_j F(x_j),$$

for $j$ sufficiently large, where $\mu_j = \lambda_j \big(1 - \frac{1}{n}\big)$ is a sequence in $[0,1]$ with $\mu_j \to \lambda\left(1 - \frac{1}{n}\right) = \mu$, $0 \leq \mu < 1$, and $x \in \lambda T_n(x) = \mu F(x)$. By Theorem 3.6, each $T_n$ has a fixed point $x_n \in Q$, i.e., $x_n \in \big(1 - \frac{1}{n}\big) F(x_n)$ for each $n$. Choose $y_n \in F(x_n)$ with $x_n = \big(1 - \frac{1}{n}\big) y_n$. It further implies that $x_n - y_n = -\frac{1}{n} y_n \to 0$ as $F(Q)$ is bounded. By (3.15), there exists an $x_0 \in Q$ with $x_0 \in F(x_0)$. ∎

Next, we prove some Fan type approximation and fixed point theorems for $S$-KKM maps.

For $x \in E$, let

$$d_\mu(x, C) = \inf\{\mu(x - y) : y \in C\}.$$

Let $r : E \to C$ be defined by

$$r(x) = \begin{cases} x & \text{if } x \in C \\ \frac{x}{\mu(x)} & \text{if } x \notin C. \end{cases}$$

**Theorem 3.8.** *Let $C$ be a closed, convex subset of a Banach space $E$ with $0 \in \operatorname{int}(C)$. Suppose $s : C \to C$ is surjective and $F \in s$-KKM$(C, C, E)$ is a closed map satisfying the following property:*

$$A \subseteq C, A \subseteq \overline{co}(\{0\} \cup F(co(\{0\} \cup A))) \text{ implies } \overline{A} \text{ is compact.}$$

*Then there exist $x_0 \in C$ and $y_0$ such that $y_0 \in F(x_0)$ and*

$$\mu(y_0 - x_0) = d_\mu(y_0, C) = d_\mu(y_0, \overline{I_C(x_0)});$$

*here $\mu$ is the Minkowski functional of $C$ in $E$. More precisely, either (i). $F$ has a fixed point $x_0 \in C$, or (ii). there exist $x_0 \in \partial C$ and $y_0 \in F(x_0)$ with*

$$0 < \mu(y_0 - x_0) = d_\mu(y_0, C) = d_\mu(y_0, \overline{I_C(x_0)}).$$

*Proof.* Let $r : E \to C$ be as defined above. Then $r$ is continuous and

$$r(A) \subseteq co(A \cup \{0\})$$

for each bounded subset $A$ of $C$. Let $G = Fr$. Essentially the same reasoning as in Theorem 3.1 guarantees that $G \in \mathbf{1}_E$-KKM$(E, E, E)$ is closed with

$$\text{if } A \subseteq E \text{ and } A \subseteq co(\{0\} \cup G(A)), \text{ then } \overline{A} \text{ is compact.}$$

Now Theorem 2.1 guarantees that there exists a $z_0 \in E$ such that $z_0 \in Fr(z_0)$. If we let $x_0 = r(z_0) \in C$, then $x_0 \in rF(x_0)$. Thus there exists some $y_0 \in F(x_0)$ such that $x_0 = r(y_0)$. Now we consider two cases: $y_0 \in C$ or $y_0 \notin C$.

If $y_0 \in C$, then $x_0 = r(y_0) = y_0$. Therefore

$$\mu(y_0 - x_0) = 0 = d_\mu(y_0, C)$$

and $x_0$ is a fixed point of $F$. On the other hand, if $y_0 \notin C$, then

$$x_0 = r(y_0) = \frac{y_0}{\mu(y_0)}.$$

As a result, for any $x \in C$,

$$\begin{aligned}\mu(y_0 - x_0) &= \mu\Big(y_0 - \frac{y_0}{\mu(y_0)}\Big) = \Big(\frac{\mu(y_0) - 1}{\mu(y_0)}\Big)\mu(y_0)\\ &= \mu(y_0) - 1 \le \mu(y_0 - x).\end{aligned}$$

Consequently $\mu(y_0 - x_0) \le d_\mu(y_0, C)$ and so $\mu(y_0 - x_0) = d_\mu(y_0, C)$. Moreover, $\mu(y_0 - x_0) > 0$ since $\mu(y_0 - x_0) = \mu(y_0) - 1$.

It remains to show $\mu(y_0 - x_0) = d_\mu(y_0, \overline{I_C(x_0)})$. Let $z \in I_C(x_0) \setminus C$. Then there exists $y \in C$ and $c \ge 1$ with $z = x_0 + c(y - x_0)$. Suppose that

$$\mu(y_0 - z) < \mu(y_0 - x_0).$$

The convexity of $C$ implies that $\frac{1}{c}z + (1 - \frac{1}{c})x_0 \in C$. Now

$$\begin{aligned}\mu(y_0 - y) &= \mu[\frac{1}{c}(y_0 - z) + (1 - \frac{1}{c})(y_0 - x_0)]\\ &\le \frac{1}{c}\mu(y_0 - z) + (1 - \frac{1}{c})\mu(y_0 - x_0)\\ &< \mu(y_0 - x_0).\end{aligned}$$

This contradicts the choice of $y_0$. Thus we have

$$\mu(y_0 - x_0) \le \mu(y_0 - z) \text{ for all } z \in I_C(x_0).$$

Since $\mu$ is continuous, we have

$$\mu(y_0 - x_0) \le \mu(y_0 - z) \text{ for all } z \in \overline{I_C(x_0)}.$$

Hence

$$0 < \mu(y_0 - x_0) = d_\mu(y_0, C) = d_\mu(y_0, \overline{I_C(x_0)}).$$

Suppose $x_0 \in \operatorname{int}(C)$. Then $\overline{I_C(x_0)} = E$, which implies $d_\mu(y_0, \overline{I_C(x_0)}) = 0$. This shows that $x_0 \in \partial C$. ∎

Essentially the same reasoning as in Theorem 3.8 (except here we use Theorem 2.2) establishes the following result.

**Theorem 3.9.** *Let $C$ be a closed, convex subset of a Banach space $E$ with $0 \in \operatorname{int}(C)$. Suppose $s : C \to C$ is surjective and $F \in s\text{-}KKM(C, C, E)$ is a closed map satisfying the following properties:*

$$A \subseteq C, A \subseteq co(\{0\} \cup F(co(\{0\} \cup A))) \textit{ implies } \overline{A} \textit{ is compact}$$

*and*

$$F(\overline{A}) \subseteq \overline{F(A)} \textit{ for any relatively compact subset } A \textit{ of } C.$$

*Then there exist $x_0 \in C$ and $y_0$ such that $y_0 \in F(x_0)$ and*

$$\mu(y_0 - x_0) = d_\mu(y_0, C) = d_\mu(y_0, \overline{I_C(x_0)}),$$

*here $\mu$ is the Minkowski functional of $C$ in $E$. More precisely, either* (i) *$F$ has a fixed point $x_0 \in C$, or* (ii) *there exist $x_0 \in \partial C$ and $y_0 \in F(x_0)$ with*

$$0 < \mu(y_0 - x_0) = d_\mu(y_0, C) = d_\mu(y_0, \overline{I_C(x_0)}).$$

Essentially the same arguments as in Theorem 3.8 (except here we use Theorem 2.3) yields the following result.

**Theorem 3.10.** *Let $C$ be a closed, convex subset of a Banach space $E$ with $0 \in \operatorname{int}(C)$. Suppose $s : C \to C$ is surjective and $F \in s$-$KKM(C, C, E)$ is a closed map satisfying the following properties:*

$$\begin{cases} A \subseteq C, A \subseteq co(\{0\} \cup F(co(\{0\} \cup A))) \text{ with } \overline{A} = \overline{Q} \\ \text{and } Q \subseteq A \text{ countable, implies } \overline{A} \text{ is compact,} \end{cases}$$

$$F(\overline{A}) \subseteq \overline{F(A)} \text{ for any relatively compact subset } A \text{ of } C,$$

$$\begin{cases} \text{for any relatively compact subset } A \text{ of } C, \text{ there} \\ \text{exists a countable set } B \subseteq A \text{ with } \overline{B} = \overline{A}, \end{cases}$$

*and*

$$\text{if } A \text{ is a compact subset of } C, \text{ then } \overline{co}(A) \text{ is compact.}$$

*Then there exist $x_0 \in C$ and $y_0$ such that $y_0 \in F(x_0)$ and*

$$\mu(y_0 - x_0) = d_\mu(y_0, C) = d_\mu(y_0, \overline{I_C(x_0)});$$

*here $\mu$ is the Minkowski functional of $C$ in $E$. More precisely, either* (i) *$F$ has a fixed point $x_0 \in C$, or* (ii) *there exist $x_0 \in \partial C$ and $y_0 \in F(x_0)$ with*

$$0 < \mu(y_0 - x_0) = d_\mu(y_0, C) = d_\mu(y_0, \overline{I_C(x_0)}).$$

Using Theorem 2.4 and following the arguments above, we obtain the following results.

**Theorem 3.11.** *Let $C$ be a closed, convex subset of a Banach space $E$ with $0 \in \operatorname{int}(C)$. Suppose $s : C \to C$ is surjective and $F \in s$-$KKM(C, C, E)$ is a closed, countably condensing map Then there exist $x_0 \in C$ and $y_0$ such that $y_0 \in F(x_0)$ and*

$$\mu(y_0 - x_0) = d_\mu(y_0, C) = d_\mu(y_0, \overline{I_C(x_0)});$$

*here $\mu$ is the Minkowski functional of $C$ in $E$. More precisely, either* (i) *$F$ has a fixed point $x_0 \in C$, or* (ii) *there exist $x_0 \in \partial C$ and $y_0 \in F(x_0)$ with*

$$0 < \mu(y_0 - x_0) = d_\mu(y_0, C) = d_\mu(y_0, \overline{I_C(x_0)}).$$

In the case when $E$ is a Hilbert space, we have the following result. For details, we refer the reader to [13, 14, 17]. Note that, in this case, the mapping $r$ is replaced by the nearest point projection.

**Theorem 3.12.** *Let $C$ be a closed, convex subset of a Hilbert space $E$. Suppose $s : C \to C$ is surjective and $F \in s\text{-}KKM(C, C, E)$ is a closed, countably condensing map. Then there exist $x_0$ and $y_0$ such that $y_0 \in F(x_0)$ and*

$$||y_0 - x_0|| = d(y_0, C) = d(y_0, \overline{I_C(x_0)});$$

*here $||.||$ is the norm induced by the inner product. More precisely, either* (i) *$F$ has a fixed point $x_0 \in C$, or* (ii) *there exist $x_0 \in \partial C$ and $y_0 \in F(x_0)$ with*

$$0 < ||y_0 - x_0|| = d(y_0, C) = d(y_0, \overline{I_C(x_0)}).$$

Using these approximation results, we now prove some fixed point theorems.

**Theorem 3.13.** *Let $C$ be a closed, convex subset of a Banach space $E$ with $0 \in \operatorname{int}(C)$. Suppose $s : C \to C$ is surjective and $F \in s\text{-}KKM(C, C, E)$ is a closed map satisfying the following property:*

$$A \subseteq C, A \subseteq \overline{co}(\{0\} \cup F(co(\{0\} \cup A))) \text{ implies } \overline{A} \text{ is compact} \qquad (H)$$

*If $F$ satisfies any one of the following conditions for any $x \in \partial C \setminus F(x)$:*
(i) *For each $y \in F(x)$, $\mu(y - z) < \mu(y - x)$ for some $z \in \overline{I_C(x)}$;*
(ii) *For each $y \in F(x)$, there exists $\lambda$ with $|\lambda| < 1$ such that $\lambda x + (1 - \lambda)y \in \overline{I_C(x)}$;*
(iii) *$F(x) \subseteq \overline{I_C(x)}$;*
(iv) *For each $\lambda \in (0, 1)$, $x \notin \lambda F(x)$;*
(v) *For each $y \in F(x)$, $\mu(y - x) \neq \mu(y) - 1$;*
(vi) *For each $y \in F(x)$, there exists $\alpha \in (1, \infty)$ such that $\mu^\alpha(y) - 1 \leq \mu^\alpha(y - x)$;*
(vii) *For each $y \in F(x)$, there exists $\beta \in (0, 1)$ such that $\mu^\beta(y) - 1 \geq \mu^\beta(y - x)$,*
*then $F$ has a fixed point.*

*Proof.* By Theorem 3.8, either
(1) $F$ has a fixed point in $C$
or
(2) there exist $x_0 \in \partial C$ and $y_0 \in F(x_0)$ with $x_0 = r(y_0)$ such that

$$0 < \mu(y_0) - 1 = \mu(y_0 - x_0) = d_\mu(y_0, C) = d_\mu(y_0, \overline{I_C(x_0)}),$$

where $\mu$ is the Minkowski functional of $C$ in $E$.

Suppose $F$ satisfies condition (i). Assume (2) holds (with $x_0$ and $y_0$ as described above) and $x_0 \notin F(x_0)$. Then, by condition (i), we have $\mu(y_0 - z) < \mu(y_0 - x_0)$ for some $z \in \overline{I_C(x_0)}$. This contradicts $\mu(y_0 - x_0) = d_\mu(y_0, \overline{I_C(x_0)})$. Hence $F$ has a fixed point in $C$.

Suppose $F$ satisfies condition (ii). Assume (2) holds (with $x_0$ and $y_0$ as described above) and $x_0 \notin F(x_0)$. Then, by condition (ii), there exists $\lambda$ with $|\lambda| < 1$ such that $\lambda x_0 + (1 - \lambda)y_0 \in \overline{I_C(x_0)}$. This implies that

$$\begin{aligned} \mu(y_0 - x_0) &\leq \mu(y_0 - (\lambda x_0 + (1 - \lambda)y_0)) \\ &< \mu(y_0 - x_0), \end{aligned}$$

which is impossible. Hence $F$ has a fixed point in $C$.

The proof for condition (iii) is obvious.

Suppose $F$ satisfies condition (iv). Assume (2) holds (with $x_0$ and $y_0$ as described above) and $x_0 \notin F(x_0)$. Then, by condition (iv), $x_0 \notin \lambda F(x_0)$ for each $\lambda \in (0,1)$. This implies that $x_0 \neq \lambda y_0$ for each $\lambda \in (0,1)$. But we have $x_0 = \frac{y_0}{\mu(y_0)}$ with $\mu(y_0) > 1$. Hence $F$ has a fixed point in $C$.

Suppose $F$ satisfies condition (v). Assume (2) holds (with $x_0$ and $y_0$ as described above) and $x_0 \notin F(x_0)$. Then, by condition (v), $\mu(y_0 - x_0) \neq \mu(y_0) - 1$. But we have $\mu(y_0 - x_0) = \mu(y_0) - 1$. Hence $F$ has a fixed point in $C$.

Suppose $F$ satisfies condition (vi). Assume (2) holds (with $x_0$ and $y_0$ as described above) and $x_0 \notin F(x_0)$. Then, by condition (vi), there exists $\alpha \in (1,\infty)$ with $\mu^\alpha(y_0) - 1 \leq \mu^\alpha(y_0 - x_0)$. Let $\lambda_0 = \frac{1}{p(y_0)}$. Then $\lambda_0 \in (0,1)$ and

$$\frac{(\mu(y_0)-1)^\alpha}{\mu^\alpha(y_0)} < 1 - \lambda_0^\alpha \leq \frac{\mu^\alpha(y_0 - x_0)}{\mu^\alpha(y_0)}.$$

Thus $\mu(y_0 - x_0) > \mu(y_0) - 1$. This contradicts $\mu(y_0 - x_0) = \mu(y_0) - 1$. Hence $F$ has a fixed point in $C$.

Finally suppose $F$ satisfies condition (vii). Then, as above (see the proof of (vi)), it can be shown that $F$ has a fixed point in $C$. ■

*Remark 3.2.* Essentially the same reasoning as in Theorem 3.13 shows that the conclusion remains true if condition $(H)$ is replaced by any one of the following conditions:

$$A \subseteq C, A \subseteq co(\{0\} \cup F(co(\{0\} \cup A))) \text{ implies } \overline{A} \text{ is compact} \qquad \text{(H1)}$$

and

$$\begin{gathered} F(\overline{A}) \subseteq \overline{F(A)} \text{ for any relatively compact subset } A \text{ of } C. \\ \begin{cases} A \subseteq C, A \subseteq co(\{0\} \cup F(co(\{0\} \cup A))) \text{ with } \overline{A} = \overline{Q} \\ \text{and } Q \subseteq A \text{ countable, implies } \overline{A} \text{ is compact,} \end{cases} \qquad \text{(H2)} \\ F(\overline{A}) \subseteq \overline{F(A)} \text{ for any relatively compact subset } A \text{ of } C, \\ \begin{cases} \text{for any relatively compact subset } A \text{ of } C, \text{ there} \\ \text{exists a countable set } B \subseteq A \text{ with } \overline{B} = \overline{A}, \end{cases} \end{gathered}$$

and

$$\text{if } A \text{ is a compact subset of } C, \text{ then } \overline{co}(A) \text{ is compact.}$$

$F$ is countably condensing. (H3)

**Theorem 3.14.** *Let $C$ be a closed, convex subset of a Hilbert space $E$. Suppose $s : C \to C$ is surjective and $F \in s$-$KKM(C,C,E)$ is a closed, countably condensing map. If $F$ satisfies any one of the following conditions for any $x \in \partial C \backslash F(x)$:*

(i) *For each $y \in F(x)$, $||y - z|| < ||y - x||$ for some $z \in \overline{I_C(x)}$;*

(ii) *For each $y \in F(x)$, there exists $\lambda$ with $|\lambda| < 1$ such that $\lambda x + (1-\lambda)y \in \overline{I_C(x)}$;*

(iii) *$F(x) \subseteq \overline{I_C(x)}$.*

*Then $F$ has a fixed point.*

*Remark 3.3.* For $R > 0$, let $B_R = \{x \in E : ||x|| \le R\}$. Then $\mu(.) = \frac{||.||}{R}$ is the Minkowski functional of $B_R$ in $E$. One can easily obtain results using Theorems 3.8 - 3.11 and Theorem 3.13 in the case when $C$ is replaced by $B_R$. For details, we refer the reader to [13, 14, 17].

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