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A Hausdorff Moment Problem with Non-Integral Powers: Approximation by Finite Moments

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Abstract. We consider the problem of finding $u \in L^2(I)$, $I = (0,1)^2 \subset \mathbb{R}^2$, satisfying

$$\int_I u(x,y) x^{\alpha_k} y^{\alpha_l} dxdy = \mu_{kl},$$

where $k, l = 0, 1, 2, \ldots$, (α_k) is a sequence of pairwise distinct real numbers which are greater than $-1/2$, and $\boldsymbol{\mu} = (\mu_{kl})$ is a given bounded sequence of real numbers. This is an ill-posed problem. We shall regularize the problem by finite moments and then, apply the result to reconstruct a function from a sequence of its Laplace transforms.

1. Introduction

In this paper we consider the problem of finding $u \in L^2(I)$, $I = (0,1) \times (0,1)$ satisfying

$$\int_I u(x,y) x^{\alpha_k} y^{\alpha_l} dxdy = \mu_{kl}, \tag{1}$$

where $k, l = 0, 1, 2, \ldots$, (α_k) is a sequence of distinct real numbers such that

$$\alpha_k > -\frac{1}{2} \quad \text{for all } k = 0, 1, 2, \ldots$$

and (μ_{kl}) is a given bounded sequence of real numbers.

As it is known, the problem (1) is ill-posed, i.e., solutions do not always exist, and in the case of existence, they do not depend continuously on the given data (which are represented by the right hand side of (1)). In [1], the authors considered a particular case, in which (α_i) is a sequence of positive integers

$$\alpha_i = i, \quad i = 0, 1, 2, \ldots$$

The remainder of this paper consists of two sections. In Sec. 2, using the solution of the finite moment problem

$$\int_I u(x, y) x^{\alpha_k} y^{\alpha_l} dxdy = \mu_{kl}, \quad k, l = 0, \ldots, n, \tag{2}$$

we shall give a regularization of problem (1). Sec. 3 deals with a moment problem associated with the Laplace transform.

2. Regularized Approximation of (1) by Finite Moments

Let L_m be the polynomial

$$L_m(x) = \sum_{j=0}^{m} C_{mj} x^{\alpha_j}, \tag{3}$$

where

$$C_{mj} = \sqrt{2\alpha_m + 1}\left(\frac{\prod_{r=0}^{m-1}(\alpha_j + \alpha_r + 1)}{\prod_{r=0, r\neq j}^{m}(\alpha_j - \alpha_r)}\right). \tag{4}$$

Recall that the α_i's are pairwise distinct. We have

Proposition (A).
a) *The set $\{L_n\}$ is orthonormal in $L^2[0, 1]$.*
b) *The set $\{L_n\}$ is complete in $L^2[0, 1]$ if and only if*

$$\sum_{i=0}^{\infty} \frac{2\alpha_i + 1}{(2\alpha_i + 1)^2 + 1} = \infty. \tag{5}$$

See, e.g., [2] for the proof.

Put

$$L_{kl}(x, y) = L_k(x).L_l(y) = \sum_{p=0}^{k}\sum_{q=0}^{l} C_{kp} C_{lq} x^{\alpha_p} y^{\alpha_q}.$$

If the assumptions in the above proposition hold, then the sequence $\{L_{kl}\}$ forms a complete orthonormal set in $L^2(I)$.

For each $\boldsymbol{\mu} = (\mu_{kl})$, we define the sequence

$$\boldsymbol{\lambda} = \boldsymbol{\lambda}(\boldsymbol{\mu}) = (\lambda_{kl}) \tag{6}$$

as follows

$$\lambda_{kl} = \lambda_{kl}(\boldsymbol{\mu}) = \sum_{p=0}^{k}\sum_{q=0}^{l} C_{kp} C_{lq} \mu_{pl}.$$

Then Problem (2) is equivalent to that of finding $u \in L^2(I)$ satisfying

$$\int_I u(x,y)L_{kl}(x,y)dxdy = \lambda_{kl}, \quad k,l = 0,\dots,n. \tag{7}$$

It is known from the elementary functional analysis that

$$p^n = p^n(\boldsymbol{\mu}) = \sum_{k,l=0}^{n} \lambda_{kl}(\boldsymbol{\mu})L_{kl} \tag{8}$$

is a minimal norm solution of (2).

We have

Theorem 1. *Let* $\boldsymbol{\mu} = (\mu_{kl})$ *be a given sequence of real numbers and let the assumption of Proposition A hold. Then Problem* (1) *has at most one solution. Moreover, the solution exists if and only if*

$$\sum_{k,l=0}^{\infty} \lambda_{kl}^2 = \sum_{k,l=0}^{\infty} \Big(\sum_{p,q=0}^{\infty} C_{kp}C_{lq}\mu_{kl} \Big)^2 < \infty, \tag{9}$$

where C_{ij} *is as in (4) if* $i \geq j$ *and* $C_{ij} = 0$ *if* $i < j$.

If u *is the solution of* (1) *then*

$$p^n(\boldsymbol{\mu}) \to u \text{ in } L^2(I).$$

Proof. By the completeness and the orthonormality of $\{L_{kl}\}$, if (1) has a solution u then

$$\sum_{k,l=0}^{\infty} \lambda_{kl}^2 = \|u\|^2 < \infty,$$

Here, $\|.\|$ is the $L^2(I)$-norm. Conversely, if (9)is satisfied then, by completeness of $\{L_{kl}\}$, the sum u of the series

$$\sum_{k,l=0}^{\infty} \lambda_{kl}L_{kl}$$

is the solution of (1). ■

The following is the main result

Theorem 2. *Let* $C > 0$ *and let* $c(t)$ *be a decreasing positive function on* $[0,\infty)$ *satisfying* $C > c(t)$ *for every* $t \geq 0$. *Suppose the sequence* (α_i) *satisfies* (5), *and*

$$\alpha_i = \varphi(i), \quad i = 0,1,2,\dots$$

where φ *is a real function defined in* $[0,\infty)$ *such that*

$$0 < c(t) \leq |\varphi'(t)| \leq C, \quad \forall t \in [0,\infty), \tag{10}$$

(we can suppose $C > 1 + 2\varphi(0)$). Let $u_0 \in L^2(I)$ be the solution of (1) corresponding to the exact data $\boldsymbol{\mu}^0 = (\mu^0_{kl})$ in the right hand side. Put

$$f(t) = \frac{3^6 C}{2^4\sqrt{5}}\Big(\frac{9C}{c(2t)}\Big)^{2t}(2t+1),$$

and for each $0 < \varepsilon < 1$, put

$$n(\varepsilon) = \big[f^{-1}(\varepsilon^{-1/2})\big],$$

where $[x]$ is the largest integer $\leq x$.
Then, there exists a function $\eta(\varepsilon)$, $0 < \varepsilon < 1$, such that $\eta(\varepsilon) \to 0$ as $\varepsilon \to 0$ and that for all sequences $\boldsymbol{\mu}$ satisfying

$$\|\boldsymbol{\mu} - \boldsymbol{\mu}^0\|_\infty \equiv \sup_{k,l} |\mu_{kl} - \mu^0_{kl}| < \varepsilon,$$

we have

$$\|p^{n(\varepsilon)}(\boldsymbol{\mu}) - u_0\| \leq \eta(\varepsilon).$$

Proof. We have

$$\|p^n(\boldsymbol{\mu}) - u_0\| \leq \|p^n(\boldsymbol{\mu}) - p^n(\boldsymbol{\mu}^0)\| + \|p^n(\boldsymbol{\mu}^0) - u_0\|. \tag{11}$$

Using (8), we have

$$\|p^n(\boldsymbol{\mu}) - p^n(\boldsymbol{\mu}^0)\| = \sum_{k,l=0}^{n}\sum_{p,q=0}^{\infty} C_{kp}C_{lq}(\mu_{kl} - \mu^0_{kl})L_{pq},$$

where $C_{ij} = 0$ if $i < j$.

Hence,

$$\begin{aligned}\|p^n(\boldsymbol{\mu}) - p^n(\boldsymbol{\mu}^0)\|^2 &= \sum_{k,l=0}^{n}\Big(\sum_{p,q=0}^{\infty} C_{kp}C_{lq}(\mu_{kl} - \mu^0_{kl})\Big)^2 \\ &\leq \varepsilon^2 \sum_{k,l=0}^{n}\Big(\sum_{p,q=0}^{\infty} C_{kp}C_{lq}\Big)^2.\end{aligned} \tag{12}$$

From (4), (10) and the Lagrange's theorem about the mean-value of differentiation, one has

$$\begin{aligned}|C_{mj}| &= \sqrt{2\alpha_m + 1}\left|\frac{\prod_{r=0}^{m-1}(\alpha_j + \alpha_r + 1)}{\prod_{r=0,r\neq j}^{m}(\alpha_j - \alpha_r)}\right| \\ &\leq \frac{C^{m+1/2}}{c^j(j)c^{m-j}(m)}\sqrt{2m+1}\frac{\prod_{r=0}^{m-1}(j+r+1)}{\prod_{r=0,r\neq j}^{m}|j-r|} \\ &< \frac{C^{m+1/2}}{c^m(m)}\sqrt{2m+1}\frac{(m+j)!}{(j!)^2(m-j)!} \\ &\leq \frac{C^{m+1/2}}{c^m(m)}\sqrt{2m+1}(1+1+1)^{m+j} = 3^{m+j}\frac{C^{m+1/2}}{c^m(m)}\sqrt{2m+1}.\end{aligned}$$

Therefore

$$\begin{aligned}\sum_{j=0}^{m}|C_{mj}| &\le \Big(\frac{C}{c(m)}\Big)^m\sqrt{C(2m+1)}\sum_{j=0}^{m}3^{m+j}\\ &= \Big(\frac{C}{c(m)}\Big)^m\sqrt{C(2m+1)}(3^m)\frac{3^{m+1}-1}{3-1}\\ &< \Big(\frac{C}{c(m)}\Big)^m\sqrt{C(2m+1)}\Big(\frac{3}{2}\Big)3^{2m}.\end{aligned}\tag{13}$$

This implies, since $C_{ij}=0$ if $i<j$ and the function $c(t)$ is decreasing, that

$$\begin{aligned}\Big(\sum_{p,q=0}^{\infty}|C_{kp}C_{lq}|\Big)^2 &= \Big\{\Big(\sum_{p=0}^{k}|C_{kp}|\Big)\sum_{q=0}^{l}|C_{lq}|\Big\}^2\\ &< \Big\{\frac{C^{k+l+1}}{c^k(k)c^l(l)}\sqrt{(2k+1)(2l+1)}\Big(\frac{3}{2}\Big)^2 3^{2(k+l)}\Big\}^2\\ &< \Big\{\frac{C}{c(k+l)}\Big\}^{2(k+l)}C^2(2k+1)(2l+1)\Big(\frac{3}{2}\Big)^4 3^{4(k+l)},\end{aligned}\tag{14}$$

and then (noting that $0<c(t)<C$ for every $t\ge 0$)

$$\begin{aligned}\sum_{k,l=0}^{n}\Big(\sum_{p,q=0}^{\infty}|C_{kp}C_{lq}|\Big)^2 &< \Big(\frac{3}{2}\Big)^4\Big(\frac{C}{c(2n)}\Big)^{4n}C^2(2n+1)^2\Big(\sum_{j=0}^{n}3^{4j}\Big)^2\\ &< \Big(\frac{3}{2}\Big)^4\Big(\frac{C}{c(2n)}\Big)^{4n}C^2(2n+1)^2\Big(\frac{3^{4(n+1)}-1}{3^4-1}\Big)^2\\ &< \Big(\frac{3}{2}\Big)^4\Big(\frac{C}{c(2n)}\Big)^{4n}C^2(2n+1)^2\Big(\frac{3^4}{3^4-1}\Big)^2 3^{8n}\\ &= \frac{3^{12}C^2}{2^8.5}\Big(\frac{9C}{c(2n)}\Big)^{4n}(2n+1)^2.\end{aligned}\tag{15}$$

Put

$$f(t)=\frac{3^6C}{2^4\sqrt{5}}\Big(\frac{9C}{c(2t)}\Big)^{2t}(2t+1).$$

From [12] we have

$$\|p^{n(\varepsilon)}(\boldsymbol{\mu})-p^{n(\varepsilon)}(\boldsymbol{\mu}^0)\|\le\sqrt{\varepsilon}.\tag{16}$$

If we put

$$\eta(\varepsilon)=\sqrt{\varepsilon}+\Big\{\sum_{\max(k,l)>n(\varepsilon)}\Big(\sum_{p,q=0}^{\infty}C_{kp}C_{lq}\mu^0_{pq}\Big)^2\Big\}^{1/2},$$

then by (11), (16), it follows that

$$\|p^{n(\varepsilon)}(\boldsymbol{\mu})-u_0\|\le\eta(\varepsilon).$$

As $\varepsilon \to 0$, we have $f^{-1}(\varepsilon^{-1/2}) \to \infty$ and $n(\varepsilon) \to \infty$. By (9)

$$\sum_{\max(k,l)>n(\varepsilon)} \Big(\sum_{p,q=0}^{\infty} C_{kp} C_{lq} \mu^0_{pq} \Big)^2 \to 0 \text{ as } \varepsilon \to 0.$$

Hence $\eta(\varepsilon) \to 0$ for $\varepsilon \to 0$. ■

3. A Moment Problem from Laplace Transform

We consider the problem of approximating $u_0 \in L^2(Q)$, $Q = (0, \infty) \times (0, \infty)$, such that

$$\int_Q u_0(x,y) e^{-\beta_k x - \beta_l y} dxdy = \mu^0_{kl}, \quad k = 0, 1, 2, \dots \tag{17}$$

where (β_i) is a sequence of distinct real numbers. Put $s = e^{-x}$, $t = e^{-y}$ and $w_0(s,t) = u_0(-\ln s, -\ln t)$. It follows from (17) that

$$\int_I w_0(s,t) s^{\alpha_k} t^{\alpha_l} dsdt = \mu^0_{kl}, \quad k, l = 0, 1, 2, \dots \tag{18}$$

where $\alpha_j = \beta_j - 1$, $j = 0, 1, 2, \dots$

Note that

$$\int_I |w_0(s,t)|^2 dsdt = \int_Q |u_0(x,y)|^2 e^{-x-y} dxdy.$$

Then we have $w_0 \in L^2(I)$, since $u_0 \in L^2(Q)$. According to Theorem 2 in the previous section, we have

Theorem 3. *Let $u_0 \in L^2(Q)$ be the solution of (17) corresponding to $\boldsymbol{\mu}^0 = (\mu^0_{kl}) \in \boldsymbol{\ell}^2$ in the right hand side of (17). Suppose the sequence (β_i) satisfies*

$$\forall i, \beta_i > \frac{1}{2}, \quad \sum_{i=0}^{\infty} \frac{2\beta_i - 1}{(2\beta_i - 1)^2 + 1} = \infty,$$

and $\beta_i = \varphi(i) + 1$, where φ is a function as in Theorem 2. Put

$$f(t) = \frac{3^6 C}{2^4 \sqrt{5}} \Big(\frac{9C}{c(2t)} \Big)^{2t} (2t+1),$$

and for each $0 < \varepsilon < 1$, put

$$n(\varepsilon) = \big\{ f^{-1}(\varepsilon^{-1/2}) \big\},$$
$$q^{n(\varepsilon)}(\boldsymbol{\mu})(x,y) = p^{n(\varepsilon)}(\boldsymbol{\mu})(e^{-x}, e^{-y})$$

where $p^{n(\varepsilon)}(\boldsymbol{\mu})$ is as in Theorem 2. Then there exists a function $\eta(\varepsilon)$, $0 < \varepsilon < 1$, such that $\eta(\varepsilon) \to 0$ as $\varepsilon \to 0$ and that for all sequences $\boldsymbol{\mu}$ satisfying

$$\|\boldsymbol{\mu} - \boldsymbol{\mu}^0\|_\infty \equiv \sup_{k,l} |\mu_{kl} - \mu^0_{kl}| < \varepsilon,$$

we have

$$\|q^{n(\varepsilon)}(\boldsymbol{\mu}) - u_0\|_\rho \leq \eta(\varepsilon),$$

where the norm $\|\cdot\|_\rho$ *is defined by*

$$\|h\|_\rho^2 = \int_Q |h(x,y)|^2 e^{-x-y} dxdy.$$

Proof. From Theorem 2, we get

$$\|p^{n(\varepsilon)}(\boldsymbol{\mu}) - w_0\| \leq \eta(\varepsilon), \quad \text{where } w_0(s,t) = u_0(-\ln s, -\ln t). \tag{19}$$

Moreover,

$$\begin{aligned} \|p^{n(\varepsilon)}(\boldsymbol{\mu}) - w_0\|^2 &= \int_I |p^{n(\varepsilon)}(\boldsymbol{\mu}(s,t) - w_0(s,t)|^2 dsdt \\ &= \int_Q |p^{n(\varepsilon)}(\boldsymbol{\mu}(e^{-x}, e^{-y}) - u_0(x,y)|^2 e^{-x-y} dxdy \\ &= \|q^{n(\varepsilon)}(\boldsymbol{\mu}) - u_0\|_\rho^2, \end{aligned} \tag{20}$$

which completes the proof. ∎

References

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