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Stability and Convergence of Implicit Iteration Processes

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Abstract. The global convergence of locally convergent and stable (quasi-stable) implicit iterations $x_n = T(x_n, x_{n-1})$ in connected (pathwise connected) metric spaces are established. A sufficient condition for the local convergence of implicit iteration processes is also given.

1. Introduction

The aim of this paper is to study the global convergence of locally convergent and stable (quasi-stable) implicit iteration processes

$$x_n = T(x_n, x_{n-1}) \ \ (n \geqslant 1) \tag{1}$$

to the set of solutions of equation

$$x = T(x, x) \tag{2}$$

of mappings of two variables. Implicit iterative methods are widely used in many applications when ordinary (explicit) iterative methods are inefficient. At each step, an implicit iterative method requires the solution of an equation. Thus, implicit iterative methods in general are more expensive than ordinary ones, but in many cases they have other advantages, e.g. higher order of convergence and stability. In this note we extend the results obtained by Anh [1] on the global convergence for explicit iteration processes to implicit ones. The main difference between explicit and implicit iterations is that for implicit iterations, on each step, the set of x_n defined by (1) is not necessarily a singleton. Thus the stability and the convergence of implicit iteration processes need to be studied.

2. Main Results

Consider an operator equation (2) where $T : X \times X \to X$ is a possibly nonlinear operator, and (X, d) is a metric space. Together with equation (2) we consider the implicit iteration process (1).

Definition 1. *A sequence $\{x_n\}_{n \geqslant 0}$ is called an admissible trajectory of* (1) *starting from $\alpha \in X$ if it satisfies relations* (1) *for all $n \geqslant 1$ and $x_0 = \alpha$.*

The set of all admissible trajectories of (1) starting from α will be denoted by $J(\alpha)$. Assume that for every fixed $\zeta \in X$ the set of all solutions of the equation $x = T(x, \zeta)$ is not empty and bounded, and is denoted by $[\zeta]$. The set of all solution of the equation (2) is also assumed to be nonempty and is denoted by $F(T)$. By $C(T)$ we denote the convergence domain of the process (1), i.e.

$$C(T) = \{\alpha \in X : \exists \{x_n\} \in J(\alpha),\ \beta([x_n], F(T)) \longrightarrow 0,\ \text{as } n \to \infty\},$$

where $\beta(A, B)$ is the so-called β-distance, defined by

$$\beta(A, B) = \sup_{a \in A} d(a, B) \ \text{ for bounded subsets } \ A, B \subset X.$$

By $B(x, \delta)$ ($B[x, \delta]$) we denote the open (closed) ball centered at x with a radius δ.

Definition 2. *The iterative process* (1) *is said to be locally (globally) convergent if: $\exists \epsilon > 0 : F_\epsilon = \{x \in X : d(x, F(T)) < \epsilon\} \subset C(T)$ ($C(T) = X$).*

Definition 3. *The process* (1) *is called stable w.r.t. initial approximations at $x^0 \in X$ if*

$$\forall \epsilon > 0\ \exists \delta = \delta(\epsilon, x^0) > 0 :\ \forall x \in B(x^0, \delta),$$
$$\forall \{x_n\} \in J(x^0),\ \forall \{y_n\} \in J(x) \Longrightarrow \beta([x_n], [y_n]) < \epsilon\ \forall n \geqslant 1.$$

Definition 4. *The process* (1) *is said to be quasi-stable w.r.t. initial approximations at $x^0 \in X$ if*

$$\forall \epsilon > 0\ \exists \delta = \delta(\epsilon, x^0) > 0 :\ \forall x \in B(x^0, \delta),\ \exists \{x_n\} \in J(x^0),\ \forall \{y_n\} \in J(x),$$
$$\forall N > 0\ \exists n = n(\epsilon, \{x_n\}, \{y_n\}) \geqslant N \Longrightarrow \beta([x_n], [y_n]) < \epsilon.$$

The process (1) *is said to be stable (quasi-stable) if it is stable (quasi-stable) w.r.t. initial approximations at every $x^0 \in X$.*

Lemma 1. *Suppose that the process* (1) *is locally convergent and stable. Then its convergence domain $C(T)$ is open.*

Proof. Suppose $x^0 \in C(T)$ and $F_\epsilon \subset C(T)$ for some $\epsilon > 0$. Since $x^0 \in C(T)$, there exist $\{x_n\} \in J(x^0)$ and $n_0 > 0$ such that for all $n \geqslant n_0$

$$\beta([x_n], F(T)) < \epsilon/2.$$

The stability of the process (1) w.r.t. initial approximations at x^0 implies that $\exists\delta = \delta(\epsilon, x^0) > 0,\ \forall x \in B(x^0, \delta),\ \forall\{y_n\} \in J(x),\ \exists n_1 \geqslant n_0 : \beta([x_{n_1}], [y_{n_1}]) < \epsilon/2$. Hence, $d(\overline{x}_{n_1+1}, [y_{n_1}]) < \epsilon/2$ for some $\overline{x}_{n_1+1} \in [x_{n_1}]$ and then there exists $\overline{y}_{n_1+1} \in [y_{n_1}]$ such that $d(\overline{x}_{n_1+1}, \overline{y}_{n_1+1}) < \epsilon/2$. Further,

$$\begin{aligned} d(\overline{y}_{n_1+1}, F(T)) &\leqslant d(\overline{x}_{n_1+1}, \overline{y}_{n_1+1}) + d(\overline{x}_{n_1+1}, F(T)) \\ &< \epsilon/2 + \beta([x_{n_1}], F(T)) < \epsilon. \end{aligned}$$

Thus $\overline{y}_{n_1+1} \in F_\epsilon \subset C(T)$, therefore there exists an admissible trajectory $\{z_n\} \in J(\overline{y}_{n_1+1})$ such that $\beta([z_n], F(T)) \to 0$. Now define

$$x_n^* = \begin{cases} y_n & \text{if } n < n_1 + 1, \\ z_{n-n_1-1} & \text{if } n \geqslant n_1 + 1. \end{cases}$$

Obviously, $\{x_n^*\} \in J(x)$ and $\beta([x_n^*], F(T)) \to 0$ as $n \to \infty$, which means $x \in C(T)$, hence $B(x^0, \delta) \subset C(T)$. The proof of the lemma is complete. ∎

Theorem 2. *Assume either*

i) *The metric space* (X, d) *is connected and the process* (1) *is stable; or*

ii) (X, d) *is pathwise connected and process* (1) *is quasi-stable.*

Then the method (1) *is locally convergent if and only if it is globally convergent.*

Proof. The proof of this theorem follows the same line as the proofs given in [1] for the case of explicit iteration methods.

i) Suppose the process (1) is stable and locally convergent. Lemma 1 ensures the openess of $C(T)$. If we can prove that $C(T)$ is also closed then since (X, d) is connected and $\emptyset \neq F(T) \subset C(T)$, it will follow $C(T) = X$. We argue by contradiction, assuming that $C(T)$ is not closed. Then there exists $x^0 \in \overline{C}(T) \setminus C(T)$. Fix a sequence $\{x_n\} \in J(x^0)$ such that $\beta([x_n], F(T)) \nrightarrow 0$. Thus,

$$\exists\epsilon > 0,\ \exists\{n_k\} :\ \beta([x_{n_k}], F(T)) \geqslant \epsilon.$$

From the stability of the process (1) w.r.t. initial approximations at x^0, it follows that

$$\exists\delta = \delta(\epsilon, x^0),\ \forall x \in B(x^0, \delta) :\ \beta([x_n], [y_n]) < \epsilon/2\ \ \forall n \geqslant 0$$

for any fixed $\{y_n\} \in J(x)$. Now $x^0 \in \overline{C}(T)$ implies that there exists $x^1 \in C(T)$ such that $d(x^0, x^1) < \delta$, hence $\exists\{y_n^1\} \in J(x^1)\ \exists N : \beta([y_n^1], F(T)) < \epsilon/2\ \ \forall n \geqslant N$. Choosing $n_k \geqslant N$ we have

$$\epsilon \leqslant \beta([x_{n_k}], F(T)) \leqslant \beta([x_{n_k}], [y_{n_k}^1]) + \beta([y_{n_k}^1], F(T)) < \epsilon,$$

which leads to a contradiction. Thus $C(T)$ is closed, which was to be proved.

ii) Now let the process (1) be quasi-stable and suppose that (X, d) is pathwise connected. Let $z \in X$ be any fixed element and $x^0 \in C(T)$ be an arbitrarily chosen element. Since (X, d) is pathwise connected, there exists a continuous mapping $x = x(t)$ such that $x(0) = x^0, x(1) = z$ and $x \in C([0, 1], X)$. Let

$$\tau = \sup\{t \in [0, 1] : x(t) \in C(T)\}.$$

Since $x(0) = x^0 \in C(T)$, τ is well-defined. We are going to prove that $\tau = 1$, hence $z = x(1) \in C(T)$, thus $C(T) = X$. The proof is devided into two steps.
a) The local convergence of the process (1) implies that $F_\epsilon \subset C(T)$ for some $\epsilon > 0$. Since the process (1) is quasi-stable w.r.t. initial approximations at $x(\tau)$, we have

$$\exists\delta = \delta(\epsilon, x(\tau)) > 0,\ \forall x \in B(x(\tau), \delta),\ \exists\{x_n(\tau)\} \in J(x(\tau)),\ \forall\{y_n\} \in J(x),$$
$$\forall N > 0\ \exists n = n(\epsilon, \{x_n(\tau)\}, \{y_n\}) \geqslant N \Longrightarrow \beta([x_n(\tau)], [y_n]) < \epsilon/2. \tag{3}$$

From the definition of τ and continuity of $x(t)$, it follows that there exists $t \in [0, \tau]$ such that $x(t) \in C(T)$ and $d(x(t), x(\tau)) < \delta$. As $x(t) \in C(T)$ there exists n_0 such that $\beta([y_n(t)], F(T)) < \epsilon/2$ for some $\{y_n(t)\} \in J(x(t))$ and $n \geqslant n_0$. Besides, according to (3) $x(t) \in B(x(\tau), \delta)$ implies that $\beta([x_{n_1}(\tau)], [y_{n_1}(t)]) < \epsilon/2$ for some $n_1 \geqslant n_0$. We have

$$\beta([x_{n_1}(\tau)], F(T)) \leqslant \beta([x_{n_1}(\tau)], [y_{n_1}(t)]) + \beta([y_{n_1}(t)], F(T)) < \epsilon,$$

hence $\overline{x}_{n_1+1}(\tau) \in F_\epsilon \subset C(T)$ for some $\overline{x}_{n_1+1}(\tau) \in [x_{n_1}(\tau)]$. By the same argument as in the proof of Lemma 1, we come to the conclusion that $x(\tau) \in C(T)$.
b) If $\tau < 1$ then by the continuity of $x(.)$ there exists $s \in (\tau, 1]$ such that $d(x(\tau), x(s)) < \delta$, where δ was chosen in part a) so that (3) holds. Since $x(\tau) \in C(T)$ there exist $\{x_n(\tau)\} \in J(x(\tau))$ and n_1 such that $\beta([x_n(\tau)], F(T)) < \epsilon/2$ for all $n \geqslant n_1$. The quasi-stability of the process (1) at $x(s)$ implies that

$$\exists\{y_n(s)\} \in J(x(s));\ \forall\{x_n(\tau)\} \in J(x(\tau));\ \exists n_2 = n_2(\epsilon, \{y_n(s)\}, \{x_n(\tau)\}) \geqslant n_1$$
$$\Longrightarrow \beta([y_{n_2}(s)], [x_{n_2}(\tau)]) < \epsilon/2.$$

So

$$\beta([y_{n_2}(s)], F(T)) \leqslant \beta([y_{n_2}(s)], [x_{n_2}(\tau)]) + \beta([x_{n_2}(\tau)], F(T)) < \epsilon,$$

hence there exists $\overline{y}_{n_2+1} \in [y_{n_2}(s)]$ such that $\overline{y}_{n_2+1} \in F_\epsilon \subset C(T)$. Thus $x(s) \in C(T)$, we come to a contradiction with the definition of τ. Therefore $\tau = 1$, hence $z = x(1) \in C(T)$. Theorem 2 is proved. ∎

Example 1. We describe a class of mappings for which the process (1) is stable. Let D be a closed bounded convex subset of an uniformly convex Banach space X, T a mapping of $D \times D$ into D satisfying the following nonexpansive condition: for all $x, y, z, t \in D$

$$\|T(x, y) - T(z, t)\| \leqslant \max\{\|x - z\|, \|y - t\|\};\text{ and}$$
$$\text{the strict inequality holds when } \|x - z\| \neq \|y - t\|. \tag{4}$$

(The operator $T(x, y) = \lambda U(x) + (1 - \lambda)S(y)$, $\lambda \in (0, 1)$, where U and S are nonexpansive mappings, satisfies (4). A mapping U is called nonexpansive if $\|U(x) - U(z)\| \leqslant \|x - z\|$). Set $T_\zeta = T(\cdot, \zeta)$, then $T_\zeta : D \to D$ and

$$\|T_\zeta(x) - T_\zeta(z)\| < \|x - z\|, \quad \forall x \neq z. \tag{5}$$

By Browder's Fixed Point Theorem [2] for nonexpansive mappings the set $F(T) \neq \emptyset$ and every set $[\zeta]$ is nonempty and single by (5), which means x_n satisfying $x_n = T(x_n, x_{n-1})$ is uniquely defined by x_{n-1}. Assume $\{x_n\}$ and $\{y_n\}$ are any trajectories defined by (1) from x^0 and x, respectively. For given ϵ, choose $\delta = \epsilon/2 \geqslant \|x^0 - x\|$. Then from (4) we can see that

$$\begin{aligned}\|x_n - y_n\| &= \|T(x_n, x_{n-1}) - T(y_n, y_{n-1})\| \leqslant \|x_{n-1} - y_{n-1}\| \\ &\leqslant \cdots \leqslant \|x^0 - x\| \leqslant \epsilon/2 < \epsilon, \quad \forall n \geqslant 1.\end{aligned}$$

Thus the process (1) is stable.

Example 2. [3, Example 2], which was given for another purpose, provides an example of a mapping T for which the process (1) is quasi-stable but not stable w.r.t. initial approximations. Let M be the subset of the plane defined as follows:

$$M = M_0 \cup \{(n/2^{k+1}, 1/2^k) : k = 0, 1, ...; n = 2, ..., 2^{k+1}\},$$

where $M_0 = \{(x, 0) : 0 \leqslant x \leqslant 1\}$. The distance on M is the Euclidean distance of $\mathbb{R}^2$. Define $f : M \to M$ as follows:

(1) If k is odd, then

$$\begin{aligned}&f(n/2^{k+1}, 1/2^k) = ((n+1)/2^{k+1}, 1/2^k) \;\text{ if }\; n < 2^{k+1}; \\ &f(1, 1/2^k) = (1, 1/2^{k+1}).\end{aligned}$$

(2) If k is even, then

$$\begin{aligned}&f(n/2^{k+1}, 1/2^k) = ((n-1)/2^{k+1}, 1/2^k) \;\text{ if }\; n > 2; \\ &f(1, 1/2^k) = (1/2^{k+1}, 1/2^{k+1}).\end{aligned}$$

(3) $f(x, 0) = (x, 0)$. Let $(x, y), (u, v) \in M$, then the mapping T is defined by

$$T((x, y), (u, v)) = \frac{1}{2}\{(x, y) + f(u, v)\}.$$

Clearly the implicit iteration process (1) for T is an explicit iteration process for f. It is not difficult to see that process (1) is quasi-stable on M, but not stable w.r.t. initial approximations at each point of M_0.

Example 3. The following example given in [1] provides a mapping T for which the process (1) is locally convergent, but not globally convergent. Let $X = [0, 1]$ with the distance of $\mathbb{R}^1$. Define f as follows:

$$f(x) = \begin{cases} -(3/2)x + 1 & \text{if } x \in [0, 1/3], \\ 1/2 & \text{if } x \in [1/3, 2/3], \\ -(3/2)x + 3/2 & \text{if } x \in [2/3, 1]. \end{cases}$$

The mapping T is defined by $T(x, y) = \frac{1}{2}[x + f(y)]$. It is clear that $F(T) = \{\frac{1}{2}\}$. For $x_0 \in (1/3, 2/3)$ we get $x_n = f(x_{n-1}) = \frac{1}{2}$ $(n \geqslant 1)$, hence process (1) is locally convergent. On the other hand, for $x_0 = 0$ we have $x_{2n} = 0, x_{2n-1} = 1$ ($n \geqslant 1$). Thus process (1) is not globally convergent. Clearly the process (1) is not quasi-stable w.r.t. initial approximations at $x_0 = 0$ or $x_0 = 1$.

Before providing a sufficient condition for the local convergence of implicit iteration processes, we recall the so-called H'adamard theorems on homeomorphism between two Banach spaces X and Y.

Theorem 3. [4, p.137] *Suppose $F \in C^1(X,Y)$ is a local homeomorphism and $\|[F'(x)]^{-1}\| \leqslant \gamma$. Then F is a global homeomorphism of X onto Y.*

Theorem 4. [4, p.138] *Suppose $F : X \to Y$ is continuously differentiable on an open set containing a closed ball $B[x_0, r]$ and $\|[F'(x)]^{-1}\| \leqslant \gamma,\ \forall x \in B[x_0, r]$. Then if $r > \gamma\|F(x_0)\|$, then equation $F(x) = 0$ has a solution in the open ball $B(x_0, r)$.*

Theorem 5. *Let the operator $T : X \times X \to Y$, where X and Y are Banach spaces, be continuously differentiable in the first variable in X and Lipschitz continuous in the second one in an open ball $B(0, R)$. Moreover, let $\|[I-\partial_1 T(x, y)]^{-1}\| \leqslant \gamma$; $\forall x \in X,\ \forall y \in B(0, R)$ and $\|\partial_1 T(x, y)\| \leqslant M$, $\|T(x, y)-T(x, \tilde{y})\| \leqslant L\|y-\tilde{y}\|$, $\forall x, y, \tilde{y} \in B(0, R)$ where $\partial_1 T$ denotes the Fréchet derivative of T w.r.t. the first variable. Further, suppose the set of fixed points $F(T)$ is nonempty and $F(T) \subset B(0, r)$, where $r < R/2$. If $\gamma L < 1$ then the implicit iteration process* (1) *is well-defined and locally convergent.*

Proof. Fix a positive number $\epsilon < \min\{R - 2r, \frac{r(1-\gamma L)}{\gamma(1+L+M)}\}$. We will show that $F_\epsilon \subset C(T)$, which means the local convergence of the process (1). Obviously, for any fixed $x_0 \in F_\epsilon$, $B[x_0, r] \subset B(0, R)$. Indeed, there exists $\xi \in F(T)$ such that $\|x_0 - \xi\| < \epsilon$, hence

$$\|x\| \leqslant \|x - x_0\| + \|x_0 - \xi\| + \|\xi\| < 2r + \epsilon < R, \quad \forall x \in B[x_0, r].$$

Further

$$\begin{aligned}\|x_0 - T(x_0, x_0)\| &\leqslant \|x_0 - \xi\| + \|T(x_0, x_0) - T(x_0, \xi)\| + \|T(x_0, \xi) - T(\xi, \xi)\| \\ &\leqslant \|x_0 - \xi\|\Big\{1 + L + \int_0^1 \partial_1 T(\xi + t(x_0 - \xi), \xi)dt\Big\} \\ &< (1 + L + M)\epsilon < r(1 - \gamma L)/\gamma.\end{aligned}$$

Set $F_0(x) = x - T(x, x_0)$ and $r_0 = r(1 - \gamma L) < r$. Then $B[x_0, r_0] \subset B[x_0, r] \subset B(0, R)$ and $\|[F_0'(x)]^{-1}\| = \|[I - \partial_1 T(x, x_0)]^{-1}\| \leqslant \gamma,\ \forall x \in B[x_0, r_0]$. Further, $\gamma\|F_0(x_0)\| = \gamma\|x_0 - T(x_0, x_0)\| < r(1 - \gamma L) = r_0$. By Theorem 4, there exists $x_1 \in X$ such that $\|x_1 - x_0\| < r_0$ and $F_0(x_1) = x_1 - T(x_1, x_0) = 0$. Now suppose by induction that there are $\{x_k\}$ ($k = 1, ..., n$) such that

$$\|x_{k+1} - x_k\| < r_k,$$

where $r_k = (\gamma L)^k r_0$, and

$$F_k(x_{k+1}) = x_{k+1} - T(x_{k+1}, x_k) = 0 \ \ (k = 0, ..., n-1).$$

Define $F_n(x) = x - T(x, x_n)$ and $r_n = (\gamma L)^n r_0$. We have for all $x \in B[x_n, r_n]$

$$\begin{aligned}\|x - x_0\| &\leqslant \|x - x_n\| + \|x_n - x_{n-1}\| + \cdots + \|x_1 - x_0\| \\ &< \sum_{i=0}^{n} r_i = r_0(1 + \gamma L + \cdots + (\gamma L)^n) < r_0/(1 - \gamma L) = r.\end{aligned}$$

Thus $B[x_n, r_n] \subset B(x_0, r) \subset B(0, R)$, hence $\|[F_n'(x)]^{-1}\| = \|[I - \partial_1 T(x, x_n)]^{-1}\| \leqslant \gamma$ for all $x \in B[x_n, r_n]$. Moreover,

$$\begin{aligned}\gamma\|F_n(x_n)\| &= \gamma\|x_n - T(x_n, x_n)\| \\ &= \gamma\|T(x_n, x_{n-1}) - T(x_n, x_n) + \{x_n - T(x_n, x_{n-1})\}\| \\ &\leqslant \gamma L\|x_n - x_{n-1}\| < (\gamma L)^n r_0 = r_n.\end{aligned}$$

Theorem 4 again ensures the existence of $x_{n+1} \in B(x_n, r_n)$ such that $F_n(x_{n+1}) = x_{n+1} - T(x_{n+1}, x_n) = 0$. Thus the process (1) is well-defined. Further, since the mapping $F_y(x) = x - T(x, y)$, where $y \in B(0, R)$, satisfies Theorem 3 on global homeomorphism, it follows that the equation $F_n(x) = 0$ has the only solution x_{n+1}. Thus, the set $[x_n]$ is a singleton. Since $\|x_{n+1} - x_n\| < (\gamma L)^n r_0$, $\{x_n\}$ is a Cauchy sequence, hence it converges to some x^*. Obviously $x^* \in F(T)$. Thus we have proved the inclusion $x_0 \in C(T)$ for any fixed $x_0 \in F_\epsilon$, therefore $F_\epsilon \subset C(T)$. The proof of Theorem 5 is complete. ■

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