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Structure of Ann-Categories of Type (R, N)

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Abstract. Ann-category is called almost strict if its natural equivalences, except a natural equivalence of commutativity and of the distributivity, are identities. The purpose of this paper is to prove that every Ann-category is Ann-equivalent to an almost strict Ann-category of the type (R, M) and to give new interpretations of the cohomology groups $H^3(R, M)$ of the rings R.

The present paper consists, in a certain sense, of an extension of our results in [4 - 6]. Reading the present paper requires certain knowledge of the main results, which were announced in [3]. For completeness, we briefly recall some of the material that will be indispensable for the understanding of this paper.

Throughout this paper, for the tensorial product of two objects A and B, we write AB instead of $A \otimes B$, but for the morphisms we still write $f \otimes g$ to avoid confusion with composition.

1. Preliminaries

1.1. The First Two Invariants of Ann-Category

Let us consider an Ann-category,

$$\mathcal{A} = (\mathcal{A}, a^+, c, (0, g, d), a, (I, l, r), \mathcal{L}, \mathcal{R}), \tag{1}$$

in which (a^+, c, g, d) is a system of natural isomorphisms of associativity, commutativity, unity, respectively, for the Picard category $(\mathcal{A}, \oplus)$, and (a, l, r) is a system of natural isomorphisms of associativity, unity for the monoidal category

$(\mathcal{A}, \otimes)$ and $(\mathcal{L}, \mathcal{R})$ is a pair of natural isomorphisms of distributivity

$$\mathcal{L} = \mathcal{L}_{A,B,C} : A(B \oplus C) \longrightarrow AB \oplus AC,$$
$$\mathcal{R} = \mathcal{R}_{A,B,C} : (A \oplus B)C \longrightarrow AC \oplus BC.$$

For each object $A \in \mathcal{A}$, functors $L^A = A \otimes \_$, $R^A = \_ \otimes A$ induce isomorphisms

$$\widehat{L}^A : A \otimes 0 \longrightarrow 0\,,\ \widehat{R}^A : 0 \otimes A \longrightarrow 0.$$

The set $\Pi_0(\mathcal{A})$ of the isomorphic classes of objects of $\mathcal{A}$ is a ring with the operations induced by $\oplus$ and $\otimes$ in $\mathcal{A}$, and $\Pi_1(\mathcal{A}) = Aut(0)$ is an abelian group whose composition law is denoted by $+$. Moreover $\Pi_1(A)$ becomes an $\Pi_0(\mathcal{A})$-bimodule where the left and right operations of the ring $\Pi_0(\mathcal{A})$ on the abelian group $\Pi_1(A)$ are defined by

$$ru = \lambda_X(u),\ ur = \rho_X(u),\ X \in r \in \Pi_0(\mathcal{A}),\ u \in \Pi_1(\mathcal{A}),$$

where λ_X, ρ_X are the two maps $Aut(0) \to Aut(0)$ given by the following commutative diagrams

$$\begin{array}{ccc} X0 & \xrightarrow{\widehat{L}^X} & 0 \\ {\scriptstyle id\otimes u}\downarrow & & \downarrow{\scriptstyle \lambda_X(u)} \\ X0 & \xrightarrow{\widehat{L}^X} & 0 \end{array} \qquad \begin{array}{ccc} X0 & \xrightarrow{\widehat{R}^X} & 0 \\ {\scriptstyle id\otimes u}\downarrow & & \downarrow{\scriptstyle \rho_X(u)} \\ 0X & \xrightarrow{\widehat{R}^X} & 0 \end{array}$$

The ring $R = \Pi_0(\mathcal{A})$ and R-bimodule $N = \Pi_1(\mathcal{A})$ are the first two invariants of Ann-categories.

1.2. The Almost Strict Ann-Category $M(\mathcal{A})$

Definition 1. *An Ann-category $\mathcal{A}$ is called almost strict if its natural equivalences, except the natural isomorphism of commutativity and one isomorphism of distributivity* (*left or right*)*, are identities.*

Definition 2. *Let $\mathcal{A}$ be an Ann-category with the system natural equivalences* (1)*. A M-functor $(F, \breve{F}, \overline{F}) : \mathcal{A} \to \mathcal{A}$ is an AC-functor $(F, \breve{F})$ with respect to the operation $\oplus$ together with isomorphism $\overline{F}$,*

$$\overline{F}_{X,Y} : F(XY) \longrightarrow (FX)Y,$$

such that the following diagram is commutative

$$\begin{array}{ccccc} F(A(BC)) & & \xrightarrow{\quad\quad\overline{F}\quad\quad} & & FA(BC) \\ {\scriptstyle F(a)}\downarrow & & & & \downarrow{\scriptstyle a} \\ F((BC)C) & \xrightarrow{\overline{F}} & F(AB)C & \xrightarrow{\overline{F}\otimes id} & ((FA)B)C \end{array}$$

and the following conditions are satisfied:

(i) *the family $(\overline{F}_{X,Y})_Y$ is $\oplus$-morphism from $F \circ L^X$ to L^{FX},*

(ii) *the family* $(\overline{F}_{X,Y})_X$ *is* $\oplus$*-morphism from* $F \circ R^Y$ *to* $L^Y \circ F$.

A M-morphism from $(F, \breve{F}, \overline{F})$ to $(G, \breve{G}, \overline{G})$ is $\oplus$-morphism $\varphi : F \to G$ such that the following diagram is commutative

$$\begin{array}{ccc} F(AB) & \xrightarrow{\overline{F}} & (FA)B \\ \varphi\downarrow & & \downarrow\varphi\otimes id \\ G(AB) & \xrightarrow{\overline{G}} & (GA)B \end{array}$$

The category of M-morphisms and M-functors of Ann-category $\mathcal{A}$ is denoted by $M(\mathcal{A})$.

Theorem 3.
1) $M(\mathcal{A})$ *is an almost strict Ann-category in which the commutativity constraint and the left distributivity constraint are defined by*

$$c^*_{F,G}(X) = c_{FX,GX},$$
$$\mathcal{L}^*_{F,G,H}(X) = \breve{F}_{GX,HX},$$

where $X \in \mathcal{A}$.
2) *Any Ann-category* $\mathcal{A}$ *is Ann-equivalent to the almost strict Ann-category* $M(\mathcal{A})$.

1.3. The Structure of Ann-Category of Type (R, N)

Given an Ann-category $\mathcal{A}$. The reduced category $\mathcal{S}$ is constructed as follows: its objects are the elements of ring $R = \Pi_0(\mathcal{A})$, and its morphisms are automorphisms of the form (r, u) with $r \in R,\ u \in N = \Pi_1(\mathcal{A})$, i.e.,

$$Aut(r) = \{r\} \times N.$$

The composition law of morphisms is induced by addition in N. We shall use the transportation of structures (see [3]) to transform $\mathcal{S}$ into an Ann-category equivalent to $\mathcal{A}$.

Choose for every $r \in R$ a representative $X_r \in \mathcal{A}$ such that $X_0 = 0,\ X_1 = 1$, and then, for every pair $r, s \in R$, two families of isomorphisms

$$\varphi_{r,s} : X_r \oplus X_s \longrightarrow X_{r+s},$$
$$\psi_{r,s} : X_r X_s \longrightarrow X_{rs},$$

such that

$$\varphi_{0,s} = g_{X_s}, \quad \varphi_{r,0} = d_{X_r}$$
$$\psi_{1,s} = l_{X_s}, \quad \psi_{r,1} = r_{X_r},\ \psi_{0,s} = \widehat{R}^{X_s},\ \psi_{r,0} = \widehat{L}^{X_r}.$$

The family (X_r, φ, ψ) is called a *stick* of Ann-category $\mathcal{A}$.

After that, define the functor $H : \mathcal{S} \longrightarrow \mathcal{A}$ by

$$H(r) = X_r, \quad H(r, u) = \gamma_{X_r}(u),$$

and put $\breve{H} = \varphi^{-1}$, $\widetilde{H} = \Phi^{-1}$, in which

$$\begin{aligned} \gamma_X : \mathrm{End}(O) &\longrightarrow \mathrm{End}(X) \\ u &\longmapsto \gamma_X(u) \end{aligned}$$

is the isomorphism defined by the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\gamma_X(u)} & X \\ g\Big\uparrow & & \Big\uparrow g \\ 0 \oplus X & \xrightarrow{u \oplus id} & 0 \oplus X \end{array}$$

(The fact that γ_X is an isomorphism follows from the equivalence of the functors $F = - \oplus X$ in a Gr-category $(\mathcal{A}, \oplus)$).

By means of transportation of structures, the category $\mathcal{S}$ has the structure of an Ann-category with respect to which $(H, \breve{H}, \widetilde{H})$ is Ann-equivalence from $\mathcal{S}$ to $\mathcal{A}$. In $\mathcal{S}$ the two operations have the explicit form:

$$\begin{aligned} & r \oplus s = r + s \quad (\text{sum in ring } R), \\ & r \otimes s = rs \quad (\text{product in ring } R), \\ & (r, u) \otimes (s, v) = (rs, rv + us). \end{aligned}$$

The natural equivalences of the Ann-category structure in $\mathcal{S}$ are induced from those of $\mathcal{A}$ by $(H, \breve{H}, \widetilde{H})$. For example natural associativity α of operation $\otimes$ is defined by the commutative diagram

$$\begin{array}{ccccc} X_r(X_sX_t) & \xleftarrow{id \otimes \widetilde{H}} & X_rX_{st} & \xleftarrow{\widehat{H}} & X_{rst} \\ a\Big\downarrow & & & & \Big\downarrow H(\alpha) \\ (X_rX_s)X_t & \xleftarrow{\widetilde{H} \otimes id} & X_{rs}X_t & \xleftarrow{\widehat{H}} & X_{rst} \end{array}$$

with $H(\alpha(r, s, t)) = \gamma_{X_{rst}}(\alpha(r, s, t))$.

This commutative diagram means the compatibility of $(H, \widetilde{H})$ with respect to the pair $(\alpha,\ a)$ of natural isomorphism of associatitity.

In the Ann-category $\mathcal{S}$, the natural equivalences of unity of the two operations $\oplus$, $\otimes$ are identities. We denote a system of natural equivalences of $\mathcal{S}$ by

$$(\xi, \eta, (0, id, id), \alpha, (1, id, id), \lambda, \rho).$$

$\mathcal{S}$ is called *Ann-category of type* (R, N) and the family of functions $(\xi, \eta, \alpha, \lambda, \rho)$ is called a *structure* of Ann-category of type (R, N). They satisfy 17 relations (see [4]).

If an Ann-category $\mathcal{A}$ satisfies *the regular condition* $c_{X,X} = id$, the function η induced by the commutativity constraint c satisfies condition $\eta(x, x) = 0$. Then the family $(\xi, \eta, \alpha, -\lambda, \rho)$ is a 3-cocycle of the ring $R = \Pi_0(\mathcal{A})$ with coefficients in the R-bimodule $N = \Pi_1(\mathcal{A})$ in the sense of Mac Lane-Shukla. The main result in [5] is as follows.

Theorem 4. *Any regular Ann-category is Ann-equivalent to a reduced Ann-category of type* (R, N) *having structure isomorphisms* $(\xi, \eta, \alpha, \lambda, \rho)$ *where* ξ, η *are identities.*

This result will be enhanced in Corollary 8.

2. Main Results

Now we shall present and prove the main result of this paper.

Theorem 5. *Any Ann-category* $\mathcal{A}$ *is Ann-equivalent to an almost strict Ann-category of type* (R, N).

In order to prove this theorem, we need some lemmas. First, for any two choices of sticks (X_r, φ, ψ) and (X'_r, φ', ψ') we prove the following:

Lemma 6. *Suppose* $\mathcal{S}$ *a stick* (X_r, φ, ψ) *and* $\mathcal{S}'$ *a stick* (X'_r, φ', ψ') *are two reduced Ann-categories of* $\mathcal{A}$. *Then*

(i) *There exists an Ann-equivalence* $(F, \breve{F}, \widetilde{F}) : \mathcal{S} \longrightarrow \mathcal{S}'$ *with* $F = id$.

(ii) *The two structures* $(\xi, \eta, \alpha, \lambda, \rho)$ *and* $(\xi', \eta', \alpha', \lambda', \rho')$, *of* $\mathcal{S}$ *and* $\mathcal{S}'$ *respectively, satisfy the following relations*

$$
\begin{aligned}
&(\xi' - \xi)(x, y, z) = \mu(y, z) - \mu(x + y, z) + \mu(x, y + z) - \mu(x, y),\\
&(\eta' - \eta)(x, y) = \mu(x, y) - \mu(y, x) = ant\ \mu(x, y),\\
&(\alpha' - \alpha)(x, y, z) = x\nu(y, z) - \nu(xy, z) + \nu(x, yz) - \nu(x, y)z,\\
&(\lambda' - \lambda)(x, y, z) = \nu(x, y + z) - \nu(x, y) - \nu(x, z) + x\mu(y, z) - \mu(xy, xz),\\
&(\rho' - \rho)(x, y, z) = \nu(x + y, z) - \nu(x, z) - \nu(y, z) + \mu(x, y)z - \mu(xz, yz),
\end{aligned}
$$

where $\mu,\ \nu : R \times R \longrightarrow N$ *are two functions satisfying the conditions*

$$
\begin{gathered}
\mu(0, y) = \mu(x, 0) = 0,\\
\nu(0, y) = \nu(x, 0) = \nu(1, y) = \nu(x, 1) = 0.
\end{gathered}
$$

Proof.

(i) We have canonical Ann-equivalences:

$$
\begin{gathered}
(H, \breve{H}, \widetilde{H}) : \mathcal{S} \longrightarrow \mathcal{A},\\
(H', \breve{H}^{,}, \widetilde{H}^{,}) : \mathcal{S}' \longrightarrow \mathcal{A},
\end{gathered}
$$

where $\breve{H} = \varphi^{-1}$, $\widetilde{H} = \psi^{-1}$, $\breve{H}^{,} = \varphi^{,-1}$, $\widetilde{H}^{,} = \psi^{,-1}$.

Let $(K, \breve{K}, \widetilde{K})$ be the Ann-functor inverse to the Ann-equivalence $(H, \breve{H}, \widetilde{H})$. We put

$$F = K \circ H', \quad \breve{F} = \breve{KH}', \quad \widetilde{F} = \widetilde{KH}',$$

and then $(F, \breve{F}, \widetilde{F}) : \mathcal{S}' \longrightarrow \mathcal{S}$ is an Ann-equivalence where $F = id$.
(ii) We put $\mu = \breve{KH}', \nu = \widetilde{KH}'$. Then the compatibility of Ann-functor $(F, \breve{F}, \widetilde{F}) = (id, \mu, \nu)$ for natural equivalences of $\mathcal{S}$ and $\mathcal{S}'$ implies the relations to be proved. Moreover, we can check that

$$\mu(0, y) = \mu(x, 0) = 0,$$
$$\nu(0, y) = \nu(x, 0) = \nu(1, y) = \nu(x, 1) = 0.$$

by the "normality" of functions $\xi, \mu, \alpha, \lambda, \rho$. ∎

Lemma 7. *Let $\mathcal{S}$ be the reduced Ann-category of $\mathcal{A}$, with the structure $(\xi, \eta, \alpha, \lambda, \rho)$ and let $\mu, \nu : R \times R \longrightarrow N$ be any pair of functions satisfying the conditions*

$$\mu(0, y) = \mu(x, 0) = 0,$$
$$\nu(0, y) = \nu(x, 0) = \nu(1, y) = \nu(x, 1) = 0.$$

Then the family $(\xi', \eta', \alpha', \lambda', \rho')$ satisfying the rules in Lemma 6 is a structure of the reduced Ann-category $\mathcal{S}'$ of $\mathcal{A}$.

Proof. Let μ, ν be as in Lemma 6. After choosing the representation (X'_r), where $X'_0 = 0$, $X'_1 = 1$, we construct functor $H' : \mathcal{S} \longrightarrow \mathcal{A}$. Then $\breve{H}'$ and $\widetilde{H}'$ are chosen for $\breve{HK}' = \mu$ and $\widetilde{KH}' = \nu$ with $(K, \breve{K}, \widetilde{K})$ an Ann-functor in the proof of Lemma 6. Now we put $\varphi' = (\breve{H}')^{-1}$ and $\psi' = (\widetilde{H}')^{-1}$. With the stick (X'_r, φ', ψ') we construct the reduced Ann-category $\mathcal{S}'$. The rest follows from Lemma 6.

Proof of Theorem 5. Let $\mathcal{S} = (R, N)$ be the reduced Ann-category of an Ann-category $\mathcal{A}$ and $(\xi, \eta, \alpha, \lambda, \rho)$ is its structure. We shall prove $\mathcal{S}$ is Ann-equivalent to an almost strict reduced Ann-category $\mathcal{S}'$ of $\mathcal{A}$ in the sense of Lemma 7.

Fisrt, we describe the structure of reduced Ann-category $\mathcal{S}$ due to the Ann-equivalences between $\mathcal{S}$ and an almost strict Ann-category $M(\mathcal{S})$ mentioned in Theorem 3. Let

$$\Phi : \mathcal{S} \longrightarrow M(\mathcal{S}),$$

be an Ann-equivalence and Γ be the Ann-functor inverse to Φ mentioned in Theorem 3. They have the explicit forms:

$$\Phi(x) = \Phi_x = (L^x, \breve{L}^x, \widetilde{L}^x), \quad x \in R, \ L^x = x \otimes -,$$
$$\Phi(x, u) = u^* : L^x \longrightarrow L^x, \ u^*_y = (x, u) \otimes (y, 0) = (xy, uy),$$
$$\breve{\Phi}_{x,y}(z) = ((x + y)z, \rho(x, y, z)) = (\bullet, \rho(x, y, z)),$$
$$\widetilde{\Phi}_{x,y}(z) = (xyz, -\alpha(x, y, z)) = (\bullet, -\alpha(x, y, z)),$$

and

$$\Gamma : M(\mathcal{S}) \longrightarrow \mathcal{S},$$
$$(F, \breve{F}, \widetilde{F}) \longmapsto F(1).$$

For $\varphi : F \longrightarrow G$, an M-morphism, we set

$$\Gamma(\varphi) = \varphi_1 : F(1) \longrightarrow G(1).$$

Then we have an isomorphism

$$r : \Gamma \circ \Phi \simeq id_{\mathcal{S}},$$

where r is the natural isomorphism for right unity of $\mathcal{S}$. As $r = id$, so $\Gamma \circ \Phi = id_{\mathcal{S}}$.

Now we shall prove some properties of the Ann-equivalence $(\Gamma, \breve{\Gamma}, \widetilde{\Gamma})$, and describe natural isomorphisms of $\mathcal{S}$. The isomorphism

$$\breve{\Gamma}_{F,G} : \Gamma(F \oplus G) \longrightarrow \Gamma F \oplus \Gamma G,$$

where $F, G \in M(\mathcal{S})$, induces a function $f : R \times R \longrightarrow N$,

$$(x + y, f(x, y)) = \breve{\Gamma}_{\Phi_x, \Phi_y} : \Gamma(\Phi_x \oplus \Phi_y) \longrightarrow \Gamma\Phi_x \oplus \Gamma\Phi_y (= x + y).$$

For the isomorphism $\breve{\Gamma}$, we have the following relations:

$$\breve{\Gamma}_{\Phi_x, \Phi_y \oplus \Phi_z} = \breve{\Gamma}_{\Phi_x, \Phi_{y+z}} = (x + y + z, f(x, y + z)), \tag{2}$$
$$\breve{\Gamma}_{\Phi_x \oplus \Phi_y, \Phi_z} = \breve{\Gamma}_{\Phi_{x+y}, \Phi_z} = (x + y + z, f(x + y, z)). \tag{3}$$

Actually, according to the naturality of $\breve{\Gamma}$, we have the commutative diagram

$$\begin{array}{ccc}
\Gamma(\Phi_x \oplus (\Phi_y \oplus \Phi_z)) & \xrightarrow{\quad \breve{\Gamma} \quad} & \Gamma\Phi_x \oplus \Gamma(\Phi_y \oplus \Phi_z) \\
\Gamma(id \oplus \breve{\Phi}) \Big\uparrow & & \Big\uparrow id \oplus \Gamma\breve{\Phi} \\
\Gamma(\Phi_x \oplus \Phi_{y+z}) & \xrightarrow{(\bullet, f(x, y+z)) \oplus id} & \Gamma\Phi_x \oplus \Gamma\Phi_{y+z}
\end{array}$$

According to the proof of Theorem 3 (see [4]) we get

$$\Gamma(\breve{\Phi}_{y,z}) = (y + z, \rho(y, z, 1)) = (y + z, 0).$$

Then by definition,

$$\Gamma(id \oplus \breve{\Phi}) = id,$$

we get equation (2). Similary we have equation (3).

The coherence condition for $\breve{\Gamma}$ (for distributivity) gives us the commutative diagram

$$\begin{array}{ccccc}
x + (y + z) & \xleftarrow{\ id \oplus \breve{\Gamma}\ } & x + \Gamma(\Phi_y \oplus \Phi_z) & \xleftarrow{\quad \breve{\Gamma} \quad} & \Gamma(\Phi_x \oplus (\Phi_y \oplus \Phi_z)) \\
\Big\downarrow (\bullet, \xi(x, y, z)) & & & & \Big\downarrow \Gamma(id) \\
(x + y) + z & \xleftarrow{\ x + (y + z)\ } & \Gamma((\Phi_x \oplus \Phi_y) \oplus \Phi_z) & \xleftarrow{\quad \breve{\Gamma} \quad} & \Gamma((\Phi_x \oplus \Phi_y) \oplus \Phi_z)
\end{array}$$

Hence

$$\xi(x,y,z) = -f(y,z) + f(x+y,z) - f(x,y+z) + f(x,y). \tag{4}$$

Now, the isomorphism $\widetilde{\Gamma}$ defines a function $g : R \times R \to N$ as follows:

$$(xy, g(x,y)) = \widetilde{\Gamma}_{\Phi_x,\Phi_y} : \Gamma(\Phi_x\Phi_y) \longrightarrow \Gamma\Phi_x\Gamma\Phi_y = xy.$$

The natural property of the isomorphism $\breve{\Gamma}$ implies the commutative diagram

$$\begin{array}{ccc} \Gamma(\Phi_x(\Phi_y\Phi_z)) & \xrightarrow{\widetilde{\Gamma}} & \Gamma\Phi_x\Gamma(\Phi_y\Phi_z) \\ \Gamma(id \otimes \widetilde{\Phi})\Big\uparrow & & \Big\uparrow id \otimes \Gamma\widetilde{\Phi} \\ \Gamma(\Phi_x\Phi_{yz}) & \xrightarrow{\widetilde{\Gamma}} & \Gamma\Phi_x\Gamma\Phi_{yz} \end{array}$$

Also from the proof of Theorem 3, we have

$$\widetilde{\Phi}_{y,z} = \big(-\alpha(y,z,t)\big)_{t\in R},$$
$$\Gamma\widetilde{\Phi_{y,z}} = (yz, -\alpha(y,z,1)) = (yz, 0)$$

and

$$\Gamma(id \otimes \widetilde{\Phi}) = id.$$

Finally

$$\widetilde{\Gamma}_{\Phi_x,\Phi_y\Phi_z} = (xyz, g(x,yz)), \tag{2'}$$

and

$$\widetilde{\Gamma}_{\Phi_x\Phi_y,\Phi_z} = (xyz, g(xy,z)). \tag{3'}$$

Now the compatibility of isomorphism $\widetilde{\Gamma}$ for a pair of natural isomorphisms of associativity $(a = id, \alpha)$ of products in $M(\mathcal{S})$ and $\mathcal{S}$ give us:

$$\alpha(x,y,z) = -xg(y,z) + g(xy,z) - g(x,yz) + g(x,y)z. \tag{5}$$

Similarly, the compatibility of pair $(\breve{F}, \widetilde{F})$ for natural isomorphisms of right distributivity $(R^* = id, \rho)$ gives us the relation

$$\rho(x,y,z) = f(xz,yz) - f(x,y)z + g(x,y) + g(y,z) - g(x+y,z). \tag{6}$$

In particular, for a pair of natural isomorphisms of left distributivity $(\mathcal{L}^*, \lambda)$, the compatibility of $(\Gamma, \breve{\Gamma}, \widetilde{\Gamma})$ for them gives the following commutative diagram

$$\begin{array}{ccccc} \Gamma(\Phi_x(\Phi_y \oplus \Phi(z))) & \xrightarrow{\widetilde{\Gamma}} & \Gamma\Phi_x.\Gamma(\Phi_y \oplus \Phi_z) & \xrightarrow{id \oplus \breve{\Gamma}} & \Gamma\Phi_x\Gamma\Phi_y \oplus \Gamma\Phi_x\Gamma\Phi_z \\ \Gamma(\mathcal{L}^*)\Big\downarrow & & & & \Big\downarrow(\bullet, \lambda(x,y,z)) \\ \Gamma(\Phi_x\Phi_y \oplus \Phi_x\Phi_z) & \xrightarrow{\breve{\Gamma}} & \Gamma(\Phi_x\Phi_y) \oplus \Gamma(\Phi_x\Phi_z) & \xrightarrow{\widetilde{\Gamma} \oplus \widetilde{\Gamma}} & \Gamma\Phi_x\Gamma\Phi_y \oplus \Gamma\Phi_x\Gamma\Phi_z \end{array}$$

Since

$$\Gamma(\mathcal{L}^*) = \mathcal{L}^*(1) = (\breve{\Phi}_x)_{\Phi_y(1),\Phi_z(1)}$$
$$= (\breve{\Phi}_x)_{y,z} = (x(y+z), \lambda(x,y,z)),$$

we obtain the relation

$$g(x, y+z) - g(x, y) - g(x, z) + xf(y, z) - f(xy, xz) = 0. \tag{7}$$

Now, according to Lemma 7, we can choose the new stick with $\mu = f$ and $\nu = g$. Then according to Lemma 6 and Relations (4)–(7), the reduced Ann-category $\mathcal{S}'$ has the structure $(\xi', \eta', \alpha', \lambda', \rho')$ satisfying

$$\xi' = 0,\ \eta' = \eta + ant\mu,\ \alpha' = 0,\ \rho' = 0,\ \lambda' = \lambda,$$

which means $\mathcal{S}'$ is almost strict. The proof is complete. ∎

According to Theorem 5, each structure of a reduced Ann-category of type (R, N) is a pair of functions

$$\eta : R \times R \longrightarrow N,\ \lambda : R \times R \times R \longrightarrow N$$

satisfying the following relations

1. $\eta(x + y, z) - \eta(x, z) - \eta(y, z) = 0$.
2. $\eta(x, y) + \eta(y, x) = 0$.
3. $\eta(x, y)z - \eta(xz, yz) = 0$.
4. $x\eta(y, z) - \eta(xy, xz) = \lambda(x, y, z) - \lambda(x, z, y)$.
5. $\lambda(x, z, t) - \lambda(x, y + z, t) + \lambda(x, y, z + t) - \lambda(x, y, z) = 0$.
6. $\lambda(x, z, t) + \lambda(y, z, t) - \lambda(x + y, z, t) = -\eta(xt, yz)$.
7. $x\lambda(y, z, t) + \lambda(x, yz, yt) - \lambda(xy, z, t) = 0$.
8. $\lambda(x, yt, zt) - \lambda(x, y, z)t = 0$.
9. $\lambda(1, y, z) = \lambda(0, y, z) = \lambda(x, 0, z) = \lambda(x, y, 0) = 0$.

The above relations can be derived from 17 relations for a structure in [4].

Corollary 8. *For a regular Ann-category $\mathcal{A}$, its reduced Ann-category $\mathcal{S}$ has the structure function $\lambda : R \times R \times R \longrightarrow N$ satisfying the following relations*

$$\begin{aligned}
&\lambda(x, y, z) - \lambda(x, z, y) = 0,\\
&\lambda(x, z, t) - \lambda(x, y + z, t) + \lambda(x, y, z + t) - \lambda(x, y, z) = 0,\\
&\lambda(x, z, t) + \lambda(y, z, t) - \lambda(x + y, z, t) = 0,\\
&x\lambda(y, z, t) + \lambda(x, yz, yt) - \lambda(xy, z, t) = 0,\\
&\lambda(x, yt, zt) = \lambda(x, y, z)t,\\
&\lambda(1, y, z) = \lambda(0, y, z) = \lambda(x, 0, z) = \lambda(x, y, 0) = 0.
\end{aligned}$$

Proof. Since an Ann-category $\mathcal{A}$ is regular, so $c_{X,X} = id$ and hence $\eta(x, x) = 0$ for each $x \in R$. Then we can choose a new stick with $\eta = id$ (see [6]). ∎

Since any structure of regular Ann-category of type (R, N) corresponds to an element of the cohomology group $H^3(R, N)$ of the ring R, the above result gives us a new interpretation of the cohomology group $H^3(R, M)$ and this suggests us a construction of a complex which is simpler than those in [6, 7].

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