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On the Existence, Uniqueness and Stability of Solutions of Stochastic Volterra–Ito Equation

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Abstract. The aim of this paper is to prove the existence and uniqueness for solution of a stochastic Volterra–Ito equation and to give conditions for the asymptotic mean square stability of the trivial solution.

1. Introduction

In this paper, we consider a class of stochastic integral equations of the Volterra-Ito type driven by a Brownian motion. This type of equation is of following form

$$x(t,\omega) = h(t,\omega) + \int_0^t K_1(t,\tau,\omega) f(\tau, x(\tau,\omega)) d\tau + \int_0^t K_2(t,\tau,\omega) g(\tau, x(\tau,\omega)) dB_\tau, \tag{1.1}$$

$$x(t_0,\omega) = x_0,$$

where $(B_t, t \geq 0)$ is a Brownian motion on R^+, and K_1, K_2 are two random kernels.

We study the existence and uniqueness for solution of the equation (1.1), under some conditions imposed on random kernels K_1, K_2 and functions h, f, g. Also the problem of asymptotic stability for the random solution of (1.1) is investigated.

This class of stochastic integral equations of the Volterra–Ito type is of great importance in various applications to engineering and biology.

The paper is organized as follows. The next section is devoted to preliminary notions and results needed to own study. The Sec. 3 presents own main results consisting of the existence and uniqueness and the asymptotic stability of the solution.

2. Priliminaries

In this section, some basic definitions and lemmas are given. All stochastic processes here are supposed to be defined in a complete probability space $(\Omega, \mathcal{F}, P)$. We suppose that $x_t \equiv 0$ is a solution of (1.1), that is

$$0 = h(t,\omega) + \int_0^t K_1(t,\tau,\omega) f(\tau,0) d\tau + \int_0^t K_2(t,\tau,\omega) g(\tau,0) dB_\tau.$$

Definition 2.1. *The solution $x_t \equiv 0$ of Equation (1.1) is said to be p-stable $(p > 0)$ if for any $\epsilon > 0$ there exists $r > 0$ such that $||x_0|| < r$ and $t > t_0$*

$$E||x(t,\omega,t_0,x_0)||^p < \epsilon.$$

Definition 2.2. *The solution $x_t \equiv 0$ of Equation (1.1) is said to be p-asymptotically stable if it is p-stable and if x_0 is small enough then*

$$E\|x(t,\omega,t_0,x_0)\|^p \to 0 \ \text{ as } \ t \to \infty.$$

Definition 2.3. *Let $C_b = C_b(\mathbb{R}^+, L_2(\Omega, \mathcal{F}, P))$ denote the space of all random processes $x(t)$ defined on $\mathbb{R}^+$ such that*
(i) $\|x(t)\|_{L_2} < \infty, \forall t \in \mathbb{R}^+$,
(ii) $x(t), t \in \mathbb{R}^+$, *is continuous in mean square.*

Thus, we see that C_b is a Banach space with the norm defined by

$$||x||_{C_b} = \big(\sup_{t\geq 0}||x(t)||^2_{L_2}\big)^{1/2}.$$

Definition 2.4. *A random variable $x(\omega)$ is said to belong to $L_\infty(\Omega, \mathcal{F}, P)$ if there exists a constant $z > 0$ such that*

$$P\{x(\omega) > z\} = 0.$$

We can define the norm of $x(\omega)$ by

$$|||x(\omega)||| = P\text{-}\operatorname*{esssup}_{\Omega}|x(\omega)| = \inf_{\Omega_0}\{\sup_{\Omega-\Omega_0}|x(\omega)|\},$$

and $P(\Omega_0) = 0$.

Definition 2.5. *Let $\xi(t,\omega)$ be an n-dimensional measuarable random process. Define*

$$\chi_p[\xi] = \chi_p[\xi(t,\omega)] = \overline{\lim_{t\to\infty}}\frac{1}{t}\ln E||\xi(t,\omega)||^p,$$

then $\chi_p[\xi]$ is called the Liapunov p-exponent of the process $\xi(t,\omega)$.

The following result can be found in [4].

Lemma 2.1. *For any fixed $p > 0$, denote by $\phi(p)$ the set of p-Liapunov exponents of all non-trivial solutions of the following equation*

$$\begin{cases} dx = G(t, x, \xi)dt, \\ G(t, 0, \xi) = 0, \\ x(t_0) = x_0, \ \ t \geq 0, \end{cases}$$

where ξ is a random process.

If $\sup\limits_{t_0, x_0} \phi(p) < 0$ then the solution $x \equiv 0$ of the above equation is asymptotically p-stable.

Lemma 2.2. *Denoting by $\chi_p[\xi_i]$ the p-Liapunov exponent of processes $\xi_i(t, \omega), 0 \leq i \leq n$, we have the following assertions:*

(i)
$$\chi_p\Big[\sum_{0 \leq i \leq n} \xi_i\Big] \leq \max_{0 \leq i \leq n} \chi_p[\xi_i],$$

(ii) *if $\xi(\omega), \eta(\omega)$ are independent random processes then*

$$\chi_p(\xi\eta) \leq \chi_p(\xi) + \chi_p(\eta),$$

(iii) *if c is a positive constant then*

$$\chi_p(c\xi) = \chi_p(\xi).$$

The proof of Lemma 2.2 follows directly from the above definitions.

3. Main Results

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space. Suppose that $(\mathcal{F}_t, t \in \mathbb{R}^+)$ is a family of increasing subalgebras of $\mathcal{F}$, i.e. $\mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F}$ for $0 < s \leq t < \infty$. B_t dentotes a standard Brownian motion adapted to $\mathcal{F}_t$.

We now prove the existence and uniqueness of random solution and asymptotic stability of random solution of Equation (1.1).

Theorem 3.1. *Assume that in Equation* (1.1)

(i) *$K_1(t, \tau, \omega), K_2(t, \tau, \omega)$ belong to $L_\infty(\Omega, \mathcal{F}, P)$ and $K_2(t, \tau, \omega)$ is adapted to $\mathcal{F}_t$ and there exists a constant $z > 0$ such that*

$$\int_0^t |||K_i(t, \tau, \omega)|||d\tau < z \quad \text{for every } t \in \mathbb{R}^+, \ i = 1, 2,$$

(ii) *the functions $f(t, x)$, and $g(t, x)$ are continuous with repect to t and* *uniformly continuous with respect to* $x \in \mathbb{R}$*, and satisfy Lipschitz conditions of the same constant* $l > 0$*:*

$$|f(t,x) - f(t,y)| < l|x-y|,$$
$$|g(t,x) - g(t,y)| < l|x-y|.$$

Moreover $f(t,0), g(t,0)$ *are continuous and bounded functions on* $\mathbb{R}^+$*,*

(iii) $h(t,\omega) \in C_b(\mathbb{R}^+, L_2(\Omega, \mathcal{F}, P))$,

(iv) $lz < 1/2$.

Then with respect to the initial condition x_0*, there exists a unique solution* $x(t,\omega) \in C_b(\mathbb{R}^+, L_2(\Omega, \mathcal{F}, P))$ *of Equation* (1.1).

Proof. It is easy to see from the property of composite functions that f and g map C_b into C_b.

Let us define an operator S on C_b as

$$Sx(t,\omega) = h(t,\omega) + \int_0^t K_1(t,\tau,\omega) f(\tau, x(\tau,\omega)) d\tau + \int_0^t K_2(t,\tau,\omega) g(\tau, x(\tau,\omega)) dB_\tau.$$

Then we have

$$\begin{aligned}
\|Sx(t,\omega)\|^2_{L_2} &\le 3\|h(t)\|^2_{L^2} + 3\Big(\Big\|\int_0^t K_1(t,\tau,\omega) f(\tau, x(\tau,\omega)) d\tau\Big\|_{L^2}\Big)^2 \\
&\quad + 3\Big(\Big\|\int_0^t K_2(t,\tau,\omega) g(\tau, x(\tau,\omega)) dB_\tau\Big\|_{L_2}\Big)^2 \\
&\le 3\|h(t)\|^2_{L^2} + 3\int_0^t |||K_1(t,\tau,\omega)|||^2 . \big\|f(\tau, x(\tau,\omega))\big\|^2_{L_2} d\tau \\
&\quad + 3\int_0^t |||K_2(t,\tau,\omega)|||^2 . \big\|g(\tau, x(\tau,\omega))\big\|^2_{L_2} d\tau \\
&\le 3\|h(t)\|^2_{L^2} + 3z^2\big(\sup_{t\ge 0}\|f(t,x(t))\|^2_{L_2} + \sup_{t\ge 0}\|g(t,x(t))\|^2_{L_2}\big) \\
&\le 3\|h(t)\|^2_{L^2} + 3z^2\big(\|f(t,x(t))\|^2_{C_b} + \|g(t,x(t))\|^2_{C_b}\big) < \infty.
\end{aligned}$$

Therefore $S(C_b) \subset C_b$. (1)

Next, we prove that there exists a fixed point for the operator S. For any $x_1(t), x_2(t) \in C_b$, we have

$$
\begin{aligned}
\|Sx_1(t)-Sx_2(t)\|^2{}_{L_2} &\le 2\Big\|\int\limits_0^t K_1(t,\tau,\omega)\big[f(\tau,x_1(\tau))-f(\tau,x_2(\tau))\big]d\tau\Big\|^2_{L_2}\\
&\quad + 2\Big\|\int\limits_0^t K_2(t,\tau,\omega)\big[g(\tau,x_1(\tau))-g(\tau,x_2(\tau))\big]dB_\tau\Big\|^2_{L_2}\\
&\le 2\int\limits_0^t |||K_1(t,\tau,\omega)|||^2.\big\|f(\tau,x_1(\tau))-f(\tau,x_2(\tau))\big\|^2_{L_2}d\tau\\
&\quad + 2\int\limits_0^t |||K_2(t,\tau,\omega)|||^2.\big\|g(\tau,x_1(\tau))-g(\tau,x_2(\tau))\big\|^2_{L_2}d\tau\\
&\le 4l^2z^2\big\|x_1(t)-x_2(t)\big\|^2_{L_2},
\end{aligned}
$$

so

$$
\big\|Sx_1(t)-Sx_2(t)\big\|_{C_b} \le 2lz\big\|x_1(t)-x_2(t)\big\|_{C_b}.
$$

Now the condition (iv) $2lz < 1$ implies that the operator S is a contraction operator. (2)

Due to (1) and (2), an application of the Banach's fixed point theorem shows that there exists a unique fixed point of S in C_b. This means also that there exists a unique solution of (1.1). The proof of Theorem 3.1 is thus complete.

■

Remark. Suppose that $|||K_1(t,\tau,\omega)|||, |||K_2(t,\tau,\omega)|||$ are continuous in t,τ on a finite interval $[0,T]$, for almost all $\omega\in\Omega$ and the functions $f(t,x)$, $g(t,x)$ satisfy conditions (ii) stated in Theothem 3.1.

Then there exists a unique solution of Equation (1.1).

Indeed, we firstly observe that

$$
\sup_{t,\tau}\big\{|||K_1(t,\tau,\omega)|||, |||K_2(t,\tau,\omega)|||\big\}
$$

exists, because $|||K_1(t,\tau,\omega)|||$ and $|||K_2(t,\tau,\omega)|||$ are continuous in $t,\tau\in[0,T]$. Now by putting

$$
k=\max\big\{\sup_{t,\tau}\{|||K_1(t,\tau,\omega)|||\},\sup_{t,\tau}\{|||K_2(t,\tau,\omega)|||\}\big\},
$$

we have

$$
\begin{aligned}
\|Sx_1(t)-Sx_2(t)\|^2_{L_2} &\le 2\Big\|\int\limits_0^t K_1(t,\tau,\omega)\big[f(\tau,x_1(\tau))-f(\tau,x_2(\tau))\big]d\tau\Big\|^2_{L_2}\\
&\quad + 2\Big\|\int\limits_0^t K_2(t,\tau,\omega)\big[g(\tau,x_1(\tau))-g(\tau,x_2(\tau))\big]dB_\tau\Big\|^2_{L_2}
\end{aligned}
$$

$$\leq 2\int_0^t |||K_1(t,\tau,\omega)|||^2.\big\|f(\tau,x_1(\tau))-f(\tau,x_2(\tau))\big\|_{L_2}^2 d\tau$$

$$+\,2\int_0^t |||K_2(t,\tau,\omega)|||^2.\big|g(\tau,x_1(\tau))-g(\tau,x_2(\tau))\big|_{L_2}^2 d\tau$$

$$\leq 4l^2k^2\|x_1(t)-x_2(t)\|_{C_b}^2 t^2.$$

Then we can write

$$\|Sx_1(t)-Sx_2(t)\|_{C_b}\leq 2lk\|x_1(t)-x_2(t)\|_{C_b}T. \tag{3}$$

We define $S^n x = S(S^{n-1}x),\ n\geq 1,$ since

$$\|S^n x_1(t)-S^n x_2(t)\|_{L_2}^2 \leq 4l^2k^2\int_0^t \|S^{n-1}x_1(t)-S^{n-1}x_2(t)\|_{L_2}^2 T$$

$$\leq \dots \leq \frac{(2lk)^{2n}t^{2n}}{n!}||x_1(t)-x_2(t)||_{L_2}^2,$$

it flows from (3) that

$$\|S^n x_1(t)-S^n x_2(t)\|_{C_b}\leq \frac{(2lk)^n T^n}{\sqrt{n!}}\|x_1(t)-x_2(t)\|_{C_b}.$$

Because $(2lk)^nT^n/\sqrt{n!}\to 0$, there exists $n_0\in N$ such that $(2lk)^{n_0}T^{n_0}/\sqrt{n_0!}=q<1$.

Thus, the operator S^{n_0} is a contraction, and the operator S is a map from C_b into C_b. Therefore, there exists a unique fixed point for the operator S, so the equation (1.1) has a unique solution $x(t,\omega)$ on C_b.

Theorem 3.2. *We consider the stochastic integral equation of the Volterra–Ito type* (1.1). *Assume that, the equation* (1.1) *satisfies the condition* (ii) *and* (iii) *of Theorem* 3.1. *Moreover we assume also that:*

(i) *There exists constants α,β, such that for all $0\leq\tau\leq t<\infty$ and for almost all $\omega\in\Omega$*

$$|||K_1(t,\tau,\omega)|||\leq \beta e^{-\alpha(t-\tau)},$$
$$|||K_2(t,\tau,\omega)|||\leq \beta e^{-\alpha(t-\tau)}.$$

(ii) *The functions $h(t,\omega),f(t,0),g(t,0)$ satisfy the following inequalities*

$$\overline{\lim_{t\to\infty}}\frac{1}{t}\ln\int_0^t e^{2\alpha\tau}\|h(\tau,\omega)\|_{L_2}^2 d\tau<0,$$

$$\overline{\lim_{t\to\infty}}\frac{1}{t}\ln\int_0^t e^{2\alpha\tau}\|f(\tau,0)\|_{L_2}^2 d\tau<2\alpha,$$

$$\overline{\lim_{t\to\infty}}\frac{1}{t}\ln\int_0^t e^{2\alpha\tau}\|g(\tau,0)\|_{L_2}^2 d\tau<2\alpha,$$

where $l > 0$ is small enough, then the trivial solution $x \equiv 0$ of the equation (1.1) *is asymptotically stable in mean square.*

Proof. We have

$$E|x(t,\omega)|^2 =$$
$$E\Big(\Big|h(t) + \int_0^t K_1(t,\tau,\omega)f(\tau,x(\tau,\omega))d\tau + \int_0^t K_2(t,\tau,\omega)g(\tau,x(\tau,\omega))dB_\tau\Big|\Big)^2$$
$$\leq 3E|h(t)|^2 + 3\int_0^t |\|K_1(t,\tau,\omega)\||^2 E\big|f(\tau,x(\tau,\omega))\big|^2 d\tau$$
$$+ 3\int_0^t |\|K_2(t,\tau,\omega)\||^2 E\big|g(\tau,x(\tau,\omega))\big|^2 d\tau.$$

It follows from (ii) of Theorem 3.1 that

$$\big\|f(\tau,x(\tau,\omega))\big\|_{L_2} \leq l\big\|x(t,\omega)\big\|_{L_2} + |f(t,0)|,$$
$$\big\|g(\tau,x(\tau,\omega))\big\|_{L_2} \leq l\|x(t,\omega)\|_{L_2} + |g(t,0)|.$$

By consequence,

$$E\big|f(\tau,x(\tau,\omega))\big|^2 \leq 2l^2 E|x(t,\omega)|^2 + 2|f(t,0)|^2,$$

and

$$E|g(\tau,x(\tau,\omega))|^2 \leq 2l^2 E|x(t,\omega)|^2 + 2|g(t,0)|^2.$$

Therefore

$$E|x(t,\omega)|^2 \leq 3E|h(t)|^2 + 6\beta^2 l^2 \int_0^t e^{-2\alpha(t-\tau)} E|x(t,\omega)|^2 d\tau$$
$$+ 6\beta^2 \int_0^t e^{-2\alpha(t-\tau)}|f(\tau,0)|^2 d\tau + 6\beta^2 l^2 \int_0^t e^{-2\alpha(t-\tau)} E|x(t,\omega)|^2 d\tau$$
$$+ 6\beta^2 \int_0^t e^{-2\alpha(t-\tau)}|g(\tau,0)|^2 d\tau,$$

$$E|x(t,\omega)|^2 \leq 3E|h(t)|^2 + 12\beta^2 l^2 \int_0^t e^{-2\alpha(t-\tau)} E|x(t,\omega)|^2 d\tau$$
$$+ 6\beta^2 \int_0^t e^{-2\alpha(t-\tau)}(|f(\tau,0)|^2 + |g(\tau,0)|^2) d\tau,$$

$$e^{2\alpha t}E|x(t,\omega)|^2 \leq 3e^{2\alpha t}E|h(t)|^2 + 12\beta^2 l^2 \int_0^t e^{2\alpha\tau}E|x(t,\omega)|^2 d\tau$$
$$+ 6\beta^2 \int_0^t e^{2\alpha\tau}(|f(\tau,0)|^2 + |g(\tau,0)|^2)d\tau.$$

If we put

$$\phi(t) = e^{2\alpha t}E|x(t,\omega)|^2,$$
$$\psi(t) = 3e^{2\alpha t}E|h(t)|^2 + 6\beta^2 \int_0^t e^{2\alpha\tau}(|f(\tau,0)|^2 + |g(\tau,0)|^2)d\tau,$$

then we see that

$$\phi(t) \leq 12l^2\beta^2 \int_0^t \phi(\tau)d\tau + \psi(t).$$

An application of Gronwall lemma yields

$$\phi(t) \leq \psi(t) + 12\beta^2 l^2 \int_0^t e^{12\beta^2 l^2 \tau}\psi(\tau)d\tau$$
$$\leq 3e^{2\alpha t}E|h(t)|^2 + 6\beta^2 \int_0^t e^{2\alpha\tau}(|f(\tau,0)|^2 + |g(\tau,0)|^2)d\tau$$
$$+ 12\beta^2 l^2 \int_0^t e^{12\beta^2 l^2 \tau}\big(3e^{2\alpha\tau}E|h(\tau)|^2 + 6\beta^2 \int_0^\tau e^{2\alpha s}(|f(s,0)|^2 + |g(s,0)|^2)ds\big)d\tau.$$

Denote $k = 12\beta^2 l^2$, then

$$E|x(t,\omega)|^2 \leq 3E|h(t)|^2 + 6\beta^2 e^{-2\alpha t} \int_0^t e^{2\alpha\tau}(|f(\tau,0)|^2 + |g(\tau,0)|^2)d\tau$$
$$+ 3ke^{-2\alpha t} \int_0^t e^{(k+2\alpha)\tau}E|h(\tau)|^2 d\tau$$
$$+ 6\beta^2 k e^{-2\alpha t} \int_0^t e^{k\tau}\Big(\int_0^\tau e^{2\alpha s}(|f(s,0)|^2 + |g(s,0)|^2)ds\Big)d\tau.$$

Also if we denote M_1, M_2, M_3 and M_4 as follows

$$M_1 = \overline{\lim_{t\to\infty}} \frac{1}{t} \ln \|h(t)\|_{L_2},$$
$$M_2 = \overline{\lim_{t\to\infty}} \frac{1}{t} \ln e^{-2\alpha t} \int_0^t e^{2\alpha\tau}(|f(\tau,0)|^2 + |g(\tau,0)|^2)d\tau,$$

$$M_3 = \overline{\lim_{t\to\infty}} \frac{1}{t} \ln e^{-2\alpha t} \int_0^t e^{(k+2\alpha)\tau} E|h(\tau)|^2 d\tau,$$

$$M_4 = \overline{\lim_{t\to\infty}} \frac{1}{t} \ln e^{-2\alpha t} \int_0^t e^{k\tau} \Big(\int_0^\tau e^{2\alpha s}(|f(s,0)|^2 + |g(s,0)|^2) ds \Big) d\tau,$$

then it follows from Lemma 2.1 that

$$\overline{\lim_{t\to\infty}} \frac{1}{t} \ln E|x(t,\omega)|^2 \le \max(M_1, M_2, M_3, M_4).$$

So $M_1, M_2, M_3, M_4 < 0$ for l small enough, by the condition (2).

Therefore

$$\overline{\lim_{t\to\infty}} \frac{1}{t} \ln E|x(t,\omega)|^2 < 0.$$

The solution of Equation (1.1) is thus asymptotically stable in mean square.

Example. We consider the following equation

$$x(t) = e^{-2t} + \int_0^t e^{-(t-s)} e^{-s.cos(x(s))} ds + \int_0^t e^{-(t-s)} e^{x(s)-s} dB_s,$$

where

$$f(t,x) = e^{-t.cos(x)},$$
$$g(t,x) = e^{x-t},$$
$$h(t,\omega) = e^{-2t}.$$

We see that the functions $h(t,\omega), f(t,x), g(t,x)$ satisfy condition (ii) of Theorem 3.2.

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