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A Fractional Hull-White Model

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Abstract. In this paper we consider a fractional Hull-White model driven by a fractional Brownian motion. We use an approximate approach to find the solution of this model that exhibits a long-range behavior of the interest.

1. Introduction

It is well-known in mathematical finance that the Hull-White model for interest r_t has the following form

$$dr_t = \big(b(t) - a(t)r_t\big)dt + \sigma(t)dW_t, \tag{1.1}$$

where $a(t)$, $b(t)$ and $\sigma(t)$ are deterministic continuous functions of t and $a(t) > 0$, $\sigma(t) > 0$, and W_t is a standard Brownian motion. This model is very useful in practice of financial markets, it gives also the price of zero-coupon bonds corresponding to each value of the rate r_t.

But each value of r_t can influence upon its behavior in some time range. Correspondingly, the prices of bonds at a time t can have some consequences on their price some time later. In this context, the ordinary Hull-White model is not suitable since its solution is always a Markov process that has no memory.

The purpose of this paper is to introduce a fractional Hull-White model for the interest rate r_t for which the driving process is replaced by a fractional Brownian motion, a process of long memory. A fractional Brownian motion B_t^H, $H \in (0,1)$, is a centered Gaussian process with the covariance function $R(t,s)$ given by

$$R(t,s) = \frac{1}{2}\Big[t^{2H} + s^{2H} - |t-s|^{2H}\Big].$$

Notice that if $H = 1/2$ the fractional Brownian motion is a standard Brownian motion. However, increments of a fractional Brownian motion are not independent except for the standard Brownian case. Moreover, for $H < 1/2$ the increments are negatively correlated and for $H > 1/2$ they are positively correlated. In the latter case, B_t^H is a long memory process since the correlation between two observations that are far apart decay to zero very slowly.

From [6] and the references therein, it is known that for $H \in (0, 1)$ the fractional Brownian motion B_t^H has a representation as follows:

$$B_t^H = \frac{1}{\Gamma(\alpha+1)}\Big[Z_t + \int_0^t (t-s)^\alpha dW_s\Big],$$

where $\alpha = H - \frac{1}{2}, \Gamma(\cdot)$ is the gamma function, $(W_t)_{t\geq 0}$ is a standard Brownian motion and

$$Z_t = \int_{-\infty}^0 \left[(t-s)^\alpha - (-s)^\alpha\right] dW_s.$$

Since Z_t is of absolutely continuous trajectories, the long-range property of B_t^H is essentially expressed by the term

$$B_t = \int_0^t (t-s)^\alpha dW_s.$$

Now, let us consider the fractional Hull-White model for the interest r_t of the form:

$$dr_t = \big(b(t) - a(t)r_t\big)dt + \sigma(t)dB_t, \tag{1.2}$$

where B_t is a fractional Brownian motion of Hurst index $H(0 < H < 1)$ defined by

$$B_t = \int_0^t (t-s)^{H-\frac{1}{2}} dW_s \tag{1.3}$$

and $\sigma(t)$ is a differentiable function of $t \in [0, T], \sigma(t) > 0$. A solution r_t of (1.2) is a long memory process satisfying the following relation:

$$r_t = r_0 + \int_0^t [b(s) - a(s)r_s]ds + \sigma(t)B_t. \tag{1.4}$$

2. Approximate Model

Starting from (1.2) we introduce in this section a so-called *approximate model*, driven by a semimartingale and we give the solution for this model. And the convergence to the solution of (1.1) will be proved in the next section.

The equation (1.2) is rewritten again as

$$dr_t = (b(t) - a(t)r_t)dt + \sigma(t)dB_t, \quad 0 \leq t \leq T, \tag{2.1}$$
$$r_{t(t=0)} = r_0,$$

where

$$B_t = \int_0^t (t-s)^\alpha dW_s, \quad -\frac{1}{2} < \alpha < \frac{1}{2} \tag{2.2}$$

and r_0 is a given square integrable random variable. Now define, for every $\varepsilon > 0$, a process B_t^ε as follows:

$$B_t^\varepsilon = \int_0^t (t-s+\varepsilon)^\alpha dW_s. \tag{2.3}$$

We can notice that

$$\begin{aligned}\int_0^t \int_0^s (s-u+\varepsilon)^{\alpha-1} dW_u ds &= \int_0^t \int_u^t (s-u+\varepsilon)^{\alpha-1} ds dW_u \\ &= \frac{1}{\alpha}\Big[\int_0^t (t-u+\varepsilon)^\alpha dW_u - \varepsilon^\alpha \int_0^t dW_u\Big] \\ &= \frac{1}{\alpha}\big[B_t^\varepsilon - \varepsilon^\alpha W_t\big].\end{aligned}$$

By the above computation we get

$$B_t^\varepsilon = \alpha \int_0^t \varphi_s^\varepsilon ds + \varepsilon^\alpha W_t, \tag{2.4}$$

where

$$\varphi_t^\varepsilon = \int_0^t (t-s+\varepsilon)^{(\alpha-1)} dW_s.$$

Since $\alpha \int_0^t \varphi_s^\varepsilon ds$ is of bounded variation and $\varepsilon^\alpha W_t$ is a martingale, then B_t^ε is a semimartingale. This result was stated in [6].

Furthermore, from [6] it was also proved that B_t^ε converges to B_t in $L^2(\Omega)$ uniformly with respect to $t \in [0,T]$. We now consider an approximate model defined for each $\varepsilon > 0$ as follows:

$$\begin{aligned} dr_t^\varepsilon &= \big(b(t) - a(t) r_t^\varepsilon\big) dt + \sigma(t) dB_t^\varepsilon, \quad 0 \le t \le T \\ r_{t(t=0)}^\varepsilon &= r_0, \end{aligned} \tag{2.5}$$

where B_t^ε is defined by (2.3) and the random variable r_0 is exactly equal to the value given in the initial condition of (2.1). From (2.4) we have

$$dB_t^\varepsilon = \alpha \varphi_t^\varepsilon dt + \varepsilon^\alpha dW_t.$$

Hence, after substituting dB_t^ε by its above-mentioned expression, (2.5) becomes

$$\begin{aligned} dr_t^\varepsilon &= \left(b(t) + \alpha \varphi_t^\varepsilon \sigma(t) - a(t) r_t^\varepsilon\right) dt + \varepsilon^\alpha \sigma(t) dW_t \\ r_{t(t=0)}^\varepsilon &= r_0. \end{aligned} \tag{2.5'}$$

Theorem 1. *Suppose that $a(t)$ and $b(t)$ are continuous functions on $[0,T]$. Then for each $\varepsilon > 0$ there exists a unique solution r_t^ε for $(2.5')$ given by*

$$r_t^\varepsilon = e^{-\int_0^t a(s)ds}\Big[r_0 + \varepsilon^\alpha \int_0^t \sigma(s) e^{\int_0^s a(u)du} dW_s + \int_0^t \big(b(u) + \sigma(u)\alpha\varphi_u^\varepsilon\big) e^{\int_0^u a(s)ds} du\Big]. \tag{2.6}$$

Proof. We split the equation (2.5') into two equations:

$$\begin{aligned} dr_t^{(1)} &= -a(t) r_t^{(1)} dt + \sigma(t)\varepsilon^\alpha dW_t, \quad 0 \le t \le T, \\ r_{t(t=0)}^{(1)} &= r_0^{(1)}, \end{aligned} \tag{2.7}$$

and

$$\begin{aligned} dr_t^{(2)} &= \big(b(t) - a(t) r_t^{(2)}\big) dt + \sigma(t)\alpha\varphi_t^\varepsilon dt, \\ r_{t(t=0)}^{(2)} &= r_0^{(2)}, \end{aligned} \tag{2.8}$$

where $r_t^{(1)} + r_t^{(2)} = r_t^\varepsilon$ satisfies (2.5') and $r_0^{(1)}$ and $r_0^{(2)}$ are two square integrable random variables such that $r_0^{(1)} + r_0^{(2)} = r_0^\varepsilon$ (given initial condition). We see that (2.7) is a linear stochastic differential equation of the form

$$dr(t) = \big(\alpha(t) + \beta(t) r(t)\big) dt + \big(\gamma(t) + \delta(t) r(t)\big) dW_t.$$

It is known that if coefficients $\alpha, \beta, \gamma, \delta$ are continuous functions of t, then the existence and uniqueness for solution of (2.7) are assured. Moreover, its solution has the form (see, [3] for example)

$$r(t) = U(t)\Big[r(0) + \int_0^t \frac{\alpha(s) - \delta(s)\gamma(s)}{U(s)} ds + \int_0^t \frac{\gamma(s)}{U(s)} dW_s\Big],$$

where

$$U(t) = U(0)\exp\Big[\int_0^t \Big(\beta(s) - \frac{1}{2}\delta^2(s)\Big) ds + \int_0^t \delta(s) dW_s\Big].$$

Here we have $\alpha(t) = 0, \beta(t) = -a(t), \gamma(t) = \sigma(t)\varepsilon^\alpha$, and $\delta(t) = 0$. Then, with $U(0) = 1$,

$$r_t^{(1)} = e^{-\int_0^t a(s)ds}\Big[r_0^{(1)} + \varepsilon^\alpha \int_0^t \sigma(s) e^{\int_0^s a(r)dr} dW_s\Big]. \tag{2.9}$$

Now let us consider the equation (2.8) that can be rewritten in the form:

$$\frac{dr_t^{(2)}}{dt} + a(t) r_t^{(2)} = b(t) + \sigma(t)\alpha\varphi_t^\varepsilon, \quad 0 \le t \le T.$$

This is an ordinary linear differential equation whose solution can be given by:

$$r_t^{(2)} = e^{-\int_0^t a(s)ds}\Big[r_0^{(2)} + \int_0^t (b(u) + \sigma(u)\alpha\varphi_u^\varepsilon)e^{\int_0^t a(s)ds} du\Big]. \quad (2.10)$$

Combining (2.9) and (2.10) yields the expression (2.6) for the solution r_t^ε of (2.5′). ■

3. Convergence

We note that the equation (1.2) is a fractional linear stochastic differential equation whose solution is defined by (1.4). Under the regularity assumptions on $a(t)$ and $b(t)$, it is easy to verify that there exists such a unique solution for (1.2). Denote this solution by r_t and suppose that r_t^ε is the solution of the corresponding approximate model (2.5'). Thus r_t and r_t^ε satisfy the following equations:

$$\begin{aligned} dr_t &= (b(t) - a(t)r_t)dt + \sigma(t)dB_t, \quad 0 \le t \le T, \\ dr_t^\varepsilon &= (b(t) - a(t)r_t^\varepsilon)dt + \sigma(t)dB_t^\varepsilon, \quad 0 \le t \le T. \end{aligned}$$

Theorem 2. *r_t^ε converges to r_t in $L^2(\Omega)$ uniformly with respect to $t \in [0,T]$ as $\varepsilon \to 0$.*

Proof. We have

$$r_t - r_t^\varepsilon = -\int_0^t a(s)\,(r_s - r_s^\varepsilon)ds + \sigma(t)(B_t - B_t^\varepsilon) - \int_0^t (B_s - B_s^\varepsilon)d\sigma(s). \quad (3.1)$$

Denote by $\|\cdot\|$ the norm in $L^2(\Omega)$. Then

$$\|r_t - r_t^\varepsilon\| \le \int_0^t |a(s)|\,\|r_s - r_s^\varepsilon\|ds + |\sigma(t)|\|B_t - B_t^\varepsilon\| + \|B_t - B_t^\varepsilon\|\,|\sigma(t) - \sigma(0)|\,. \quad (3.2)$$

Since $a(t)$ and $\sigma(t)$ are continuous, hence bounded, on $[0,T]$ then there exist positive constants M_1, M_2 such that

$$|a(t)| \le M_1 = \max_{0\le t\le T} |a(t)| \text{ and } |\sigma(t)| \le M_2 = \max_{0\le t\le T} |\sigma(t)|.$$

Hence,

$$\|r_t - r_t^\varepsilon\| \le M_1 \int_0^t \|r_s - r_s^\varepsilon\|ds + 2M_2\|B_t - B_t^\varepsilon\|. \quad (3.3)$$

We know that $B_t^\varepsilon \to B_t$ in $L^2(\Omega)$ uniformly with respect to $t \in [0,T]$ and we have also the following estimate (see [6])

$$\|B_t - B_t^\varepsilon\|^2 = E\,|B_t - B_t^\varepsilon|^2 \le C(\alpha)\,\varepsilon^{1+2\alpha}, \quad (3.4)$$

where $C(\alpha)$ is a constant depending only on α. Then

$$\|B_t - B_t^\varepsilon\| \le K(\alpha)\,\varepsilon^{\frac{1}{2}+\alpha}, \tag{3.5}$$

where $K(\alpha) = \sqrt{C(\alpha)}$. It follows from (3.2), (3.3) and (3.4) that

$$\|r_t - r_t^\varepsilon\| \le M_1 \int_0^t \|r_s - r_s^\varepsilon\| ds + M\,\varepsilon^{\frac{1}{2}+\alpha}, \tag{3.6}$$

where $M = 2M_2K(\alpha)$. A standard application of Gronwall Lemma will give us:

$$\|r_t - r_t^\varepsilon\| \le e^{M_1 t} M\,\varepsilon^{\frac{1}{2}+\alpha}. \tag{3.7}$$

Hence

$$\sup_{0\le t\le T} \|r_t - r_t^\varepsilon\| \le e^{M_1 T} M\,\varepsilon^{\frac{1}{2}+\alpha} \to 0 \tag{3.8}$$

as $\varepsilon \to 0$. The proof of Theorem 3.1 is thus complete. ∎

Remark 1. It is known that a fractional stochastic dynamical system driven by a fractional Brownian motion exhibits a long-range behavior of system states. In spite of the fact that the interest rate is in general a short rate, its behavior in some considerably long time has no more Markov property. In this context, a fractional Hull-White model is needed to understand realistic dynamics of interest rate.

Remark 2. The interest rate is strictly related to bond prices. And as we know, for a bond market driven by a fractional Brownian motion, in general, the absence of arbitrage opportunity cannot be guaranteed [5]. But by the approximate approach given by the authors of [6], the fractional bond price model can be approximated by a model driven by a semimartingale where there is no more any arbitrage opportunity.

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