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Extending Holomorphic Maps Into Hartogs Domains

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Abstract. Let X be a complex space and $\varphi : X \to [-\infty; \infty)$ be a plurisubharmonic function on X. In this article we study the problem of extending holomorphic maps from $\Delta^n \setminus S$ into the Hartogs domain $\Omega_\varphi(X)$, where S is a closed subset of the open unit polydisc Δ^n in $\mathbb{C}^n$ with the locally finite (real) d-dimensional Hausdorff measure.

1. Introduction

This article is a continuation of our studies in [9 - 12] of extensions of holomorphic mappings through subsets with locally finite Hausdorff measure in some (real) dimension. As well known, the above-mentioned problem is a generalization of the problem of extending holomorphic mappings through pluripolar sets and it has been studied intensively recently. However, regrettfully that the present knowledge of this problem has remained limited.

In this article, we study the problem of determining when a holomorphic map f from $\Delta^n \setminus S$ into a Hartogs domain $\Omega_\varphi(X)$ extends holomorphically over a closed subset S of the open unit polydisc Δ^n of $\mathbb{C}^n$ with $H_d(S) = 0$, where d is a real number such that $0 < d < 2n - 1$ and H_d is the Hausdorff measure in (real) dimension d.

Namely, we are going to prove the following

Main Theorem. *Let d be a real number such that $0 < d < 2n - 1$. Let X be a complex space having the d-EP and ρ a Hermitian metric on X and φ a plurisubharmonic function on X. Let S be a closed subset of Δ^n with $H_d(S) = 0$. Let $f := (f_1, f_2) : \Delta^n \setminus S \to \Omega_\varphi(X)$ be a holomorphic mapping such that the following are satisfied*

(i) $\limsup_{x\to a}\big(\varepsilon\log\rho(x,a)-\varphi(x)\big)<\infty$ *for every* $a\in X$ *and every* $\varepsilon>0$*;*
(ii) *The mapping* f_1 *is non-constant on* $\Delta^n\setminus S$.
Then f *extends holomorphically to* Δ^n.

Moreover, we also give two counterexamples to show that the condition (ii) cannot be dropped in Main theorem and the extendibility of f to Δ^n does not imply the condition (i). On the other hand, it is easy to see that the extendibility of f over Δ^n also cannot imply the condition (ii), hence, the converse assertion of Main theorem is not true in general case.

2. Main Results

We now recall some definitions

Definition 2.1. *Let* φ *be an upper-semicontinuous function on a complex space* X*. Define*

$$\Omega_\varphi(X)=\big\{(x,\lambda)\in X\times\mathbb{C}:|\lambda|<e^{-\varphi(x)}\big\}.$$

The domain $\Omega_\varphi(X)$ *is called a Hartogs domain.*

Definition 2.2. *Let* Δ^n *be the open unit polydisc in* $\mathbb{C}^n$ *and* d *a real number such that* $0<d<2n-1$.

Let X *be a complex space. We say that* X *has the d-extension property for* Δ^n *(shortly* X *has the d-EP) if and only if for any closed set* $S\subset\Delta^n$*, which is of locally finite* H_d *measure , for any holomorphic map* f *from* $\Delta^n\setminus S$ *to* X*, there exists a map* $\widetilde{f}\in H(\Delta^n,X)$ *such that* $\widetilde{f}|_{\Delta^n\setminus S}=f$.

Definition 2.3.
(i) *A domain* $D\subset\mathbb{C}^n$ *is called to be hyperconvex if there exists a continuous plurisubharmonic exhaustion function* $\rho:D\to(-\infty,0)$.
(ii) *A bounded domain* $D\subset\mathbb{C}^n$ *is said to be strictly hyperconvex if there exist a bounded domain* Ω *and a function* $\rho\in\mathcal{C}(\Omega,(-\infty,1))\cap PSH(\Omega)$ *such that* $D=\{z\in\Omega:\rho(z)<0\}$*,* ρ *is exhaustive for* Ω *and for all real number* $c\in[0,1]$*, the open set* $\{z\in\Omega:\rho(z)<c\}$ *is connected.*

It is easy to prove the following implications for any bounded domain in $\mathbb{C}^n$:

Strictly pseudoconvex $\Longrightarrow$ strictly hyperconvex $\Longrightarrow$ hyperconvex $\Longrightarrow$ pseudoconvex

The converse implications are not true in general.

For details concerning the above-mentioned notions, we refer the reader to [2-4, 7].

The following shows that the class of complex spaces having the *d-EP* is rather large.

Proposition 2.4. *Every strictly hyperconvex domain in $\mathbb{C}^N$ has the d-EP, in particular, the open unit disc Δ of $\mathbb{C}$ has the d-EP.*

Proof. To prove Proposition 2.4, recall the following result of extension.

Theorem. [1, Th.A1.4, p.299]. *Let D be a domain in $\mathbb{C}^n$, and let E be a closed subset of D of Hausdorff measure $H_{2n-1}(E) = 0$. Then every function f, holomorphic and uniformly bounded in $D\backslash E$, has a holomorphic continuation into D.*

Let Ω be a strictly hyperconvex domain in $\mathbb{C}^N$. Let $f : \Delta^n \setminus S \to \Omega$ be any holomorphic mapping, where S is any closed subset of Δ^n with $H_d(S) = 0$. Put $f = (f_1, f_2, ..., f_N)$. By the above-mentioned theorem , f_j extends to the holomorphic function $\hat{f}_j$ on Δ^n. Then the mapping $\hat{f} = (\hat{f}_1, \hat{f}_2, ..., \hat{f}_N) \in H(\Delta^n, \bar{\Omega})$. Let ρ be a plurisubharmonic exhaustion function of $\tilde{\Omega}$ such that $\Omega = \{z \in \tilde{\Omega} : \rho(z) < 0\}$, where $\tilde{\Omega}$ is a bounded neighborhood of $\overline{\Omega}$ in $\mathbb{C}^N$. Put $h = \rho \circ \hat{f}$. Then h is plurisubharmonic on Δ^n. Since h is negative on $\Delta^n \setminus S$, it follows that $h \leq 0$ on Δ^n. Suppose that there exists $z_0 \in S$ such that $\hat{f}(z_0) \in \partial\Omega$. Then $h(z_0) = 0$. The maximum principle implies $h = 0$ on Δ^n. This is a contradiction and hence, $f \in H(\Delta^n, \Omega)$. ∎

Proof of Main theorem

By the hypothesis, f_1 extends to $\hat{f}_1 \in H(\Delta^n, X)$. We now prove that f_2 is extended to a holomorphic function over Δ^n.

Let $z_0 \in S$ be an arbitrary point. Choose a sufficiently small neighborhood U of $\hat{f}_1(z_0)$ in X such that U is isomorphic to an analytic set in an open ball of $\mathbb{C}^m$. Without loss of generality we may assume that $U \subset \mathbb{C}^m$.

Put $W = \hat{f}_1^{-1}(U)$, and

$$h(z) := \hat{f}_1(z) - \hat{f}_1(z_0) := (h_1(z), \ldots, h_m(z))$$

for all $z \in W$, where h_j are holomorphic on W.

Since f_1 is non-constant, $h \not\equiv 0$ on W. Without loss of generality we also may assume that $h_1 \not\equiv 0$ on W. By [8] there exist C, $p > 0$ such that

$$|h_1(z)| \geq C\rho(z, \mathrm{Zer}\, h_1)^p$$

for every z belonging to a neighborhood of z_0 in W. Since $d < 2n - 1$, we can find $r > 0$ such that $2n - \dfrac{p+r}{p+r-1} \geq d$. Take $\varepsilon > 0$ such that $\varepsilon p(p + r) < 1$.

By the hypothesis, there exists $\delta > 0$ such that $\varepsilon \log\rho(x, \hat{f}_1(z_0)) - \varphi(x) < 0$ for all $x \in U$ with $\rho(x, \hat{f}_1(z_0)) < \delta$, or equivalently,

$$e^{-\varphi(x)} < \frac{1}{(\rho(x, \hat{f}_1(z_0)))^\varepsilon}$$

for all $x \in U$ such that $\rho(x, \hat{f}_1(x_0)) < \delta$.

Choose a neighborhood W_1 of z_0 in W such that

$$\rho(\hat{f}_1(z), \hat{f}_1(z_0)) < \delta, \quad \forall z \in W_1.$$

Then for every $z \in W_1 \setminus S$, we have

$$|f_2(z)| < e^{-\varphi(\hat{f}_1(z))} < \frac{1}{(\rho(\hat{f}_1(z), \hat{f}_1(z_0)))^{\varepsilon}} \le \frac{1}{|h_1(z)|^{\varepsilon}} \le \frac{1}{C^{\varepsilon}\rho(z, \operatorname{Zer} h_1)^{\varepsilon p}}.$$

This inequality implies that $f_2 \in L^{p+r}_{loc}(W_1)$.

Let p' be the conjugate exponent of $p+r$, i.e. $\dfrac{1}{p+r} + \dfrac{1}{p'} = 1$. Then $2n - p' = 2n - \dfrac{p+r}{p+r-1} \ge d$, and hence, $H_{2n-p'}(S) = 0$.

By the theorem of Harvey–Polking [5], f_2 extends holomorphically over W_1. This implies that f_2 is extended to a holomorphic function $\hat{f}_2$ on Δ^n. Thus f is extended to the holomorphic mapping

$$\hat{f} = (\hat{f}_1, \hat{f}_2) : \Delta^n \to X \times \mathbb{C}.$$

By the mean value inequality for the plurisubharmonic function $\log|\hat{f}_2| + \varphi(\hat{f}_1)$, we get the inequality

$$\log|\hat{f}_2(z)| + \varphi(\hat{f}_1(z)) < 0 \quad \text{for all } z \in \Delta^n.$$

This implies that $\hat{f}(\Delta^n) \subset \Omega_\varphi(X)$. ■

We now present some counterexamples to show the following.

• The extendibility of f over Δ^n does not imply the condition (i) in Main theorem.

• The condition (ii) of being non-constant of f_1 cannot be omitted in Main theorem.

Proposition 2.6. *There exists a subharmonic function φ on Δ such that $\Omega_\varphi(\Delta)$ has the d-EP ($0 < d < 1$), but $\limsup_{z \to 0} \big(\varepsilon \log|z| - \varphi(z)\big) = \infty$ for every $\varepsilon > 0$ sufficiently small.*

Proof. The main ideas in the proof are taken from [9, Lemmas 2,3]. For the reader's convenience we repeat the details.

Let $z_k = 2^{-k}$, $\alpha_k = k^{-3}$, $\delta_k = e^{-\frac{k}{2\alpha_k}}$ for all $k \ge 1$.

Then $\varphi(z) := \sum\limits_{k=1}^{\infty} \alpha_k \cdot \log(\delta_k^2 + |z - z_k|^2)$ defines a function in $SH(\Delta)$, where $SH(\Delta)$ is the set of subharmonic functions on Δ.

Indeed, each term of the series is subharmonic, and

$$\log(\delta_k^2 + |z - z_k|^2) - \log 5 < 0, \quad \sum_{k=1}^{\infty} \alpha_k \log 5 < \infty,$$

so that $\varphi \in SH(\Delta) \cup \{-\infty\}$ as the limit of the decreasing sequence of subharmonic functions

$$\sum_{k=1}^{n} \alpha_k \log\Big(\frac{\delta_k^2 + |z - z_k|^2}{5}\Big) + \Big(\sum_{k=1}^{\infty} \alpha_k\Big)\log 5.$$

Since

$$\begin{aligned}\varphi(0) &= \sum_k \alpha_k \log(\delta_k^2 + |z_k|^2) \geq 2\sum_k \alpha_k \log|z_k| \\ &= -2(\log 2)\Big(\sum_{k\geq 1} \frac{1}{k^2}\Big) > -\infty,\end{aligned}$$

we have $\varphi \not\equiv -\infty$. This also proves that $\varphi \in SH(\Delta)$.

On the other hand, for every $k \geq 1$ we have

$$\begin{aligned}\varphi(z_k) &= \alpha_k \log(\delta_k^2) + \sum_{j\neq k} \alpha_j \log(\delta_j^2 + |z_j - z_k|^2) \\ &\geq -k + 2\sum_{j\neq k} \alpha_j \log|z_j - z_k| \\ &\geq -k + 2\Big(\sum_{j\neq k} \alpha_j\Big)\log\Big(\frac{1}{2^{k+1}}\Big) > -\infty.\end{aligned}$$

Let $z \in \Delta^* = \Delta \setminus \{0\}$ be such that $z \neq z_k$ for all $k \geq 1$. Then

$$\begin{aligned}\varphi(z) &= \sum_{k=1}^{\infty} \alpha_k \log(\delta_k^2 + |z - z_k|^2) \geq 2\sum_{k=1}^{\infty} \alpha_k \log|z - z_k| \\ &\geq 2\Big(\sum_{k=1}^{\infty} \alpha_k\Big)\log C > -\infty,\end{aligned}$$

where $C = \inf_{k\geq 1} |z - z_k| > 0$.

Thus $\varphi \in SH(\Delta)$ with $\varphi(z) > -\infty$, $\forall z \in \Delta$. Since Δ has the d-EP, by a result in [10], $\Omega_\varphi(\Delta)$ has the d-EP.

It remains to show that $\limsup_{z\to 0}(\varepsilon \log|z| - \varphi(z)) = \infty$ for $\varepsilon > 0$ sufficiently small.

Indeed, for every $k > 2$ we have

$$\begin{aligned}-\varphi(z_k) &= -\alpha_k |\log(\delta_k^2)| - \sum_{j\neq k} \alpha_j |\log(\delta_j^2 + |z_j - z_k|^2)| \\ &\geq -\alpha_k |\log(\delta_k^2)| = k.\end{aligned}$$

Thus $-\varphi(z_k) + \varepsilon \log|z_k| = -\varphi(z_k) - (\varepsilon \log 2)k \geq (1 - \varepsilon \log 2)k$. This implies that

$$\lim_{k\to\infty}(-\varphi(z_k) + \varepsilon \log|z_k|) = \infty \quad \text{for every } \varepsilon \in \Big(0, \frac{1}{\log 2}\Big). \qquad \blacksquare$$

Proposition 2.7. *There exists a subharmonic function φ on Δ such that*

$$\limsup_{x\to a}\big(\varepsilon\, log|x-a| - \varphi(x)\big) < \infty$$

for every $a \in \Delta$ *and every* $\varepsilon > 0$*, but* $\Omega_\varphi(\Delta)$ *does not have the d-EP for every* $0 < d \le 1$.

Proof. Let u be an increasing convex function on $[-\infty, 0]$ such that $u(-k^2) = -k$ for all $k \ge 0$ and u is linear on $[-k^2, -(k-1)^2]$ for all $k \ge 1$.

By a result of Hörmander [6, Th. 3.2.18, p.156], the function $\varphi(x) = u(\log|x|)$ is subharmonic on Δ.

It is easy to see that

$$\limsup_{x\to a}(\varepsilon \log|x-a| - \varphi(x)) < \infty$$

for every $a \in \Delta$ and every $\varepsilon > 0$, but $\varphi(0) = -\infty$.

Consider the holomorphic map

$$\sigma : \Delta^* = \Delta \setminus \{0\} \to \Omega_\varphi(\Delta)$$

given by $\sigma(z) = \left(0, \dfrac{1}{z}\right)$ for each $z \in \Delta^*$. Then σ cannot extend holomorphically over Δ. ∎

Remark 1. Using the same argument in the proof of Main theorem, we can prove the following

Let d be a real number such that $0 < d < 2n-1$.

Let X be a complex space with a Hermitian metric ρ and $\varphi : X \to [-\infty; \infty)$ a plurisubharmonic function on X. Let S be a closed subset of Δ^n with $H_d(S) = 0$.

Let $f := (f_1, f_2) : \Delta^n \setminus S \to \Omega_\varphi(X)$ be a holomorphic mapping such that the following are satisfied

(i) $\limsup_{x\to a}\big(\varepsilon \log \rho(x,a) - \varphi(x)\big) < \infty$ for every $a \in X$ and every $\varepsilon > 0$;

(ii) the mapping f_1 extends to a non-constant holomorphic mapping $\hat{f}_1 : \Delta^n \to X$.

Then f extends holomorphically over Δ^n.

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References

1. E. M. Chirka, *Complex Analytic Sets*, Kluwer Academic Published, 1989.
2. J. P. Demailly, Mesures de Monge-Ampère et mesures pluriharmoniques, *Math. Z.* **194** (1987) 519–564.
3. K. Diederich and J. E. Fornaess, Pseudoconvex domains: An example with non-trivial Nebenhülle, *Math. Ann.* **225** (1977) 275–292.

4. J. E. Fornaess and R. Narasimhan, The Levi problem on complex spaces with singularities, *Math. Ann.* **248** (1980) 47–72.
5. R. Harvey and J. Polking, Extending analytic objects, *Comm. Pure and Appl. Math.* **28** (1975) 701–727.
6. L. Hörmander, Notions of Convexity, *Progress in Math.* **127**, Birkhäuser, 1994.
7. N. Kerzman and J. P. Rosay, Fonctions plurisousharmoniques d'exhaustion bornees et domaines taut, *Math. Ann.* **257** (1981) 171–184.
8. S. Lojasiewicz, *Ensembles Semi-analytiques*, I.H.E.S, Bures-sur-Yvette, 1965.
9. D. D. Thai and P. J. Thomas, D^*-extension property without hyperbolicity, *Indiana Univ. Math. Jour.* **47** (1998) 1125–1130.
10. D. D. Thai and P. J. Thomas, On D^*-extension property of the Hartogs domains, *Pub. Math.* **45** (2001) 421–429 (Spain).
11. D. D. Thai, N. T. T. Mai, and N. T. Son, Noguchi-type convergence - extension theorems for (n, d)-sets, *Ann. Math. Pol.* **82** (2003) 189-201.
12. D. D. Thai and P. N. Mai, Convergence and extension theorems in geometric function theory, *Kodai Math. J.* **26** (2003) 179–198.