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Basic Analogue of I- Function of Several Matrix Arguments

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Abstract. In the present paper the authors define a new basic analogue of I-function of several matrix arguments. As $q \to 1^-$, it reduces to the Lauricella functions studied by Mathai. Two integral relations for this function, multiplication formula, convolution for the M-transform for the function of several matrix arguments and application to integral equation are investigated.

1. Introduction

The H-function of matrix arguments is defined as follows [9, 10]

$$\int\limits_{Z>0} (\det Z)^{\rho-\frac{m+1}{2}} H(Z, p, \overline{n}, \overline{a}, \overline{\alpha}) dZ = X^p_{\overline{n},\overline{a},\overline{\alpha}}(\rho),$$

where

$$X^p_{\overline{n},\overline{a},\overline{\alpha}}(\rho) = \prod_{i=1}^{p} \Gamma^{n_i}_m (A_i + \alpha_i \rho),$$

$$A_i = \frac{m+1}{4} + \Big(a_i - \frac{m+1}{4}\Big) \mathrm{sign}(\alpha_i),$$

$\Gamma_m(\delta)$ is the generalized gamma - function,
p is a natural number, ρ is a complex number,
$\overline{n} = (n_1, n_2, ..., n_p)$, n_i are integers,
$\overline{a} = (a_1, a_2, ..., a_p)$, a_i are complex numbers,
$\overline{\alpha} = (\alpha_1, \alpha_2, ..., \alpha_p)$, α_i are real numbers,

$\operatorname{Re}(A_i + \alpha_i\rho) > \frac{m-1}{2}\alpha_i$, $i = 1, ..., p$.

$Z > 0$ means that Z is a $m \times m$ real symmetric positive definite matrix, $dZ = \prod\limits_{i \geq j} dx_{ij}$.

All gamma - functions are assumed to exist, the poles of the functions $X^p_{\overline{n},\overline{a},\overline{\alpha}}(\rho)$, $\alpha_i > 0, n_i > 0, i = 1, ..., p$, and those of $X^p_{\overline{n},\overline{a},\overline{\alpha}}(\rho)$, $\alpha_i < 0, n_i > 0$, $i = 1, ..., p$, are assumed to be separated from each other.

The special cases of H-function with matrix argument are known as H-function (see Fox [2]), G-function (Meijer [2]).

In 1995, Mathai studied some functions of several matrix arguments: Lauricella functions [7], Φ_2 and ψ_2-functions, Humbert's functions [8], Appell's and Kampe-de-Feriet's functions [8].

Let us consider the integral transform $K : U(X) \to V(Y)$, where $U(X)$ is a linear space, $V(Y)$ is an algebraic one. The convolution of two functions f, g for the transform K is defined by the symbol $f * g$, such that the following factorization property is valid:

$$K(f * g)(y) = (Kf)(y)(Kg)(y), \quad y \in Y.$$

In 1942, for the first time Churchill initiated the convolution of functions f, g for the Fourier transform in the famous form as follows [1]:

$$(f * g)(x) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} f(x-t)g(t)dt.$$

Later on, convolutions for the Mellin, Laplace, cosine-Fourier, Hilbert, Stieltjes and I-transform have been investigated [4, 15 - 17]:

$$(f * g)(x) = \int\limits_0^{+\infty} f\Big(\frac{x}{t}\Big) g(t) \frac{dt}{t},$$

$$(f * g)(x) = \int\limits_0^{x} f(x-t)g(t)dt,$$

$$(f * g)(x) = \frac{1}{\sqrt{2\pi}} \int\limits_0^{+\infty} \big[f(|x-t|) + f(x+t)\big] g(t)dt,$$

$$(f * g)(x) = \frac{1}{\pi} \int\limits_{-\infty}^{+\infty} \frac{f(x)g(t) + f(t)g(x) + f(t)g(t)}{x-t} dt,$$

$$(f * g)(x) = f(x) \int\limits_0^{+\infty} \frac{g(t)dt}{t-x} + g(x) \int\limits_0^{+\infty} \frac{f(t)}{t-x} dt.$$

In the present paper, we will assume the basic analogue of I-function of several matrix arguments, including, for example, Lauricella function [8], hypergeometric function [8], Fox's H-function [1]. Integral relation, contiguous correlations, convolution for M-transform and its application to integral equations are established.

2. I-Function of Several Matrix Arguments

Definition 1. *A basic analogue of I-function of several matrix arguments is defined as follows:*

$$\int\limits_{Z^N>0} (detZ)^{\rho-\frac{m+1}{2}} I_q(Z\,|\,r, p_i, \overline{n}_i, \overline{a}_i, \alpha_i)dZ = \Theta(\rho), \tag{1}$$

where

$$\frac{1}{\Theta(\rho)} = \sum_{i=1}^{r} Q^{p_i}_{\overline{n}_i,\overline{a}_i,\alpha_i}(\rho) \neq 0,$$

$$Q^{p_i}_{\overline{n}_i,\overline{a}_i,\alpha_i}(\rho) = \pi^{\frac{m}{2}(m-1)\sum_{j=1}^{p} n_{ij}} \prod_{j=1}^{p}\prod_{t=1}^{m} \Gamma^{n_{ij}}_{q,m}\Big(a_{ij} - \frac{t-1}{2} + \sum_{k=1}^{N} \alpha_{ijk}\rho_k\Big),$$

$\Gamma_{q,m}(\delta)$ is a product of the q-gamma functions [5]:

$$\Gamma_{q,m}(\delta) = \pi\frac{m(m-1)}{4}\Gamma_q\Big(\delta - \frac{1}{2}\Big)...\Gamma_q\Big(\delta - \frac{m-1}{2}\Big),$$

$$\Gamma_q(s) = \frac{(q,q)_\infty}{(q^s,q)_\infty}(1-q)^{1-s}, \quad (a;q)_\infty = \prod_{n=0}^{\infty}(1-aq^n), \quad |q|<1$$

r, p_i, m, N are natural numbers, ρ_j are complex numbers,
$\overline{n}_i = (n_{i1}, n_{i2}, ..., n_{ip_i})$, n_{ij} are integers,
$\overline{a}_i = (a_{i1}, a_{i2}, ..., a_{ip_i})$, a_{ij} are complex numbers,
$\alpha_i = (\alpha_{ijk})_{p_i\times N}$, α_{ijk} are real numbers, $i = 1, ..., r$, $j = 1, ..., p_i$, $k = 1, ..., N$,
$Z = (Z_1, ..., Z_N)$, $(\det Z)^{\rho-\frac{m+1}{2}} = \prod\limits_{k=1}^{N} (\det Z_k)^{\rho_k-\frac{m+1}{2}}$,
$\{Z^N > 0\} = \{Z_k > 0, \quad k = 1, ..., N\}$, $Z_k > 0$ means that Z_k is an $m\times m$ real symmetric positive definite matrix.

$$dZ = \prod_{k=1}^{N} dZ_k, \ \ \mathrm{Re}\Big(a_{ij} + \sum_{k=1} \alpha^N_{ijk}\rho_k\Big) > \frac{m-1}{2}, \quad i = 1, ..., r, \ \ j = 1, ..., p_i.$$

All gamma-functions are assumed to exist, the poles of funciton $\Theta(\rho)$, $\alpha_{ijk} > 0$, $n_{ij} > 0$, $i = 1, ..., r$, $j = 1, ..., p_i$, $k = 1, ..., N$, and those of $\Theta(\rho)$, $\alpha_{ijk} < 0$, $n_{ij} > 0$, $i = 1, ..., r$, $j = 1, ..., p_i$, $k = 1, ..., N$, are assumed to be separated from each other.

Remark. Basic analogue of I-functions (1) is very general, and with certain constraints on the parameters of them we arrive at well-known functions. For examples, when $N = 1, m = 1$, we have a basic analogue of I-functions [13], when $N = 2, m = 1$, we obtain that of I-functions of two variables [11], when $N = 1, m = 1, r = 1$, we have a basic analogue of Fox' H-functions [14], when $N = 2, m = 1, r = 1$, we obtain that of H-functions of two variables [6], and when $m = 1, r = 1$, we have that of H-functions of several variables [12].

Theorem 1. a) *If*

$$\begin{aligned}
&r = 1,\ \ p = 5N+2,\ \ n_1 = 1,\ \ a_1 = a, \alpha_{1j} = 0,\\
&n_2 = -1,\ \ a_2 = \frac{m+1}{2} - a,\ \ \alpha_{2j} = -1,\\
&n_{2+i} = 1, a_{2+i} = \frac{m+1}{2} - b_i,\ \ \ \alpha_{2+i,j} = 0,\\
&n_{2+N+i} = -1,\ \ a_{2+N+i} = b_i,\ \ \ \alpha_{2+N+i,j} = \begin{cases} -1 & j = i \\ 0 & j \neq i \end{cases},\\
&n_{2+2N+i} = -1,\ \ a_{2+2N+i} = c_i,\ \ \ \alpha_{2+2N+i,j} = 0,\\
&n_{2+3N+i} = 1,\ \ a_{2+3N+i} = \frac{m+1}{2} - c_i,\ \ \ \alpha_{2+3N+i,j} = \begin{cases} -1 & j = i \\ 0 & j \neq i \end{cases},\\
&n_{2+4N+i} = -1,\ \ a_{2+4N+i} = 0,\ \ \ \alpha_{2+4N+i,j} = \begin{cases} 1 & j = i \\ 0 & j \neq i \end{cases},
\end{aligned}$$

then the following statement holds

$$\lim_{q \to 1^-} I_q(Z \,|\, 1, p, \overline{n}, \overline{a}, \alpha) = f_A(a, b_1, ..., b_N; c_1, ..., c_N; -Z_1, ..., -Z_N),$$

where $f_A(\cdot)$ *and* $f_B(\,.\,), f_C(\,.\,), f_D(\,.\,)$ *are Lauricella functions* [7].
b) *If*

$$\begin{aligned}
&r = 1,\ \ p = 5N+2,\ \ n_i = 1,\ \ \alpha_{ij} = 0,\\
&n_{N+i} = \ -1,\ \ a_{N+i} = \frac{m+1}{2} - a_i,\ \ \alpha_{N+i,j} = \begin{cases} -1 & j = i \\ 0 & j \neq i \end{cases},\\
&n_{2N+i} = 1,\ \ a_{2N} = b_i,\ \ \ \alpha_{2N+i,j} = 0,\\
&n_{3N+i} = \ -1,\ \ a_{3N+i} = \frac{m+1}{2} - b_i,\ \ \alpha_{3N+i,j} = \begin{cases} -1 & j = i \\ 0 & j \neq i \end{cases},\\
&n_{4N+1} = \ -1,\ \ a_{4N+1} = c,\ \ \ \alpha_{4N+1,j} = 0,\\
&n_{4N+2} = \ -1,\ \ a_{4N+2} = \frac{m+1}{2} - c;\ \ \alpha_{4N+2,j} = -1,\\
&n_{4N+2+i} = \ -1,\ \ a_{4N+2+i} = 0,\ \ \alpha_{4N+2+i,j} = \begin{cases} -1 & j = i \\ 0 & j \neq i \end{cases},
\end{aligned}$$

then

$$\lim_{q \to 1^-} I_q(Z|1, p, \overline{n}, \overline{a}, \alpha) = f_B(a_1, ..., a_N; b_1, ..., b_N; c; -Z_1, ..., -Z_N).$$

c) *If*

$$r=1\ \ p=3N+4,\ \ n_1=1,\ \ a_1=a,\ \ \alpha_1=0,$$
$$n_2=\ -1,\ \ a_2=\frac{m+1}{2}-a,\ \ \alpha_{2j}=-1,$$
$$n_3=1,\ \ a_3=b,\ \ \alpha_{3j}=0$$
$$n_4=\ -1,\ \ a_4=\frac{m+1}{2}-b,\ \ \alpha_{4j}=-1,$$
$$n_{4+i}=\ -1,\ \ a_{4+i}=c_i,\ \ \alpha_{4+i,j}=0,$$
$$n_{N+4+i}=1,\ \ a_{N+4+i}=\frac{m+1}{2}-c_i,\quad \alpha_{N+4+i,j}=\begin{cases}-1 & j=i\\ 0 & j\neq i\end{cases},$$
$$n_{2N+4+i}=\ -1,\ \ a_{2N+4+i}=0,\quad \alpha_{2N+4+i,j}=\begin{cases}-1 & j=i\\ 0 & j\neq i\end{cases},$$

then

$$\lim_{q\to 1^-} I_q(Z|1,p,\overline{n},\overline{a}_1,\alpha)=f_C(a,b;c_1,...,c_N;-Z_1,...,-Z_N).$$

d) *If*

$$r=1\ \ p=3N+4,\ \ n_1=1,\ \ a_1=a,\ \ \alpha_{1j}=0,$$
$$n_2=\ -1,\ \ a_2=\frac{m+1}{2}-a,\ \ \alpha_{2j}=-1,$$
$$n_{2+i}=1,\ \ a_{2+i}=b_i,\ \ \alpha_{2+i,j}=0,$$
$$n_{N+2+i}=\ -1,\ \ a_{N+2+i}=\frac{m+1}{2}-b_i,\quad \alpha_{N+2+i,j}=\begin{cases}-1 & j=i\\ 0 & j\neq i\end{cases},$$
$$n_{2N+3}=\ -1,\ \ a_{2N+3}=c,\ \ \alpha_{2N+3,j}=0,$$
$$n_{2N+4}=1,\ \ a_{2N+4}=\frac{m+1}{2}-c,\ \ \alpha_{2N+4,j}=-1,$$
$$n_{2N+4+i}=\ -1,\ \ a_{2N+4+i}=0\ \ \ \alpha_{2N+4+i,j}=\begin{cases}1 & j=i\\ 0 & j\neq i\end{cases},$$

then the following statement holds

$$\lim_{q\to 1^-} I_q(Z|1,p,\overline{n},\overline{a},\alpha)=f_D(a,b_1,...,b_N;c;-Z_1,...,-Z_N).$$

Consequence 1.
a) If

$$r=1\ \ p=3N+2,\ \ n_1=1,\ \ a_1=a,\ \ \alpha_{1j}=0,$$
$$n_2=\ -1,\ \ a_2=\frac{m+1}{2}-a,\ \ \alpha_{2j}=-1,$$
$$n_{2+i}=\ -1,\ \ a_{2+i}=c_i,\ \ \alpha_{2+i,j}=0,$$
$$n_{2+N+i}=1,\ \ a_{2+N+i}=\frac{m+1}{2}-c_i,\quad \alpha_{2+N+i,j}=\begin{cases}-1 & j=i\\ 0 & j\neq i\end{cases},$$
$$n_{2+2N+i}=\ -1,\ \ a_{2+2N+i}=0,\quad \alpha_{2+2N+i,j}=\begin{cases}-1 & j=i\\ 0 & j\neq i\end{cases},\ \ i,j=1,...,N,$$

then

$$\lim_{q\to 1^-} I_q(Z|1,p,\overline{n},\overline{a},\alpha)=\psi_2(a;c_1,...,c_N;-Z_1,...,-Z_N).$$

b) If

$$\begin{aligned}
r=1\ \ p=3N+2,\ \ n_i=1,\ \ a_i=b_i,\ \ \alpha_{ij}=0,\\
n_{N+i}=\ -1,\ \ a_{N+i}=\frac{m+1}{2}-b_i,\ \ \alpha_{N+i,j}=\begin{cases}-1 & j=i\\ 0 & j\neq i\end{cases},\\
n_{2N+1}=\ -1,\ \ a_{2N+1}=c,\ \ \alpha_{2N+1,j}=0,\\
n_{2N+2}=1,\ \ a_{2N+2}=\frac{m+1}{2}-c,\ \ \alpha_{2N+2,j}=-1,\\
n_{2+2N+i}=\ -1,\ \ a_{2+2N+i}=0,\ \ \alpha_{2+2N+i,j}=\begin{cases}1 & j=i\\ 0 & j\neq i\end{cases},\ \ i,j=1,...,N,
\end{aligned}$$

then the following statement holds valid

$$\lim_{q\to 1^-} I_q(Z|1,p,\overline{n},\overline{a},\alpha)=\phi_2(b_1,...,b_N;c;-Z_1,...,-Z_N),$$

where functions ψ_2,ϕ_2 are defined in [8].

Consequence 2. If

$$\begin{aligned}
r'=1,\ \ p=N+2s+2r,\ \ n_i=1,\ \ \alpha_{ij}=0,\\
n_{r+i}=\ -1,\ \ a_{r+i}=a_i,\ \ \alpha_{r+i,j}=1,\ \ i=1,...,r,\\
n_{2r+k}=\ -1, a_{2r+k}=b_k,\ \ \alpha_{2r+k,j}=0,\\
n_{2r+s+k}=1,\ \ a_{2r+s+k}=b_k,\ \ \alpha_{2r+s+k,j}=-1,\ \ k=1,...,s,\\
n_{2r+2s+t}=\ -1,\ \ a_{2r+2s+t}=0,\ \ \alpha_{2r+2s+t,j}=\begin{cases}1 & t=j\\ 0 & t\neq j\end{cases},\ \ t,j=1,...,N,
\end{aligned}$$

then

$$\lim_{q\to 1^-} I_q(Z|1,p,\overline{n},\overline{a},\alpha)=\ {}_rF_s(a_1,...,a_r;b_1,...,b_s;-Z_1,...,-Z_N),$$

where the hypergeometric function ${}_rF_s$ is defined in [8].

Under supplemental conditions, the basic analogue of I-function of two matrix argument will coincide, for example, Humbert's function [8], Kampe de Feriet's function [8], Appell's function [8].

Theorem 2. *If $Z_i,A_i,i=1,...,N$, are $m\times m$ symmetric matrices, then the following formula is true:*

$$\int\limits_{Z^N>0}(det\,Z)^{\rho-\frac{m+1}{2}}I_q(ZA|r,p_i,\overline{n}_i,\overline{a}_i,\alpha_i)I_q(Z\,|s,q_j,\overline{t}_j,\overline{b}_j,\beta_j)dZ=$$

$$=I_q\Big(A|r,p_i,\overline{n}_i,\overline{a}_i,\alpha_i;s,q_j,\overline{t}_j,\overline{b}_j+\sum_{k=1}^{N}\Big(\rho_k-\frac{m-1}{2}\Big)\overline{\beta}_{jk},\beta_j\Big),$$

where $\overline{\beta}_j = (\beta_{1j}, ..., \beta_{pj})$.

Proof. From (1) it follows that

$$\int\limits_{A^N>0} (\det A)^{\rho-\frac{m+1}{2}} I_q(ZA\,|r, p_i, \overline{n}_i, \overline{a}_i, \alpha_i)dA =$$

$$= (\det Z)^{-\rho-\frac{m-1}{2}} \Big(\sum_{i=1}^{r} Q^{p_i}_{\overline{n}_i,\overline{a}_i,\alpha_i}(\rho)\Big)^{-1}.$$

Then we have

$$\int\limits_{A^N>0} (\det A)^{\eta-\frac{m+1}{2}}$$
$$\times \Big\{ \int\limits_{Z^N>0} (\det Z)^{\rho-\frac{m+1}{2}} I_q(ZA|r, p_i, \overline{n}_i, \overline{a}_i, \alpha_i) I_q(Z|s, q_j, \overline{t}_j, \overline{b}_j, \beta_j)dZ\Big\}dA$$
$$= \int\limits_{Z^N>0} (\det Z)^{\rho-\frac{m+1}{2}} I_q(Z|s, q_j, \overline{t}_j, \overline{b}_j, \beta_j)$$
$$\times \Big\{ \int\limits_{A^N>0} (\det A)^{\eta-\frac{m+1}{2}} I_q(ZA|r, p_i, \overline{n}_i, \overline{a}_i, \alpha_i)dA\Big\}dZ$$
$$= \Big(\sum_{i=1}^{r} Q^{p_i}_{\overline{n}_i,\overline{a}_i,\alpha_i}(\eta)\Big)^{-1}$$
$$\times \int\limits_{Z^N>0} (\det Z)^{\left(\rho-\eta-\frac{m-1}{2}\right)-\frac{m+1}{2}} I_q(Z|s, q_j, \overline{t}_j, \overline{b}_j, \beta_i)dZ$$
$$= \Big(\sum_{i=1}^{r} Q^{p_i}_{\overline{n}_i,\overline{a}_i,\alpha_i}(\eta)\Big)^{-1} \Big(\sum_{j=1}^{s} Q^{q_j}_{\overline{t}_j,\overline{b}_j,\beta_j}\Big(\rho-\eta-\frac{m-1}{2}\Big)\Big)^{-1}$$
$$= \Big(\sum_{i,j=1}^{r,s} Q^{p_i}_{\overline{n}_i,\overline{a}_i,\alpha_i}(\eta) Q^{q_j}_{\overline{t}_j,\overline{b}_j,\beta_j}\Big(\rho-\eta-\frac{m-1}{2}\Big)\Big)^{-1}.$$

Thus, Theorem 2 is proved. ∎

Theorem 3. *The following formulas are true:*

a)

$$A \int\limits_{[0,1]^m} \prod_{j=1}^{m} x_j^{\sigma-\frac{j+1}{2}} E_q(qx_j) I_q\big((x_1...x_m)^{-\delta} Z\,|r, p_i, \overline{n}_i, \overline{a}_i, \alpha_i\big) d_q x_1 ... d_q x_m$$

$$= I_q\Big((1-q)^{-m\delta} Z|r, p_i, \overline{n}_i, \overline{a}_i, \alpha_i; 1, -1, \sigma + \frac{1-m}{2}\sum_{j=1}^{N}\delta_j, \delta\Big),$$

where $E_q(x)$ *is an exponential function,* $\int\limits_0^1 f(x)d_qx$ *is q-integral* [3],

$$\operatorname{Re}\sigma > 0, \quad \delta_j > 0, \quad \delta = (\delta_1, ..., \delta_N),$$
$$(x_1...x_m)^{-\delta} Z = \big((x_1...x_m)^{-\delta_1} Z_1, ..., (x_1...x_m)^{-\delta_N} Z_N\big),$$
$$A = \pi^{\frac{m}{4}(m-1)}(1-q)^{-m\sigma - \frac{m(m-1)}{4} + \frac{m}{2}(m-1)\sum\limits_{k=1}^{N}\delta_k}.$$

b)
$$B \int\limits_{[0,1]^m} \prod_{j=1}^{m} x_j^{\sigma - \frac{j+1}{2}} (1-qx_j)_{\delta-\sigma-1}$$
$$\times I_q\big((x_1...x_m)^{-\gamma} Z | r, p_i, \overline{n}_i, \overline{a}_i, \alpha_i\big) d_q x_1 ... d_q x_m$$
$$= I_q\Big(Z | r, p_i, \overline{n}_i, \overline{a}_i, \alpha_i; 1, -1, \sigma + \frac{1-m}{2}\sum_{j=1}^{N}\gamma_j, \gamma; 1, 1, \delta + \frac{1-m}{2}\sum_{j=1}^{N}\gamma_j, \gamma\Big),$$

where

$$\operatorname{Re}\sigma > 0, \quad \operatorname{Re}\delta_j > 0, \quad \operatorname{Re}(\delta-\sigma) > 0, \quad \gamma_j > 0,$$
$$(1-y)_\beta = \prod_{n=0}^{\infty}(1-yq^n)(1-yq^{n+\beta})^{-1}, \quad B = \Gamma_q^{-m}(\delta-\sigma).$$

Proof. Utilize formulas of ([5, p. 372]):

$$\frac{G(q)}{1-q}\int\limits_0^1 t^{\beta-s-1} E_q(qt) d_q t = G(q^{\beta-s}),$$
$$G(q^\alpha) = \prod_{n=0}^{\infty}(1-q^{\alpha+n})^{-1},$$
$$\frac{1}{1-q}\int\limits_0^1 t^{\alpha-1}(1-qt)_{\beta-1} d_q t = \prod_{n=0}^{\infty}\frac{(1-q^{\alpha+\beta+n})(1-q^{1+n})}{(1-q^{\alpha+n})(1-q^{\beta+n})},$$

we have

$$\int\limits_{Z^N>0} (\det Z)^{\rho - \frac{m+1}{2}} \times \Big\{ \int\limits_{[0,1]^m} \prod_{j=1}^{m} x_j^{\sigma-\frac{j+1}{2}} (1-qx_j)_{\delta-\sigma-1} \times$$
$$\times I_q\big((x_1...x_m)^{-\gamma} Z | r, p_i, \overline{n}_i, \overline{a}_i, \alpha_i\big) d_q x_1 ... d_q x_m \Big\} dZ$$
$$= \int\limits_{[0,1]^m} \prod_{j=1}^{m} x_j^{\sigma-\frac{j+1}{2}} (1-qx_j)_{\delta-\sigma-1}$$
$$\times \Big\{ \int\limits_{Z^N>0} (\det Z)^{\rho-\frac{m+1}{2}} I_q\big((x_1...x_m)^{-\gamma} Z | r, p_i, \overline{n}_i, \overline{a}_i, \alpha_i\big) dZ \Big\} d_q x_1 ... d_q x_m$$
$$= \int\limits_{[0,1]^m} \prod_{j=1}^{m} x_j^{\sigma + \frac{1-m}{2}\sum\limits_{k=1}^{N}\gamma_k + \sum\limits_{k=1}^{N}\gamma_k\rho_k - \frac{j+1}{2}} (1-qx_j)_{\delta-\sigma-1}$$

$$\times \Big\{ \int\limits_{Z^N>0} \big(\det(x_1...x_m)^{-\gamma} Z\big)^{\rho-\frac{m+1}{2}}$$

$$\times I_q\big((x_1...x_m)^{-\gamma} Z\,|r, p_i, \overline{n}_i, \overline{a}_i, \alpha_i\big) d\big((x_1...x_m)^{-\gamma} Z\big)\Big\} d_q x_1...d_q x_m$$

$$= \Big(\sum_{i=1}^{r} Q^{p_i}_{\overline{n}_i,\overline{a}_i,\alpha_i}(\rho)\Big)^{-1} \times$$

$$\times \prod_{j=1}^{m} \int_0^1 x_j^{\sigma+\frac{1-m}{2}\sum\limits_{k=1}^{N}\gamma_k+\sum\limits_{k=1}^{N}\gamma_k\rho_k-\frac{j-1}{2}-1} (1-qx_j)_{\delta-\sigma-1} d_q x_j$$

$$= \Big(\sum_{i=1}^{r} Q^{p_i}_{\overline{n}_i,\overline{a}_i,\alpha_i}(\rho)\Big)^{-1}$$

$$\times \prod_{j=1}^{m} \Gamma_q(\delta-\sigma) \frac{\Gamma_q\Big(\sigma+\frac{1-m}{2}\sum\limits_{k=1}^{N}\gamma_k+\sum\limits_{k=1}^{N}\gamma_k\rho_k-\frac{j-1}{2}\Big)}{\Gamma_q\Big(\delta+\frac{1-m}{2}\sum\limits_{k=1}^{N}\gamma_k+\sum\limits_{k=1}^{N}\gamma_k\rho_k-\frac{j-1}{2}\Big)}$$

$$= \Gamma_q^m(\delta-\sigma)\Big(\sum_{i=1}^{r} Q^{p_i}_{\overline{n}_i,\overline{a}_i,\alpha_i}(\rho)\Big) \frac{\Gamma_{q,m}\Big(\sigma+\frac{1-m}{2}\sum\limits_{k=1}^{N}\gamma_k+\sum\limits_{k=1}^{N}\gamma_k\rho_k\Big)}{\Gamma_{q,m}\Big(\delta+\frac{1-m}{2}\sum\limits_{k=1}^{N}\gamma_k+\sum\limits_{k=1}^{N}\gamma_k\rho_k\Big)}.$$

Thus, the second part of Theorem 3 is proved. In the same way, we can verify the first part too.

From (1) and properties of basic analogue gamma-function, we have

Theorem 4. *We employ the notation $I_q(a_i - \text{sign}(n_{ij}))$, $i = 1, ..., r$, $j = 1, ..., p_i$ to denote contiguous function in (1), where $\Gamma^{n_{ij}}_{q,m}\Big(a_{ij}+\sum\limits_{k=1}^{N}\alpha_{ijk}\rho_k\Big)$ is replaced by:*

$$\frac{(1-q)^{m-1}\Gamma^{\text{sign}(n_{ij})}_{q,m}(a_{ij}-\text{sign}(n_{ij})+\sum\limits_{k=1}^{N}\alpha_{ijk}\rho_k)}{\prod\limits_{t=1}^{m-1}\Big(1-q^{\big(a_{ij}+\sum\limits_{k=1}^{N}\alpha_{ijk}\rho_k-\frac{t}{2}-\frac{1+\text{sign}(n_{ij})}{2}\big)}\Big)}$$

$$\times \Gamma^{n_{ij}-\text{sign}(n_{ij})}_{q,m}\Big(a_{ij}+\sum_{k=1}^{N}\alpha_{ijk}\rho_k\Big)$$

but with all other parameters left unchanged. Then the following equalities are true:

a)

$$I_q(a_{ik_0}+1) - I_q(a_{ij_0}+1) =$$

$$= \frac{q^{a_{ij_0}} - q^{a_{ik_0}}}{1-q} q^{\Big(\frac{1-m}{2}\sum_{t=1}^{N}\alpha_{ik_0t}\Big)} I_q\big(q^{-\alpha_{ik0}} Z|r, p_i, \overline{n}_i, \overline{a}_i, \alpha_i\big),$$

where $n_{ij_0} < 0$, $n_{ik_0} < 0$, $\alpha_{ik_0t} = \alpha_{ij_0t}$, $t = 1, ..., N$.

b)

$$I_q(a_{ik_0} - 1) - I_q(a_{ij_0} - 1) =$$
$$= \frac{q^{a_{ij_0}-1} - q^{a_{ik_0}-1}}{1-q} q^{\left(\frac{1-m}{2}\sum_{t=1}^{N}\alpha_{ik_0t}\right)} I_q\big(q^{-\alpha_{ik0}} Z | r, p, \overline{n}_i, \overline{a}_i, \alpha_i\big),$$

where $n_{ij_0} > 0$, $n_{ik_0} > 0$, $\alpha_{ik_0t} = \alpha_{ij_0t}$, $t = 1, ..., N$.

c)

$$I_q(a_{ik_0} + 1) - I_q(a_{ij_0} - 1) =$$
$$= \frac{q^{a_{ij_0}-1} - q^{a_{ik_0}}}{1-q} q^{\left(\frac{1-m}{2}\sum_{t=1}^{N}\alpha_{ik_0t}\right)} I_q\big(q^{-\alpha_{ik0}} Z | r, p_i, \overline{n}_i, \overline{a}_i, \alpha_i\big),$$

where $n_{ij_0} > 0$, $n_{ik_0} < 0$, $\alpha_{ik_0t} = \alpha_{ij_0t}$, $t = 1, ..., N$.

d)

$$I_q(a_{ik_0} - 1) - I_q(a_{ij_0} + 1) =$$
$$= \frac{q^{a_{ij_0}} - q^{a_{ik_0}-1}}{1-q} q^{\left(\frac{1-m}{2}\sum_{t=1}^{N}\alpha_{ik_0t}\right)} I_q\big(q^{-\alpha_{ik0}} Z | r, p_i, \overline{n}_i, \overline{a}_i, \alpha_i\big),$$

where $n_{ij_0} < 0$, $n_{ik_0} > 0$, $\alpha_{ik_0t} = \alpha_{ij_0t}$, $t = 1, ..., N$.

3. A Convolution for the M-Transform

Definition 2. [8] *The M-transform of $f(Z)$ with N matrix arguments $Z_1, Z_2, ..., Z_N$ is defined by:*

$$(Mf)(\rho) = \int\limits_{Z^N>0} (det\, Z)^{\rho - \frac{m+1}{2}} f(Z) dZ, \tag{2}$$

where $(\det Z)^{\rho - \frac{m+1}{2}} = \prod\limits_{j=1}^{N} (\det Z_j)^{\rho_j - \frac{m+1}{2}}$, ρ_j *are complex numbers,* $\operatorname{Re}\rho_j > \dfrac{m-1}{2}$, $Z = (Z_1, Z_2, ..., Z_N)$, $dZ = dZ_1 dZ_2 ... dZ_N$, $dZ_i = \prod\limits_{j\geq k} dx_{ijk}$, $Z_N > 0 = \{Z_j > 0,\ \ j = 1, ..., N\}$, $Z_j > 0$ *means that* Z_j *an* $m \times m$ *real symmetric positive definite matrix.*

Definition 3. *A covolution of two functions $f(Z), g(Z)$ with N matrix arguments $Z_1, ..., Z_n$ for M-transform (2) is defined as follows:*

$$_{Z^N}(f * g)(A) = \int\limits_{Z^N>0} (\det Z)^{-m} f(AZ^{-1}) g(Z) dZ. \tag{3}$$

Theorem 5. *Let $f, g \in L\big((det\, Z)^{\rho - \frac{m+1}{2}}, Z^N > 0\big)$, then the convolution (3) belongs to $L\big((det\, Z)^{\rho - \frac{m+1}{2}}, Z^N > 0\big)$, and*

$$M[_{Z^N}(f * g)](\rho) = (Mf)(\rho)(Mg)(\rho),$$

where

$$L\big((\det Z)^{\rho-\frac{m+1}{2}}, Z^N>0\big) =$$
$$=\Big\{f: \int\limits_{Z^N>0} |\det Z|^{\rho-\frac{m+1}{2}} |\det f(Z)|\,|dZ| < +\infty\Big\}.$$

Proof. We have

$$\int\limits_{A^N>0} (\det A)^{\rho-\frac{m+1}{2}}\ {}_{Z^N}(f*g)(A)dA =$$
$$= \int\limits_{A^N>0} (\det A)^{\rho-\frac{m+1}{2}} \Big\{ \int\limits_{Z^N>0} (\det Z)^{-m} f(AZ^{-1})g(Z)dZ\Big\}dA$$
$$= \int\limits_{Z^N>0} (\det Z)^{-m} \Big\{ \int\limits_{A^N>0} (\det A)^{\rho-\frac{m+1}{2}} f(AZ^{-1})dA\Big\}g(Z)dZ$$
$$= \int\limits_{Z^N>0} (\det Z)^{-m} \Big\{ \int\limits_{U^N>0} (\det UZ)^{\rho-\frac{m+1}{2}} (\det Z)^m f(U)dU\Big\}g(Z)dZ$$
$$= (Mf)(\rho)(Mg(\rho),$$

Further

$$\int\limits_{A^N>0} |\det A|^{\rho-\frac{m+1}{2}} \big|\det {}_{Z^N}(f*g)(A)\big|\,|dA|$$
$$\le \int\limits_{U^N>0} |\det U|^{\rho-\frac{m+1}{2}} \big|\det f(U)|\,|dU| \int\limits_{Z^N>0} |\det Z|^{\rho-\frac{m+1}{2}} |\det g(Z)|\,|dZ| < +\infty.$$

Therefore

$${}_{Z^N}(f*g) \in L\big((\det Z)^{\rho-\frac{m+1}{2}}, Z^N>0\big).$$

Thus, Theorem 5 is proved. ■

Theorem 6. *Let $A_1, A_2, ..., A_N$, $Z_1, Z_2, ..., Z_N$ be $m\times m$ symmetric matrices, functions g and h belong to $L\big((\det Z)^{\rho-\frac{m+1}{2}}, Z^N>0\big)$, λ a complex constant such that $1+\lambda Mg \neq 0$. Then there exists a function $k \in L\big((\det Z)^{\rho-\frac{m+1}{2}}, Z^N>0\big)$, such that $Mk = \dfrac{\lambda Mg}{1+\lambda Mg}$. Moreover, the integral equation*

$$f(A) + \lambda \int\limits_{Z^N>0} (\det Z)^{-m} g(AZ^{-1})f(Z)dZ = h(A) \tag{4}$$

has a solution

$$f(A) = h(A) - {}_{Z^N}(k*h)(A) \in L\big((\det A)^{\rho-\frac{m+1}{2}}, A^N>0\big).$$

Proof. From (4) and Theorem 5 we have

$$(Mf)(\rho) + \lambda(Mf)(\rho)(Mg)(\rho) = (Mh)(\rho).$$

Therefore

$$(Mf)(\rho) = \frac{1}{1 + \lambda(Mg)(\rho)}(Mh)(\rho).$$

On one hand, from Wiener-Levi Theorem it follows that there exists a function k, which belongs to $L\big((\det A)^{\rho - \frac{m+1}{2}}, A^N > 0\big)$ and satisfies:

$$(Mk)(\rho) = \frac{\lambda(Mg)(\rho)}{1 + \lambda(Mg)(\rho)}. \quad (6)$$

From (5) and (6) we have

$$(Mf)(\rho) = (Mh)(\rho) - (Mh)(\rho)(Mk)(\rho).$$

Consequently

$$f(A) = h(A) -_{Z^N} (k * h)(A)$$

From Theorem 5, it follows that $f(A) \in L\big((\det A)^{\rho - \frac{m+1}{2}}, A^N > 0\big)$.

Thus, Theorem 6 is proved. ■

Example. Let $A_i, Z_i, i = 1, ..., N$ be $m \times m$ symmetric matrices, function $h(Z)$ belong to $L\big((\det Z)^{\rho - \frac{m+1}{2}}, Z^N > 0\big)$, $\operatorname{Re}\rho_i > \dfrac{m-1}{2}$, the function $I_q(Z|r, p_i, \overline{n}_i, \overline{a}_i, \alpha_i)$ be dominated by the function $\exp\Big(-\sum\limits_{i=1}^{N} \operatorname{tr} Z_i\Big)$, and

$$1 + \sum_{i=1}^{r} Q^{p_i}_{\overline{n}_i, \overline{a}_i, \alpha_i}(\rho) \neq 0.$$

Then the integral equation

$$f(A) + \int\limits_{Z^N > 0} (\det Z)^{-m} I_q(AZ^{-1}|r, p_i, \overline{n}_i, \overline{a}_i, \alpha_i) f(Z) dZ = h(A)$$

has a solution

$$f(A) = h(A) -_{Z^N} \big(h * I_q(Z|r, p_i, \overline{n}_i, \overline{a}_i, \alpha; 1, 1, 0, a_1, \alpha_1)\big)(A)$$

that belongs to $L\big((\det A)^{\rho - \frac{m+1}{2}}, A^N > 0\big)$.

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