Vietnam Journal of Mathematics **32**:4 (2004) 441–449

Vietnam Journal
of
MATHEMATICS

Exchange Rings with Weakly Stable Range One*

Wei Jiaqun

Department of Mathematics,
Nanjing Normal University, Nanjing 210097, China

Received November 4, 2003
Revised January1, 2004

Abstract. As a generalization of rings with stable range one, a ring R is said to have weakly stable range one provided whenever $ax + b = 1$ in R there exists $y \in R$ such that $a + by$ is onesided invertible. Let R be an exchange ring. We show that over the ring R, every regular element is onesided unit-regular if and only if every regular element is onesided unit-regular-regular (i.e., $a = awa$ with w one-sided unit-regular) if and only if R has weakly stable range one. As a corollary we obtain that R has stable range one if and only if every regular element is unit-regular-regular. We also give necessary and sufficient conditions in terms of onesided unit-regularity over R under which R has weakly stable range one.

1. Introduction

Let R be an associative ring with nonzero identity. The ring R is said to have stable range one provided that whenever $ax + b = 1$ in R, there exist $y \in R$ such that $a + by$ is a unit in R. As well known, this definition is left-right symmetric. Originally, the stable range one condition is connected with the question of cancellation of modules. A theorem of Evans [6] showed that a right R-module M can be cancelled from direct summands (that is, $A \cong B$ whenever $M \oplus A \cong M \oplus B$ for right R-modules A, B) if $\mathrm{End}_R M$ has stable range one. In case that $\mathrm{End}_R M$ is an exchange rings, the stable range one condition of $\mathrm{End}_R M$ is also necessary by a result of [16]. However, the converse of Evans result is false in general (e.g. Z_Z). Recall that a right R-module M satisfies

*This work was partially supported by NSFC.

the weak cancellation if whenever $M \oplus A \cong M \oplus B$ for right R-modules A, B there exists a right R-module N such that $A \cong N \oplus B$ or $B \cong N \oplus A$. In other words, M satisfies the weak cancellation if and only if whenever $M \oplus A \cong M \oplus B$ there is a splitting epimorphism (equivalently, monomorphism) between A and B. Investigations of this property lead to the study of weakly stable range one, as proceeded in [8, 10, 12, 13]etc.. Following [13], we say that a ring R has weakly stable range one if whenever $ax + b = 1$ in R there exists $y \in R$ such that $a + by$ is onesided invertible. Wu showed that the right R-module M can be weakly cancelled provided $\mathrm{End}_R M$ has weakly stable range one [10]. The converse also holds if moreover $\mathrm{End}_R M$ is an exchange ring [8].

We note that the weakly stable range one condition (using other notions) was also studied in the context of C^*-algebras (see [9] etc.).

While the stable range one condition was studied extensively by many authors (cf. [1-3, 5, 7, 11, 13] etc.), one would try to extend results about rings with stable range one to rings with weakly stable range one. Such work had been proceeded by Li and Tong [8] etc. In this note we will give more characterizations of rings with weakly stable range one. It should be noted that some of these results also lead to new characterizations of rings with stable range one.

Main results in this paper are Theorems 3, 6, 7, and 10.

2. Results

We begin with the following result which will be used repeatedly in this paper. For the proof we refer to [8] or [5].

Lemma 1. *Let $ab+c = 1$ in R. Then $a+cx$ is left (respectively, right) invertible for some $x \in R$ if and only if $b+yc$ is right (respectively, left) invertible for some $y \in R$.* ∎

Let R be a ring. Recall that $x \in R$ is said to be *qu*-regular if there exist $y, u \in R$, where u is left invertible, such that $x = xux$ or $x = xyx = xyu$ (see [12]).

The following lemma is the key to obtain our main results.

Lemma 2. *Let R be a ring and $x \in R$. The following are equivalent:*

(1) *x is onesided unit-regular.*

(2) *x is qu-regular.*

(2)' *There exist $y, v \in R$, where v is right invertible, such that $x = xvx$ or $x = xyx = vyx$.*

(3) *There exist $y, w \in R$ such that w is onesided unit-regular and $x = xyx = xyw$.*

(3)' *There exist $y, w \in R$ such that w is onesided unit-regular and $x = xyx = wyx$.*

Proof.
(1) ⇒ (2). If x is onesided unit-regular, then $x = xux$ where u is left invertible,

or $x = xvx$ where v is right invertible. In the second case we obtain an element $u \in R$ such that $vu = 1$ and consequently, $x = xvx = xvu$. By the definition of qu-regular elements, we see that x is qu-regular.
(2) $\Rightarrow$ (1). If x is qu-regular then there exist $y, u \in R$, where u is left invertible, such that $x = xux$ or $x = xyx = xyu$. In the first case we have nothing to say. Now suppose that $x = xyx = xyu$.

Let $e = xy$. Then e is an idempotent of R and $x = eu$. Hence, from the identity $xy + (1 - xy) = 1$ we obtain that $(eu)y(1 - e) + (1 - xy)(1 - e) = 1 - e$. Consequently we have $e + (1 - xy)(1 - e) = 1 - euy(1 - e)$ is a unit in R. Then $x+(1-xy)(1-e)u = eu+(1-xy)(1-e)u = (e+(1-xy)(1-e))u = (1-euy(1-e))u$ is left invertible. By Lemma 1 we see that there exists $z \in R$ such that $y + z(1 - xy) = v$ is right invertible. Now we have $x = xyx = x(y + z(1 - xy))x = xvx$. Hence x is onesided unit-regular.
(2)$\Rightarrow$ (3). Suppose x is qu-regular. Then there exist $y, u \in R$, where u is left invertible, such that $x = xux$ or $x = xyx = xyu$. Obviously, we have nothing to say in the second case, since the left invertible element u is clearly onesided unit-regular. Now assume that $x = xux$. Let $y = u$ and $w = x$. Clearly we have w is onesided unit-regular and $x = xyx = xyw$.
(3)$\Rightarrow$ (2). Let $x = xyx = xyw$ where w is onesided unit-regular.

Assume first $w = wuw$ for some left invertible element $u \in R$. Let $vu = 1$ and $uw = e$. Then e is an idempotent of R and $w = vuw = ve$. Hence, from the identity $yw + (1 - yw) = 1$ we obtain $y(ve) + (1 - yw) = 1$. Then $(1 - e)yve + (1 - e)(1 - yw) = 1 - e$. Consequently we have that $e + (1 - e)(1 - yw) = 1 - (1 - e)yve$ is a unit in R. Now $w + v(1 - e)(1 - yw) = ve + v(1 - e)(1 - yw) = v(e + (1 - e)(1 - yw)) = v(1 - (1 - e)yve)$ is right invertible. By Lemma 1 $y + (1 - yw)z = p$ is left invertible for some $z \in R$. Therefore, $x = xyx = x(y + (1 - yw)z)x = xpx$ since $x = xyw$. It follows that x is qu-regular.

Now assume that $w = wvw$ for some right invertible element $v \in R$. Similarly let $vu = 1$ and $wv = e$. Then e is an idempotent of R and $w = wvu = eu$. Now from the identity $xy + (1 - xy) = 1$ we obtain that $xyw + (1 - xy)w = w$. Since $x = xyw$ we have $x + (1 - xy)w = w$. Then $xy + (1 - xy)wy = wy$. Hence $wy + (1 - xy)(1 - wy) = wy - (1 - xy)wy + 1 - xy = 1$. It follows that $euy(1 - e) + (1 - xy)(1 - wy)(1 - e) = 1 - e$. Consequently $e + (1 - xy)(1 - wy)(1 - e) = 1 - euy(1 - e)$ is a unit in R. Now we deduce that $q := w + (1 - xy)(1 - wy)(1 - e)u = eu + (1 - xy)(1 - wy)(1 - e)u = (e + (1 - xy)(1 - wy)(1 - e))u = (1 - euy(1 - e))u$ is left invertible. It is easy to check that $x = xyx = xyw = xyq$. Again we obtain that x is qu-regular.

The proofs of (1) $\Leftrightarrow$ (2)' $\Leftrightarrow$ (3)' $\Leftrightarrow$ (1) are similar to (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (1). ■

Theorem 3. *Let R be an exchange ring. The following are equivalent:*
(1) *R has weakly stable range one.*
(2) *For any regular element $x \in R$, there exist a left invertible element u and an idempotent e such that $x = eu$ or $x = xux$.*
(3) *For any regular element $x \in R$, there exist $e, u \in R$ such that $e^2 = e$, u is*

left invertible and $x = eu$, or there exist $e, v \in R$ such that $e^2 = e$, v is right invertible and $x = ve$.

(4) *Every regular element is onesided unit-regular.*

(5) *For any regular element x there exist $y, u \in R$ such that u is onesided invertible and $x = xyx = xyu$.*

(6) *For any regular element x there exist $y, w \in R$ such that w is onesided unit-regular and $x = xyx = xyw$.*

Proof.
(1) $\Rightarrow$ (5). Assume that $x = xyx$. Since $xy + (1 - xy) = 1$ and R has weakly stable range one, there exist $\alpha \in R$ and an onesided invertible element $u \in R$ such that $x + (1 - xy)\alpha = u$. Clearly we have that $x = xyx = xyu$.
(5) $\Rightarrow$ (2). Assume that $x = xyx = xyu$, where u is onesided invertible. If u is left invertible, let then $e = xy$ we have that $e^2 = e$ and $x = eu$. Now assume u is right invertible and let $uv = 1$. Then $xv = (xyu)v = xy$ and hence, $x = xyx = xvx$, where v is left invertible.
(2) $\Rightarrow$ (3). We need only check the second case in (2). Assume that $x = xux$, where u is left invertible. Accordingly we have $v \in R$ such that $vu = 1$. Let $e = ux$. Then we have $x = ve$, where $e^2 = e$ and v is clearly right invertible.
(3) $\Rightarrow$ (4). If $x = xux$ with u left invertible then we have nothing to say. Assume that $x = eu$, where $e^2 = e$ and u is left invertible. Then by the same method used in the proof of (2) $\Rightarrow$ (1) in Lemma 2, we have that $x = xvx$ for some right invertible element $v \in R$.
(4) $\Rightarrow$ (1). By [8, Theorem 2.5], R has weakly stable range one if and only if every regular element is qu-regular. Now the assertion follows from Lemma 2.
(4) $\Leftrightarrow$ (6). By Lemma 2. ∎

Corollary 4. *Let R be an exchange ring. The following are equivalent:*

(1) *R has weakly stable range one.*

(2) *Whenever $x \in R$ is regular there exists a left invertible element $u \in R$ such that ux is an idempotent, or there exists a right invertible element $v \in R$ such that xv is an idempotent.*

Proof.
(1) $\Rightarrow$ (2). By Theorem 3, whenever $x \in R$ is regular there exists an onesided invertible element $u \in R$ such that $x = xux$. Clearly, ux and xu are idempotents.
(2) $\Rightarrow$ (1). Assume $x = xyx$. If ux is an idempotent, where u is left invertible, let then $e = ux$ and $vu = 1$, we obtain that $x = vux = ve$, where v is clearly right invertible. Now assume that there exists a right invertible element $v \in R$ such that xv is an idempotent. Let then $e = xv$ and $vu = 1$, we have that $x = xvu = eu$, where u is left invertible. Now the conclusion follows by Theorem 3. ∎

In [3, Theorem 2.9], Canfell showed that R has stable range one if and only if $aR + bR = dR$ implies that there exists a unit $u \in R$ such that $a + by = du$ for some $y \in R$, by using the method of completion of diagrams. Using another technique which is based on the fact that R has stable range one if and only if

$M_n(R)$ has stable range one for any $n \geq 1$, Chen [5] showed that R has unit stable range one if and only if $aR + bR = dR$ implies that there exist units $u, v \in R$ such that $a + bv = du$. Now we generalize Canfell's result to rings with weakly stable range one. It should be noted that both methods mentioned above cannot be applied to our case, since it is still an open question whether $M_n(R)$ has weakly stable range one if R has so.

Proposition 5. *Let R be a ring. The following are equivalent:*
(1) *R has weakly stable range one.*
(2) *Whenever $aR+bR = R$ there exists some $z \in R$ such that $a+bz$ is onesided invertible.*
(3) *Whenever $aR+bR = dR$ there exist some $z \in R$ and some onesided invertible element $u \in R$ such that $a + bz = du$.*

Proof.
(3) $\Rightarrow$ (2) $\Rightarrow$ (1) are obvious.
(1) $\Rightarrow$ (3). Let $aR + bR = dR$. Then $a, b \in dR$. Hence we may assume that $a = dr$ and $b = ds$ for some $r, s \in R$. Let $ax + by = d$. Equivalently we have $drx + dsy = d$. It follows that $dg = 0$, where $g = 1 - rx - sy$. Now from the fact that $rx + sy + g = 1$ we have that $r + (sy + g)z' = u$ is onesided invertible for some $z' \in R$ because R has weakly stable range one. Hence, $du = d(r + (sy + g)z') = dr + dsyz' + dgz' = a + byz' = a + bz$ with $z = yz'$. ∎

In case R is an exchange ring, we even have the following more general result.

Theorem 6. *Let R be an exchange ring. The following are equivalent:*
(1) *R has weakly stable range one.*
(2) *Whenever $aR + bR = R$ there exist some $z \in R$ and some onesided unit-regular element $w \in R$ such that $a + bz = w$.*
(3) *Whenever $aR + bR = dR$ there exist some $z \in R$ and some onesided unit-regular element $w \in R$ such that $a + bz = dw$.*

Proof.
(1) $\Rightarrow$ (3) follows from Proposition 5 since every onesided invertible element is clearly onesided unit-regular.
(3) $\Rightarrow$ (2) is obvious.
(2) $\Rightarrow$ (1). Let $x = xyx$ for some $y \in R$. Then $xR + (1 - xy)R = R$ since $xy + (1 - xy) = 1$. By assumptions we have that $x + (1 - xy)z = w$ is onesided unit-regular for some $z \in R$. Hence, $x = xyx = xy(x + (1 - xy)z) = xyw$. Now the conclusion follows from Theorem 3. ∎

Following a similar route above we give the following characterizations of exchange rings with weakly stable range one. Our result is also a generalization of [5, Theorem 3.1].

Theorem 7. *Let R be an exchange ring. The following are equivalent:*
(1) *R has weakly stable range one.*

(2) *Whenever* $aR+bR=R$ *there exists an onesided unit-regular element* $w\in R$ *such that* $aw+by=1$ *for some* $y\in R$.
(3) *Whenever* $aR+bR=R$ *there exist onesided unit-regular elements* $w_1,w_2\in R$ *such that* $aw_1+bw_2=1$.
(4) *Whenever* $a_1R+\cdots+a_mR=R$ *there exist onesided unit-regular elements* $w_1,\dots w_m\in R$ *such that* $a_1w_1+\cdots+a_mw_m=1$, *where* $m\geq 2$.
(5) *Whenever* $aR+bR=dR$ *there exists an onesided unit-regular element* $w\in R$ *such that* $aw+by=d$ *for some* $y\in R$.
(6) *Whenever* $aR+bR=dR$ *there exist onesided unit-regular elements* $w_1,w_2\in R$ *such that* $aw_1+bw_2=d$.
(7) *Whenever* $a_1R+\cdots+a_mR=dR$ *there exist onesided unit-regular elements* $w_1,\dots,w_m\in R$ *such that* $a_1w_1+\cdots+a_mw_m=d$, *where* $m\geq 2$.

Proof.
(7) $\Rightarrow$ (4) $\Rightarrow$ (3) $\Rightarrow$ (2) and (7) $\Rightarrow$ (6) $\Rightarrow$ (5) $\Rightarrow$ (2) are obvious.
(1) $\Rightarrow$ (7). Assume that $a_1R+\cdots+a_mR=dR$. Then $a_i\in dR$, $i=1,\dots,m$. Let $a_i=du_i$, $i=1,\dots,m$. Obviously we have $du_1x_1+\cdots+du_mx_m=d$ for some $x_i\in R$, $i=1,\dots,m$. It follows that $dg=0$, where $g=1-(u_1x_1+\cdots+u_mx_m)$. Since $u_1x_1+\cdots+u_mx_m+g=1$ we obtain that $u_1R+\cdots+u_mR+gR=R$. Note that R is an exchange ring, so there exist idempotents $e_i\in R$, $i=1,\dots,m$ and an idempotent $f\in R$, where e_i, $i=1,\cdots,m$, and f are orthogonal satisfying $e_1+\cdots+e_m+f=1$, such that $e_i=u_iy_i$, $i=1,\dots,m$, and $f=gz$ for some $y_i,z\in R$, $i=1,\dots,m$. Let $w_i=y_ie_i$, $i=1,\dots,m$. Then $u_iw_i=u_iy_ie_i=e_i$ and $w_iu_iw_i=y_ie_ie_i=y_ie_i=w_i$. By assumptions and Theorem 3, every w_i is onesided unit-regular. Observing that $u_1w_1+\cdots+u_mw_m+gy=e_1+\cdots+e_m+f=1$ we obtain that $a_1w_1+\cdots+a_mw_m=d(u_1w_1+\cdots+u_mw_m+gy)=d$, as desired.
(2) $\Rightarrow$ (1). Let $x=xyx$. Since $yx+(1-yx)=1$ we have that $yR+(1-yx)R=R$. By assumptions there exists an onesided unit-regular element $w\in R$ such that $yw+(1-yx)z=1$ for some $z\in R$. Hence, $x=xyx=x(yw+(1-yx)z)=xyw$.
■

The following proposition may be viewed as a supplement of Theorem 7 in case $m=1$, which also generalizes [5, Proposition 3.5].

Proposition 8. *Let* R *be an exchange ring. The following are equivalent:*
(1) R *has weakly stable range one.*
(2) *Whenever* $aR=bR$ *there exists an onesided invertible element* $u\in R$ *such that* $b=au$.
(3) *Whenever* $aR=bR$ *there exists an onesided unit-regular element* $w\in R$ *such that* $b=aw$.

■

Proof.
(1) $\Rightarrow$ (2). Let $ax=b$ and $by=a$ for some $x,y\in R$. Since $xy+(1-xy)=1$ and R has weakly stable range one, we have that $x+(1-xy)z=u$ is onesided invertible for some $z\in R$. It follows that $b=ax=a(x+(1-xy)z)=au$.
(2) $\Rightarrow$ (3) is trivial.

(3) $\Rightarrow$ (1). For any regular element $x = xyx \in R$, we have $xR = xyR$. Hence, $x = xyw$ for some onesided regular element $w \in R$ by assumption. It follows that $x = xyx = xyw$. By Theorem 3 we obtain that R has weakly stable range one. ■

Corollary 9. *Let R be an exchange ring. The following are equivalent:*
(1) *R has weakly stable range one.*
(2) *Whenever $\psi : aR \cong bR$, where $a, b \in R$, there exists an onesided invertible element $u \in R$ such that $b = \psi(a)u$.*
(3) *Whenever $\psi : aR \cong bR$, where $a, b \in R$, there exists an onesided unit-regular element $w \in R$ such that $b = \psi(a)w$.*

Proof.
(1) $\Rightarrow$ (2). If $\psi : aR \cong bR$, then $b = \psi(ax)$ and $a = \psi^{-1}(by)$ for some $x, y \in R$. Then $b = \psi(ax) = \psi(\psi^{-1}(by)x) = byx$. Since $yx + (1 - yx) = 1$ and R has weakly stable range one, we have that $y + (1 - yx)z = u$ is onesided invertible for some $z \in R$. It follows that $\psi(a) = by = b(y + (1 - yx)z) = bu$.
(2) $\Rightarrow$(3) is trivial.
(3) $\Rightarrow$ (1). By Proposition 8. ■

Let R be a ring. A regular element $x \in R$ is said to be (onesided) unit-regular-regular if $x = xwx$ for some (onesided) unit-regular element $w \in R$. For example every element in the unit-regular ring is unit-regular-regular. It is well known that an exchange ring has stable range one if and only if every regular element in R is unit-regular [2]. Also, an exchange ring has weakly stable range one if and only if every regular element in R is onesided unit-regular by Theorem 3. Now we generalize these results by characterizing these stable range conditions in terms of the (onesided) unit-regular-regularity.

Theorem 10. *Let R be an exchange ring. The following are equivalent:*
(1) *R has weakly stable range one.*
(2) *Every regular element is onesided unit-regular-regular.*

Proof.
(1) $\Rightarrow$ (2). Of course, every onesided invertible element is onesided unit-regular and every onesided unit-regular is onesided unit-regular-regular.
(2) $\Rightarrow$ (1). By assumptions we may assume that $x = xwx$ for some onesided unit-regular element $w \in R$.

Suppose first $w = wvw$, where v is right invertible. Then $w = eu$, where $e^2 = e = wv$ and $uv = 1$. Since $wx + (1 - wx) = 1$, $eux + (1 - wx) = 1$. Then $eux(1-e) + (1-wx)(1-e) = 1-e$. Hence, $e + (1-wx)(1-e) = 1 - eux(1-e)$ is a unit of R. Note that $p := w + (1 - wx)(1 - e)u = eu + (1 - wx)(1 - e)u = (e + (1 - wx)(1 - e))u = (1 - eux(1 - e))u$ is left invertible, so we have $x = xwx = x(w + (1 - wx)(1 - e)u)x = xpx$.

Now assume that $w = wvw$, where v is left invertible. Then $w = ue$, where $e^2 = e = vw$ and $uv = 1$. Since $xw + (1 - xw) = 1$, $xue + (1 - xw) = 1$. Then $(1-e)xue + (1-e)(1-xw) = 1-e$. Hence, $e + (1-e)(1-xw) = 1-(1-e)xue$ is a unit of R. Note that $w+u(1-e)(1-xw) = ue+u(1-e)(1-xw) = u(e+(1-e)(1-xw)) = u(1-(1-e)xue)$ is right invertible, so we have $x+(1-xw)z = p$ is left invertible for some $z \in R$. It follows that $x = xwx = xw(x+(1-xw)z) = xwp$.

Combining the arguments above we then obtain the desired result by Lemma 2 and Theorem 3. ∎

Applying the above result to rings with stable range one, we have the following.

Corollary 11. *Let R be an exchange ring. The following are equivalent:*
(1) *R has stable range one.*
(2) *Every regular element is unit-regular-regular.*

Proof.
(1) ⇒ (2) is obvious.
(2) ⇒ (1). From the proof of Theorem 10 we derive that whenever $x \in R$ is unit-regular-regular there exists a unit $p \in R$ such that $x = xyx = xyp$ for some $y \in R$. Now the result follows from the following lemma. ∎

Lemma 12. *Let R be an exchange ring. The following are equivalent:*
(1) *R has stable range one.*
(2) *For every regular element $x \in R$ there exists a unit $p \in R$ such that $x = xyx = xyp$ for some $y \in R$.*
(2)' *For every regular element $x \in R$ there exists a unit $p \in R$ such that $x = xyx = pyx$ for some $y \in R$.*

Proof.
(1) ⇒ (2). Since every regular element in R is unit-regular for R with stable range one, the assertion (2) follows from the proof of Theorem 10.
(2) ⇒ (1). Let $x \in R$ be regular. By assumptions $x = xyx = xyp$ for some unit $p \in R$ and some $y \in R$. An argument similar to the second part of (2) ⇒ (1) in the proof of Lemma 2 shows that x is unit-regular. Hence the result follows from [2, Theorem 3].
(1) ⇔ (2)'. It is similar to the proof (1) ⇔ (2). ∎

References

1. P. Ara, Strongly π-regular rings have stable range one, *Proc. Amer. Math. Soc.* **124** (1996) 3293–3298.
2. V. P. Camillo and H. Yu, Stable range one for rings with many idempotents, *Trans. Amer. Math. Soc.* **347** (1995) 3141–3147.
3. M. J. Canfell, Completions of diagrams by automorphisms and Bass' first stable range condition, *J. Algebra* **176** (1995) 480–513.
4. H. Chen, Power-substituation, exchange rings and unit Π-regularity, *Comm. Algebra* **28** (2000) 5223–5233.
5. H. Chen, On stable range conditions, *Comm. Algebra* **28** (2000) 3913–3924.

6. E. G. Evans, Krull-Schmidt and cancellation over local rings, *Pacific J. Math.* **46** (1973) 115–121.
7. K. R. Goodeal and P. Menal, Stable range one for rings with many units, *J. Pure Appl. Algebra* **54** (1988) 261–287.
8. Q. Li and W. Tong, Weak cancellation of modules and weakly stable range conditions for exchange rings, *Acta Math. Sinica* **45** (2002) 1121–1126.
9. G. K. Pedersen, The λ-function in operator algebras, *J. Operator Theory* **26** (1991) 345–381.
10. T. Wu, Weak cancellation of modules and wsr1 condition, *J. of Nanjing University Math. Biquartly* **11** (1994) 109–116.
11. T. Wu, Unit 1-stable condition, *Chinese Ann. Math.* **16**(A) (1995) 760–768.
12. T. Wu, Weak cancellation of modules and qu-regular rings, *Acta Math. Sinica* **38** (1995) 746–751 (in Chinese).
13. T. Wu, Characterization wsr1 condition by completions of diagrams, *J. Math. Research and Exposition* **19** (1999) 131–134 (in Chinese).
14. T. Wu and Y. Xu, On the stable range condition of exchange rings, *Comm. Algebra* **25** (1997) 2355–2363.
15. H. Yu, Stable range one for exchange rings, *J. Pure Appl. Algebra* **98** (1995) 105–109.