Vietnam Journal of Mathematics **32**:4 (2004) 399–411

Vietnam Journal of MATHEMATICS
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Regularity of AP-Injective Rings

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Received August 6, 2003
Revised July 11, 2004

Abstract. A ring R is called right AGP-injective if, for any $0 \neq a \in R$, there exists $n > 0$ such that $a^n \neq 0$ and Ra^n is a direct summand of $lr(a^n)$. In this paper some conditions which are sufficient or equivalent for a right AGP-injective ring to be von Neumann regular (right self-injective, semisimple) are provided. It is shown that a ring R is von Neumann regular if and only if R is right AGP-injective and for any $0 \neq a \in R$ there exists a positive integer n with $0 \neq a^n$ such that $a^n R$ is a projective right R-module if and only if R is a right AGP-injective ring whose divisible and torsionfree right R-modules are GP-injective. We also show that if R is a primitively finite right AGP-injective ring, then $R \cong R_1 \times R_2$, where R_1 is semisimple and every simple right ideal of R_2 is nilpotent. In addition, it is proven that if R is a right MI and right AGP-injective ring satisfying the a.c.c on right annihilators, then R is quasi-Frobenius.

1. Introduction

Throughout, R is an associative ring with identity and all modules are unitary. The Jacobson radical, left (right) singular ideal and right socle of R are denoted by J,$Z(_RR)$ $Z(R_R)$) and $Soc(R_R)$, respectively. For a subset X of R, $r(X)$ (resp. $l(X)$) is reserved for the right (resp. left) annihilator of X in R, and we write $r(x)$ (resp. $l(x)$) for $r(\{x\})$(resp. $l(\{x\})$) when $x \in R$. A right R-module M is called principally injective (P-injective) [11] if every R-homomorphism from a principal right ideal aR to M extends to one from R_R to M. M is said to be generalized principally injective (GP-injective) [10] if, for any $0 \neq a \in R$, there exists a positive integer n with $a^n \neq 0$ such that any R-homomorphism from

a^nR to M extends to one from R_R to M. A ring R is called right P-injective (resp. GP-injective) if the right R-module R_R is P-injective (resp. GP-injective). Following [12], a ring R is called right almost principally injective (AP-injective) if every principal left ideal is a direct summand of a left annihilator, and the ring R is called right almost generalized principally injective(AGP-injective) if, for any $0 \neq a \in R$, there exists $n > 0$ such that $0 \neq a^n$ and Ra^n is a direct summand of $lr(a^n)$. The detailed discussion of right P-injective and right GP-injective rings can be found in [2-3, 7, 10-12, 17-21]. Clearly, P-injective rings are AP-injective and GP-injective rings are AGP-injective. But there exists a right AP-injective ring which is not right GP-injective [12, Example 1.5]. Recently, it was noted that GP-injective rings need not be P-injective (see [3]). Several results which are known for right P-injective (resp. right GP-injective) rings were shown to hold for right AP-injective(resp. right AGP-injective) rings in [12, 21]. In this paper, we discuss the regularity and the self-injectivity of right AGP-injective rings.

In Sec. 2, we characterize the regularity of AGP-injective rings. It is shown that R is von Neumann regular if and only if it is a right AGP-injective ring whose divisible and torsionfree right R-modules are GP-injective; R is strongly regular if and only if it is a right quasi-duo (resp. WRD) and right AGP-injective ring whose divisible and torsionfree right R-modules are GP-injective if and only if it is a right AGP-injective ring with a reduced maximal right ideal. An example is given to show that there is a primitively finite ring which is not semiperfect. We also prove that if R is a primitive finite right AGP-injective ring, then $R \cong R_1 \times R_2$, where R_1 is semisimple and every simple right ideal of R_2 is nilpotent, which extends the results in [11, Theorem 1.4] and [12, Theorem 2.16].

It is well known that a ring R is quasi-Frobenius (QF) if and only if it is right self-injective and left (or right) Noetherian. In Sec. 3, we prove that, if R is a right MI ring, then R is right self-injective if and only if it is right $(A)P$-injective if and only if it is right $(A)GP$-injective. In particular, if R is a right MI ring, then R is QF if and only if R is a right AGP-injective ring satisfying the a.c.c on right annihilators.

Recall that a ring R is called left (resp. right) PP [9] if, for any $a \in R$,Ra (aR) is a projective R-module; R is said to be a left (resp. right) GPP ring [9] if, for any $a \in R$, there exists a positive integer m such that Ra^m (a^mR) is a projective R-module. The classes of these rings were studied by many authors, for example, Hirano, Xue and Zhou (see [8, 13, 21]). In Sec. 3, we consider the class of rings satisfying $(*)$ (R is called a ring satisfying left (resp. right) $(*)$ if, for any $0 \neq a \in R$, there exists a positive integer n such that $0 \neq a^n$ and Ra^n (resp. a^nR) is a projective R-module). Clearly, PP rings are rings satisfying $(*)$ and rings satisfying $(*)$ are GPP rings. We prove that a ring R is von Neumann regular if and only if R is a right AGP-injective ring satisfying right $(*)$, which extends Zhou [21, Proposition 2.5]. An example shows that GPP rings need not be rings satisfying $(*)$, and none of the notions of PP rings, AGP-injective rings and rings satisfying $(*)$ is left-right symmetric.

2. Nonsingular Right AGP-Injective Rings

Let R be a ring. Recall that R is called (*von Neumann*) *regular* if for every $x \in R$, there exists some $y \in R$ such that $x = xyx$; R is called *strongly regular* if for every $x \in R$, there exists some $y \in R$ such that $x = x^2y$; R is called right *nonsingular* if $Z(R_R) = 0$; R is called *reduced* if it contains no nonzero nilpotent elements; R is called *semiprime* if it contains no nonzero nilpotent ideal (equivalently, for any $a \in R$, $RaRa = 0$ implies $Ra = 0$).

In this section, we characterize the regularity of AGP-injective rings.

Lemma 2.1. *Let $c \in C(R)$, where $C(R)$ is the center of R. If c is regular in R, then c is regular in $C(R)$.*

Proof. Let $c = cdc$ with $d \in R$. Put $u = dcd$. Then $c = cuc = uc^2$. We claim that $u \in C(R)$. In fact, for any $x \in R$, $ux - xu \in r(c^2) = r(c)$, so $c^2(xd^2 - d^2x) = c(xu - ux) = 0$, which implies $xd^2 - d^2x \in r(c^2) = r(c)$. Thus $xu - ux = xcd^2 - cd^2x = c(xd^2 - d^2x) = 0$. This completes the proof. ∎

Proposition 2.2. *If R is a right nonsingular right AGP-injective ring, then the center $C(R)$ of R is regular.*

Proof. By hypothesis, R has a regular maximal right quotient ring S (see [5, Corollary 2.31]). Consequently, the center $C(S)$ of S is regular by Lemma 1.1. For any $0 \neq a \in C(R)$, there exists $s \in S$ such that $a = asa = a^2s = sa^2$. Thus $r(a^n) = r(a) = l(a) = l(a^n)$ for any positive integer n. We claim that a is regular in $C(R)$. Note that $a^2 \neq 0$, so there is a positive integer m with $a^{2m} \neq 0$ such that $lr(a^{2m}) = Ra^{2m} \oplus X_{a^{2m}}$ for some left ideal $X_{a^{2m}}$ of R since R is right AGP-injective. Thus $a^{2m-1} \in lr(a^{2m-1}) = lr(a^{2m})$, and so $a^{2m-1} = da^{2m} + x$ for some $d \in R$ and some $x \in X_{a^{2m}}$. Then $a^{2m} = ada^{2m} + ax$ and $ax \in Ra^{2m} \cap X_{a^{2m}} = 0$. Hence $a^{2m} = ada^{2m}$. Therefore $1 - ad \in l(a^{2m}) = l(a)$, and so $a = ada$. This implies that $C(R)$ is regular by Lemma 2.1. ∎

Proposition 2.3. *If R is a semiprime right AGP-injective ring, then the center $C(R)$ of R is regular.*

Proof. For any $0 \neq c \in C(R)$, $Rc \cap l(c) = 0$ since R is semiprime. Therefore, $l(c^m) = l(c) = r(c) = r(c^m)$ for any positive integer m. Note that $c^2 \neq 0$ because $Rc \cap l(c) = 0$. As in the proof of Proposition 2.2, $C(R)$ is regular. ∎

Recall that (1) R is called right (resp. left) duo [1] if every right (resp. left) ideal is a two-sided ideal; (2) R is called right quasi-duo [1] if every maximal right ideal is a two-sided ideal; (3) R is said to be weakly right duo (abbreviated WRD) [15] if for any $a \in R$, there exists a positive integer n such that a^nR is a two-sided ideal. Right quasi-duo (resp. WRD) rings are non-trivial generalizations of right duo rings (see [1, 15]). Note that if R is right quasi-duo (resp. WRD) then R/J is reduced (see [17]).

A right R-module M is called torsionfree if, for any $0 \neq m \in M$, $mc \neq 0$ for every non-zero-divisor c of R. The module M is divisible if $M = Mc$ for each non-zero-divisor c of R.

Lemma 2.4. *If R is a right AGP-injective ring, then any non-zero-divisor is invertible, thus any right (resp. left) R-module is divisible and torsionfree.*

Proof. Let $a \in R$ be a non-zero-divisor, then $r(a) = l(a) = 0$. Hence $r(a^n) = r(a) = l(a) = l(a^n) = 0$ for any positive integer n. Since R is right AGP-injective, there exists a positive integer n with $a^n \neq 0$ such that $R = lr(a^n) = Ra^n \oplus X_{a^n}$ for some left ideal X_{a^n} of R. Thus $1 = ra^n + x$ with $r \in R,\ x \in X$, and so $a^n x = a^n - a^n r a^n \in X_{a^n} \cap Ra^n = 0$, which implies that $a^n = a^n r a^n$. If $n = 1$, then $1 = ra = ar$. If $n > 1$, then $1 = (ra^{n-1})a = a(a^{n-1}r)$. It follows that a is invertible, and hence any right (resp. left) R-module is divisible and torsionfree. ∎

Let M be a right R-module. We write $l_M(X) = \{m \in M \mid\ ma = 0$ for any $a \in X\}$, where X is a subset of R, and write $Z_r(M) = \{m \in M \mid mK = 0$ for some essential right ideal K of $R\}$.

Lemma 2.5. [20, Proposition 1] *Let M be a right R-module. Then M is GP-injective if and only if, for any $0 \neq a \in R$, there exists a positive integer n such that $a^n \neq 0$ and $l_M(r(a^n)) = Ma^n$.*

Theorem 2.6. *Let R be a right AGP-injective ring. The following conditions are equivalent:*
(1) *R is strongly regular;*
(2) *R is a right quasi-duo (resp. WRD) ring containing a nonsingular maximal right ideal;*
(3) *R is a right quasi-duo (resp. WRD) ring with $Z(R_R) = 0$;*
(4) *R is a right quasi-duo (resp. WRD) ring whose divisible and torsionfree right R-modules are GP-injective;*
(5) *R has a maximal right ideal M such that for any $y \in M$, there exists a central idempotent $e \in M$ and a right regular element $c \in R$ (i.e, $r(c) = 0$) such that $y = ce$;*
(6) *R is a ring with a reduced maximal right ideal;*
(7) *R is a reduced ring.*

Proof. (1)⇒(2) and (1)⇒(4) are clear.

(2)⇒(3): Let M be a nonsingular maximal right ideal. We claim $Z(R_R) = 0$. In fact, for any $a \in Z(R_R)$, $a \in J \subseteq M$ by [12, Corollary 2.3]. Thus $Z(R_R) = Z(R_R) \cap M = Z_r(M) = 0$.

(3)⇒(7): Assume that the condition (3) holds. Then R/J is reduced. By [12, Corollary 2.3], $J = Z(R_R) = 0$. Thus R is reduced.

(4)⇒(3): Let $0 \neq a \in Z(R_R)$ with $a^2 = 0$. By Lemma 2.4, aR is divisible and torsionfree. Therefore aR is GP-injective. By Lemma 2.5, $l_{aR}(r(a)) = aRa$. Note that $a \in l_{aR}(r(a))$, so $a = aba$ for some $b \in R$. Then $a = 0$ since $ba \in Z(R_R)$, a contradiction. This shows that $Z(R_R)$ is reduced, and so $Z(R_R) = 0$.

(1)⇒(5): By [4, Corollary 4.2], R is unit-regular. Let M be a maximal right ideal. For any $y \in M$, there exist unit element $c, d \in R$ such that $y = ydy$ and $d = dcd$. Note that $y = y^2d$ and $d = d^2c$, thus $y = ydc$. Let $e = yd$, then $e^2 = e \in M$ and $y = ec$. Since R is strongly regular, so e is central.

(5)$\Rightarrow$(6): If $b \in M$ with $b^2 = 0$, then $b = ce$, where e is a central idempotent in M and c is right regular. Then $r(b) = r(e) = l(e)$ and $b = be$, which implies $b = be = eb = 0$ because $b \in r(b)$. Hence M is reduced.

(6)$\Rightarrow$(7): Let M be a reduced maximal right ideal. If I is a nilpotent ideal of R, then $I \subseteq J \subseteq M$, and thus $I = 0$ because M is reduced, so R is semiprime. If M is essential, then there is $0 \neq r \in R$ such that $0 \neq ar \in M$ for any $0 \neq a \in R$, thus $0 \neq ara$ and $0 \neq a^2$ since M is reduced. This proves that R is reduced. If M is not essential, then $R = M \oplus L$. Thus $M = eR$, $U = (1-e)R$, where $e^2 = e \in R$ and U is a minimal right ideal of R. Note that $(1-e) \in l(eR) = l(M) \subseteq r(M) = r(eR)$ since M is reduced. So $eR(1-e) = 0$, and hence e is central since R is semiprime. Take $a \in R$ with $a^2 = 0$. Then $a = ec + (1-e)d$ for some $c,\ d \in R$, so $0 = a^2 = ec^2 + (1-e)d^2$, and therefore $d(1-e)d = 0$. If $(1-e)d \neq 0$, then $U = (1-e)dR$ by the minimality of U, which implies $d(1-e) = 0$. Hence $(1-e)d = 0$ since e is central, a contradiction. This gives that $a = ec \in M$. Then $a = 0$ since M is reduced, and so R is reduced.

(7)$\Rightarrow$(1): For any $0 \neq a \in R$, $a^2 \neq 0$ since R is reduced. Then there is a positive integer n with $a^{2n} \neq 0$ such that $lr(a^{2n}) = Ra^{2n} \oplus X$ for some $X \leq_R R$. But R is reduced, so $a^{2n-1} \in lr(a^{2n-1}) = lr(a^{2n})$. Hence $a^{2n} = ada^{2n}$ for some $d \in R$. Since R is reduced, we obtain $1 - ad \in l(a^{2n}) = l(a) = r(a)$. Thus $a = a^2d$, R is strongly regular. ■

By the proof "(1) $\Leftrightarrow$ (3)" of Theorem 2.6, the following result is immediate.

Corollary 2.7. *A ring R is regular if and only if R is a right AGP-injective ring whose divisible and torsionfree right R-modules are GP-injective.*

A ring Q is called a classical right quotient ring of R (see [5]) if (a) $R \subseteq Q$; (b) every non-zero-divisor of R is invertible in Q; (c) for any $q \in Q$, $q = ab^{-1}$, where $a,\ b \in R$ and b is a non-zero-divisor.

Theorem 2.8. *If R is a right AGP-injective ring and has classical right quotient ring Q, then the following conditions are equivalent:*
(1) *Q is strongly regular;*
(2) *R is reduced.*

Proof. (1)$\Rightarrow$(2): Obvious.

(2)$\Rightarrow$(1): For any $q = ab^{-1} \in Q$ with $q^2 = 0$, we see that $ab^{-1}ab^{-1} = 0$ and $ab^{-1}a = 0$. Since Q is a classical right quotient ring, there exist $c,\ d \in R$ such that $b^{-1}a = dc^{-1}$, so $ac = bd$ and $adc^{-1} = 0$. Then $dbd = dac$ and $ad = 0$, this implies $da = 0$ since R is reduced. Hence $dbd = 0$, $bd = 0$ since R is reduced. Thus $ac = 0$, $a = 0$, and hence $q = ab^{-1} = 0$. This shows that Q is reduced. By Theorem 1.6, R is regular. Take $q = ab^{-1} \in Q$ with $a,\ b \in R$. Then there exists $r \in R$ such that $a = ara = ab^{-1}bra = qbra$ since R is regular. This shows that $q = ab^{-1} = qbrab^{-1} = qbrq$, and $br \in R \subseteq Q$. Hence Q is strongly regular. ■

Let e be an idempotent element of R. If eRe is a local ring, then e is called a local idempotent. It is well known that local idempotents are primitive, but the converse is not true. For the integral ring $\mathbb{Z}$, 1 is a primitive idempotent, but it is not local idempotent since $\mathbb{Z}$ is not a local ring.

Recall that a ring R is called orthogonally finite if R has no infinite subsets consisting of orthogonal idempotents; R is called primitively finite if there exist finite orthogonal primitive idempotents $e_1, e_2 \dots, e_n$ such that $1 = e_1 + e_2 + \dots + e_n$. It is well known that R is semiperfect if and only if 1 is a sum of finite orthogonal local idempotents. Thus, every semiperfect ring is primitively finite, but the converse is not true as shown by the following example.

Example 1. There exists a primitively finite ring R which is not semiperfect. Let $R = \left\{ \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} \end{pmatrix} \mid a, b, c \in \mathbb{Z} \right\}$. Then $e = e_{11} + e_{22}$, where

$$e_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \;\; e_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

and e is a unit of $M_2(\mathbb{Z})$. Thus R is primitively finite since e_{11} and e_{22} are primitive idempotent elements of R. But $J = \begin{pmatrix} 0 & \mathbb{Z} \\ 0 & 0 \end{pmatrix}$, and $R/J \cong \mathbb{Z} \times \mathbb{Z}$, hence R is not semiperfect.

The following theorem extends the results in [11, Theorem 1.4] and [12, Theorem 2.16].

Theorem 2.9. *If R is a primitively finite right AGP-injective ring, then $R \cong R_1 \times R_2$, where R_1 is semisimple and every simple right ideal of R_2 is nilpotent.*

Proof. Since R is primitive finite, there exist orthogonal primitive idempotents $e_1, \dots, e_n$, such that $1 = e_1 + \dots + e_n$. We assume that there exists a natural number $m (1 \leq m \leq n)$ such that $e_i R$ is simple $(1 \leq i \leq m)$ and $e_j R$ is not simple $(m < j \leq n)$.

Claim (1) $e_i R e_j = 0,\ 1 \leq i \leq m,\ m < j \leq n$.

If not, then there exist $(0 \neq)\ a \in e_i R e_j$ and a nonzero right R-homomorphism

$$f : e_j R \to e_i R; e_j r \mapsto a e_j r.$$

Since $e_i R$ is simple, so f is an epimorphism. Since $e_i R$ is projective, so $e_j R = \mathrm{Ker} f \oplus T$, where $T \cong e_i R$. Hence $\mathrm{Ker} f = 0$ and $e_j R \cong e_i R$ since e_j is primitive, it is a contradiction.

Claim (2) $e_j R e_i = 0,\ \ 1 \leq i \leq m,\ m < j \leq n$.

If not, there exist $0 \neq b \in e_j R e_i$ and R-homomorphism

$$g : e_i R \to e_j R; e_i r \mapsto b e_i r.$$

Clearly, g is a nonzero homomorphism, so g is a monomorphism since $e_i R$ is simple. Hence $e_i R \cong g(e_i R)$. Since R is a right AGP-injective ring, by [12, Proposition 2.13], $g(e_i R)$ is a direct summand of R. Hence there exists $u^2 = u \in R$ such that $g(e_i R) = uR$, thus $e_j R = e_j R \cap R = e_j R \cap (uR \oplus (1-u)R) = uR \oplus (e_j R \cap (1-u)R)$, so $e_j R \cap (1-u)R = 0$ and $e_j R = uR = g(e_i R) \cong e_i R$ since e_j is primitive. This is a contradiction.

Claim (3) $R \cong R_1 \times R_2$, where R_1 is semisimple and every simple right ideal of R_2 is nilpotent.

Let $e = e_1 + \cdots + e_m$, then $e^2 = e \in R$ and $eR(1-e) = (1-e)Re = 0$ by the above proof. Hence e and $1-e$ are central idempotents of R, eR and $(1-e)R$ are two-sided ideals of R. Thus $R \cong R_1 \times R_2$, where $R_1 = eR$ and $R_2 = (1-e)R$. Obviously, R_1 is semisimple. We shall prove that every simple right ideal K of R_2 is nilpotent.

If not, then there exists a nonzero element $t \in K$ such that $tK \neq 0$, where K is a simple right ideal of R_2. Since K is simple, $K = tK$. Thus there exists $u \in K$ such that $tu = t$, and so $u^2 - u \in r(t) \cap K$. Hence $u^2 = u \in K$, $tK \neq 0$ and K is simple. So $uR_2 = K$. Since $(1-e)K = K(1-e) \neq 0$, there exists j $(m < j \leq n)$ such that $Ke_j = KRe_j \neq 0$. Let $0 \neq c \in Ke_j = uR_2e_j = uRe_j$, then there exists an R-homomorphism

$$h : e_jR \to uR; e_jr \mapsto ce_jr.$$

Since e_j is primitive and $uR = uR_2 = K$ is projective simple, $e_jR = K$. This contradicts the assumption that e_jR is not simple. Therefore $tK = 0$, and K is nilpotent. ■

3. Right *AGP*-Injective Rings with Right Chain Conditions

From [12, Example 1.5(2)], we know that right *AGP*-injective rings satisfying the a.c.c on right annihilators need not be regular. In this section, we characterize the self-injectivity of right *AGP*-injective rings.

Recall that a ring R is π-regular if, for any $a \in R$, there exists a positive integer m such that $a^m = a^m b a^m$ for some $b \in R$. Following [2], a ring R is called generalized π-regular if, for any $a \in R$, there exists a positive integer m such that $a^m = a^m ba$ for some $b \in R$. For convenience, a ring R is said to be left generalized π-regular if, for any $a \in R$, there exists a positive integer m such that $a^m = aba^m$ for some $b \in R$.

Definition 3.1. *A ring R is said to satisfy the a.c.c on the special right annihilators if, for any $0 \neq x \in R$, the chain $r(x) \subseteq r(x^2) \subseteq \cdots \subseteq r(x^n) \subseteq \cdots$ terminates.*

From [21, Theorem 1.5], it is easy to verify that if R is a right *AGP*-injective ring satisfying the a.c.c on the special right annihilators, then J is nilpotent. Now, we have the following result.

Proposition 3.2. *If R is a right AGP-injective ring satisfying the a.c.c on the special right annihilators, then R is left generalized π-regular.*

Proof. Let $0 \neq a \in J$, then there is a positive integer n such that $r(a^n) = r(a^{n+1})$ by hypothesis. If $a^n = 0$, we are done. If $0 \neq a^n$, then $0 \neq a^{n+1}$, and so there is a positive integer m such that $0 \neq a^{m(n+1)}$ and $lr(a^n) = lr(a^{m(n+1)}) = Ra^{m(n+1)} \oplus X$ with $X \leq_R R$. Thus $a^n = ra^{m(n+1)} + x$ with $r \in R$ and $x \in X$.

If $m = 1$, then $a^{n+1} = ara^{n+1}$. If $m > 1$, then $a^{m(n+1)} = a^{(m-1)(n+1)}ara^{m(n+1)}$. In all cases, R is left generalized π-regular. ∎

By Proposition 3.2 and [2, Theorem 2.2], we have the following corollary.

Corollary 3.3. *If R is a right AGP-injective ring satisfying the a.c.c on the special right annihilators and $N_1 = \{0 \neq a \in R \mid a^2 = 0\}$ is regular (every element of N_1 is von Neumann regular), then R is regular.*

The following proposition extends [18, Proposition 10].

Proposition 3.4. *Let R be a right AGP-injective ring satisfying the a.c.c on right annihilators. If K_1, K_2 are injective right R-modules, $g_1 : M_1 \to K_1$, $g_2 : M_2 \to K_2$ right R-monomorphisms of right R-modules M_1, M_2 into K_1, K_2 respectively, and $f : M_1 \to M_2$ a right R-homomorphism, then there exists a right R-homomorphism $h : K_1 \to K_2$ such that $hg_1 = g_2f$. If $g_i(r_{M_i}(J)) = r_{K_i}(J)$, $i = 1, 2$, and f is a monomorphism (resp. isomorphism), then h is a monomorphism (resp. isomorphism).*

Proof. There exists a right R-homomorphism $h : K_1 \to K_2$ such that $hg_1 = g_2f$ since K_2 is an injective right R-module. By [5, Proposition 3.31], $Z(R_R)$ is nilpotent, which shows that J is nilpotent by [12, Corollary 2.3]. The proof of [18, Proposition 10] can be used to prove the result. ∎

Corollary 3.5. *If R is a right AGP-injective ring satisfying the a.c.c on right annihilators, E, Q are injective right R-modules such that $r_E(J)$ is isomorphic to $r_Q(J)$ (as right R-modules), then $E_R \cong Q_R$.*

Following [19], a ring R is called right IF if every injective right R-module is flat, and R is called right MI if R contains an injective maximal right ideal. In [19], Yue Chi Ming gave an example, in which R is an MI-ring but not left self-injective, and proved that if R is a left MI-ring, then R is a left IF-ring satisfying the a.c.c on right annihilators if and only if R is QF. The result remains true if IF-rings are replaced by AGP-injective rings as shown by the following theorem.

Theorem 3.6. *Let R be a right AGP-injective and right MI-ring. Then R is right self-injective.*

Proof. Since R is a right MI-ring, there exists $e^2 = e \in R$ such that $R_R = M \oplus L$, where $M = eR$ is an injective maximal right ideal of R and $L = (1 - e)R$ is a minimal right ideal of R.

If $LM \neq 0$, we claim that L_R is injective, and hence $R_R = M \oplus L$ also is injective.

In fact, there exists $u \in L$ such that $uM \neq 0$, which implies that $L = uM$. Thus there is a right R-epimorphism $\varphi : M \to L$ with $\varphi(x) = ux$. Since $L = (1 - e)R$ is projective, there exists a right submodule T of M such that

$M = \mathrm{Ker}(\varphi) \oplus T$ with $T \cong L$. Hence $L \cong T$ is injective by the injectivity of M_R, so R is injective.

If $LM = 0$, then $M = r(L)$ is a two-sided ideal of R. Let $f = 1 - e$. Then for any $0 \neq u \in Rf$, $u = uf$ and $r(u) = M = r(f)$ by the maximality of M. The fact that R is right AGP-injective implies that there exists a positive integer n with $0 \neq u^n$ such that $lr(u^n) = Ru^n \oplus X_{u^n}$ for some left ideal X_{u^n} of R. Note that $M = r(u) = r(u^n)$ by the maximality of M, so $Rf = lr(f) = Ru^n \oplus X$. Therefore $u^n f = u^n = u^n r u^n$ for some $r \in R$. Thus $(f - ru^n) \in r(u^n) = r(f)$, which implies that $f = fru^n$ and $Rf \subseteq Ru$. This shows that $R(1-e)$ is a minimal left ideal of R. Consequently, Re is a maximal left ideal of R. Since $eR = M$ is a two-sided ideal of R, we see that $M = eR = Re$ by the maximality of Re. Then ${}_R(R/M) \cong R(1-e)$ is projective, and so it is flat. Let L be a right ideal of R and $f : L \to (R/M)_R$ a non-zero right R-homomorphism. Since ${}_R(R/M)$ is flat, we see that $LM = M \cap L$. If $L \subseteq M$, then $L = M \cap L = LM$, and hence $f(L) = f(LM) = f(L)M = 0$. This contradicts $f \neq 0$. Thus $L \nsubseteq M$, and so $L + M = R$ by the maximality of M. This shows that $1 = a + b$ for some $a \in L$ and some $b \in M$. Therefore for every $x \in L$, $x - ax = bx \in L \cap M = LM$, and hence $f(x - ax) \in f(LM) = f(L)M = 0$. This implies that $f(x) = f(a)x$ for all $x \in L$. Thus $L \cong (R/M)_R$ is injective, and so $R_R = M \oplus L$ is injective.

■

Corollary 3.7. *If R is a right MI ring, then R is QF if and only if R is a right AGP-injective ring satisfying the a.c.c on right annihilators.*

Proof. By Theorem 3.6, R is right self-injective. So R is QF by [6, Corollary 24.22]. ■

Corollary 3.8. *Let R be a ring with $Soc(R_R) \nsubseteq J$. The the following conditions are equivalent:*
(1) *R is right self-injective;*
(2) *R is right MI, right P-injective;*
(3) *R is right MI, right AP-injective;*
(4) *R is right MI, right GP-injective;*
(5) *R is right MI, right AGP-injective.*

Proof. (2)⇒ (3), (2)⇒ (4) and (4)⇒ (5) are clear.
(1)⇒ (2). Since $Soc(R_R) \nsubseteq J$, there exists a minimal right ideal M of R such that $M^2 \neq 0$. Thus there exists $e^2 = e \in R$ such that $M = eR$ by the minimality of M. But $R_R = (1-e)R \oplus eR$ is right self-injective, and so $(1-e)R$ is an injective maximal right ideal of R by the minimality of M.
(5)⇒ (1). By Theorem 3.6.
(3)⇒ (1). It follows from the proof of Theorem 3.6. ■

Recall that a ring R is right uniform if any two nonzero right ideals of R have nonzero intersection, equivalently, if $R \neq 0$ and every nonzero right ideal is essential in R. Yue Chi Ming [16] proved that any right Noetherian right uniform right GP-injective ring is right Artinian and local. Now we prove that the result is still true for AGP-injective rings.

Proposition 3.9. *Let R be a right uniform right AGP-injective ring. Then R is a local ring and $J = Z(R_R)$ is the unique maximal left (and right) ideal of R.*

Proof. If $Z(R_R) = 0$, then $r(a) = 0$ for any $0 \neq a \in R$ since R is right uniform. For any $0 \neq a \in R$, there is a positive integer n such that $0 \neq a^n$ and $lr(a^n) = Ra^n \oplus X_{a^n}$ for some left ideal X_{a^n} of R since R is right AGP-injective. Note that $r(a^n) = 0$, and so $R = lr(a^n) = Ra^n \oplus X_{a^n}$. Hence $1 = ra^n + x$ with $r \in R,\ x \in X_{a^n}$. This implies $a^n = a^n ra^n$, so $1 - ra^n \in r(a^n) = 0$ and $1 = ra^n$. Thus a is left invertible. Hence R is a division ring. If $Z(R_R) \neq 0$, then any proper left ideal I of R is contained in $Z(R_R)$. If not, there is $0 \neq a \in I \setminus Z(R_R)$. Then $r(a) = 0$. By the preceding proof, there is $c \in R$ such that $ca = 1$, which implies $I = R$. This is a contradiction. By [12, Corollary 2.3], $J = Z(R_R)$ is the unique maximal left (and right) ideal of R. ∎

The next corollary follows from [21, Corollary 1.2] and Proposition 3.9.

Corollary 3.10. *Any right Noetherian right uniform right AGP-injective ring is right Artinian and local.*

4. Right AGP-Injective Rings and PP Rings

We say that a ring R satisfies left (resp. right) $(*)$ if, for any $0 \neq a \in R$, there exists a positive integer n such that $0 \neq a^n$ and Ra^n (a^nR) is a projective R-module. Clearly, every right PP-ring satisfies right $(*)$.

Let us start this section with the following proposition.

Proposition 4.1. *If R is a ring satisfying right $(*)$, then $Z(R_R) = 0$, i.e., R is a right nonsingular ring.*

Proof. Suppose that $0 \neq a \in Z(R_R)$ with $a^2 = 0$. Then aR is projective, and so the short exact sequence $0 \to r(a) \to R \to aR \to 0$ splits, where $\varphi : R \to aR;\ r \mapsto ar$. This implies $R = r(a) \oplus L$ for some $L \leq R$. But $r(a) \neq R$ since $0 \neq a$, and hence $L \neq 0$, contradicting $a \in Z(R_R)$. Thus $Z(R_R)$ is reduced, $Z(R_R) = 0$. ∎

The following theorem extends [13, Theorem 3], [2, Theorem 2.9], [14, Theorem 2.3], and [21, Proposition 2.5].

Theorem 4.2. *A ring R is von Neumann regular if and only if R is a right AGP-injective ring satisfying right $(*)$.*

Proof. One direction is obvious. Conversely, suppose that R is a right AGP-injective ring satisfying right $(*)$. For any $0 \neq a \in R$, there is a positive integer m with $0 \neq a^m$ such that $lr(a^m) = Ra^m \oplus X_{a^m}$ for some left ideal X_{a^m} of R. Put $b = a^m$. Then there is a positive integer n such that $0 \neq b^n$ and b^nR is projective. Thus the short exact sequence $0 \to r(b^n) \to R \to b^nR \to 0$ splits. This implies that $r(b^n) = eR$ with $e^2 = e \in R$. Let $f = 1 - e$. Then $lr(b^n) = Rf$ and $f^2 = f \in R$, and so $b^n = b^n f$. Since $lr(b^n) \subseteq lr(b)$, $f = da^m + x$ for some $d \in R$ and some $x \in X_{a^m}$. Thus

$$b^n x = b^n f - b^n da^m = a^{mn} - a^{mn} da^m \in Ra^m \cap X_{a^m} = 0.$$

This shows that $a^{mn} = a^{mn} da^m$, and so R is generalized π-regular. If $0 \neq a$ with $a^2 = 0$, the argument above gives that a is a regular element. Thus, by [2, Theorem 2.2], R is a regular ring. ■

Corollary 4.3. [21, Proposition 2.5] *A ring R is von Neumann regular if and only if R is right PP and right AGP-injective.*

Corollary 4.4. *Suppose that R is a right PP-ring having a classical right quotient ring Q. Then Q is right AGP-injective if and only if Q is regular.*

Proof. By [8, Proposition 1], Q is a right PP ring. Thus, by Corollary 3.5, Q is regular. The converse is obvious. ■

Following [6, Definition 1.1], a ring R is called completely right p-injective (briefly, right *cp*-injective) if every ring homomorphic image of R is right P-injective.

Example 2.
(1) Let $R = \mathbb{Z}_{q^2}$, where $\mathbb{Z}_{q^2}$ is the ring of integers modulo q^2 and q is a prime number. By [7, Example 1.4(1)], R is a commutative *cp*-injective ring but it is not von Neumann regular. By [7, Proposition 1.7], R is strongly π-regular. Thus R is a GPP ring. Obviously, *cp*-injective rings are AGP-injective. This shows that R is a GPP and AGP-injective ring which is not regular. Thus, by Theorem 4.2, R is a GPP ring which is not a ring satisfying $(*)$.
(2) Let $K = \mathbb{Z}_2$ and $R = \begin{pmatrix} K & K \\ 0 & A \end{pmatrix}$ with

$$A = \{(a_1, a_2, \ldots, a_n, d, d, \ldots) \mid d, a_1, a_2, \ldots, a_n \in K, n \in \mathbb{N}\}.$$

Then $\mathbb{Z}_2$ is a left (and right) A-module by defining

$$A \times \mathbb{Z}_2 \longrightarrow \mathbb{Z}_2; (\alpha, k) \mapsto \alpha k \doteq dk$$

$$\mathbb{Z}_2 \times A \longrightarrow \mathbb{Z}_2; (k, \alpha) \mapsto k\alpha \doteq kd$$

By [2, Example 1], R is a left PP and right P-injective ring, so is a right AGP-injective ring satisfying left $(*)$. Note that R is not regular since $J \neq 0$, thus by Corollary 4.3, R is neither left AGP-injective nor right PP. By Theorem 4.2, R is not a ring satisfying right $(*)$.

Remark 1. Example 2 (1) above shows that GPP rings need not be rings satisfying $(*)$, and hence GPP rings need not be PP rings. Example 2 (2) shows that the condition "R is right AGP-injective" cannot be replaced by "R is left AGP-injective" in Theorem 3.2, and hence the notion of AGP-injective rings is not left-right symmetric. Likewise, the notions of PP rings and rings satisfying $(*)$ are not left-right symmetric. However, we do not know if every ring satisfying right $(*)$ is a right PP ring.

Proposition 3.6. *If R is a right GPP and right AGP-injective ring, then R is π-regular.*

Proof. For any $0 \neq a \in R$, there exists a positive integer m such that $a^m R$ is projective by the assumption. Thus the short exact sequence

$$0 \to r(a^m) \to R \to a^m R \to 0$$

splits, and so there exists $T \cong a^m R$ such that $R_R = r(a^m) \oplus T$. By [11, Proposition 2.13(2)], $a^m R$ is a direct summand of R_R. Thus R is π-regular. ∎

Acknowledgements. The authors would like to thank the referee for his useful suggestions. This work is supported by the Doctorate Foundation of China Education Ministry (20020284009).

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