Regularity of AP-Injective Rings

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Abstract. A ring \(R\) is called right AGP-injective if, for any \(0 \neq a \in R\), there exists \(n > 0\) such that \(a^n \neq 0\) and \(Ra^n\) is a direct summand of \(lr(a^n)\). In this paper some conditions which are sufficient or equivalent for a right AGP-injective ring to be von Neumann regular (right self-injective, semisimple) are provided. It is shown that a ring \(R\) is von Neumann regular if and only if \(R\) is right AGP-injective and for any \(0 \neq a \in R\) there exists a positive integer \(n\) with \(0 \neq a^n\) such that \(a^n R\) is a projective right \(R\)-module if and only if \(R\) is a right AGP-injective ring whose divisible and torsionfree right \(R\)-modules are GP-injective. We also show that if \(R\) is a primitively finite right AGP-injective ring, then \(R \cong R_1 \times R_2\), where \(R_1\) is semisimple and every simple right ideal of \(R_2\) is nilpotent. In addition, it is proven that if \(R\) is a right MI and right AGP-injective ring satisfying the a.c.c on right annihilators, then \(R\) is quasi-Frobenius.

1. Introduction

Throughout, \(R\) is an associative ring with identity and all modules are unitary. The Jacobson radical, left (right) singular ideal and right socle of \(R\) are denoted by \(J, Z(RR)\) and \(Soc(R_R)\), respectively. For a subset \(X\) of \(R\), \(r(X)\) (resp. \(l(X)\)) is reserved for the right (resp. left) annihilator of \(X\) in \(R\), and we write \(r(x)\) (resp. \(l(x)\)) for \(r(\{x\})\) (resp. \(l(\{x\})\)) when \(x \in R\). A right \(R\)-module \(M\) is called principally injective (\(P\)-injective) \([11]\) if every \(R\)-homomorphism from a principal right ideal \(aR\) to \(M\) extends to one from \(RR\) to \(M\). \(M\) is said to be generalized principally injective (\(GP\)-injective) \([10]\) if, for any \(0 \neq a \in R\), there exists a positive integer \(n\) with \(a^n \neq 0\) such that any \(R\)-homomorphism from
A ring $R$ is called right $P$-injective (resp. $GP$-injective) if the right $R$-module $R_R$ is $P$-injective (resp. $GP$-injective). Following [12], a ring $R$ is called right almost principally injective ($AP$-injective) if every principal left ideal is a direct summand of a left annihilator, and the ring $R$ is called right almost generalized principally injective ($AGP$-injective) if, for any $0 \neq a \in R$, there exists $n > 0$ such that $0 \neq a^n$ and $Ra^n$ is a direct summand of $lr(a^n)$. The detailed discussion of right $P$-injective and right $GP$-injective rings can be found in [2-3, 7, 10-12, 17-21]. Clearly, $P$-injective rings are $AP$-injective and $GP$-injective rings are $AGP$-injective. But there exists a right $AP$-injective ring which is not right $GP$-injective [12, Example 1.5]. Recently, it was noted that $GP$-injective rings need not be $P$-injective (see [3]). Several results which are known for right $P$-injective (resp. right $GP$-injective) rings were shown to hold for right $AP$-injective (resp. right $AGP$-injective) rings in [12, 21]. In this paper, we discuss the regularity and the self-injectivity of right $AGP$-injective rings.

In Sec. 2, we characterize the regularity of $AGP$-injective rings. It is shown that $R$ is von Neumann regular if and only if it is a right $AGP$-injective ring whose divisible and torsionfree right $R$-modules are $GP$-injective; $R$ is strongly regular if and only if it is a right quasi-duo (resp. $WRD$) and right $AGP$-injective ring whose divisible and torsionfree right $R$-modules are $GP$-injective if and only if it is a right $AGP$-injective ring with a reduced maximal right ideal. An example is given to show that there is a primitively finite ring which is not semiperfect. We also prove that if $R$ is a primitive finite right $AGP$-injective ring, then $R \cong R_1 \times R_2$, where $R_1$ is semisimple and every simple right ideal of $R_2$ is nilpotent, which extends the results in [11, Theorem 1.4] and [12, Theorem 2.16].

It is well known that a ring $R$ is quasi-Frobenius ($QF$) if and only if it is right self-injective and left (or right) Noetherian. In Sec. 3, we prove that, if $R$ is a right $MI$ ring, then $R$ is right self-injective if and only if it is right ($AP$-injective if and only if it is right ($AGP$-injective. In particular, if $R$ is a right $MI$ ring, then $R$ is $QF$ if and only if $R$ is a right $AGP$-injective ring satisfying the a.m.c on right annihilators.

Recall that a ring $R$ is called left (resp. right) $PP$ [9] if, for any $a \in R$, $Ra$ ($aR$) is a projective $R$-module; $R$ is said to be a left (resp. right) $GPP$ ring [9] if, for any $a \in R$, there exists a positive integer $m$ such that $Ra^m$ ($a^nR$) is a projective $R$-module. The classes of these rings were studied by many authors, for example, Hirano, Xue and Zhou (see [8, 13, 21]). In Sec. 3, we consider the class of rings satisfying ($\ast$) ($R$ is called a ring satisfying left (resp. right) ($\ast$) if, for any $0 \neq a \in R$, there exists a positive integer $n$ such that $0 \neq a^n$ and $Ra^n$ (resp. $a^nR$) is a projective $R$-module). Clearly, $PP$ rings are rings satisfying ($\ast$) and rings satisfying ($\ast$) are $GPP$ rings. We prove that a ring $R$ is von Neumann regular if and only if $R$ is a right $AGP$-injective ring satisfying right ($\ast$), which extends Zhou [21, Proposition 2.5]. An example shows that $GPP$ rings need not be rings satisfying ($\ast$), and none of the notions of $PP$ rings, $AGP$-injective rings and rings satisfying ($\ast$) is left-right symmetric.
2. Nonsingular Right AGP-Injective Rings

Let $R$ be a ring. Recall that $R$ is called (von Neumann) regular if for every $x \in R$, there exists some $y \in R$ such that $x = yx$; $R$ is called strongly regular if for every $x \in R$, there exists some $y \in R$ such that $x = x^2y$; $R$ is called right nonsingular if $Z(R_R) = 0$; $R$ is called reduced if it contains no nonzero nilpotent elements; $R$ is called semiprime if it contains no nonzero nilpotent ideal (equivalently, for any $a \in R$, $RaRa = 0$ implies $Ra = 0$).

In this section, we characterize the regularity of AGP-injective rings.

**Lemma 2.1.** Let $c \in C(R)$, where $C(R)$ is the center of $R$. If $c$ is regular in $R$, then $c$ is regular in $C(R)$.

**Proof.** Let $c = cdc$ with $d \in R$. Put $u = dcd$. Then $c = cec = uc^2$. We claim that $u \in C(R)$. In fact, for any $x \in R$, $ux - xu \in r(c^2) = r(c)$, so $c^2(xd^2 - d^2x) = c(xu - ux) = 0$, which implies $xd^2 - d^2x \in r(c^2) = r(c)$. Thus $ux - xu = xd^2 - cd^2x = c(xd^2 - d^2x) = 0$. This completes the proof.

**Proposition 2.2.** If $R$ is a right nonsingular right AGP-injective ring, then the center $C(R)$ of $R$ is regular.

**Proof.** By hypothesis, $R$ has a regular maximal right quotient ring $S$ (see [5, Corollary 2.31]). Consequently, the center $C(S)$ of $S$ is regular by Lemma 1.1. For any $0 \neq a \in C(R)$, there exists $s \in S$ such that $a = aas = a^2s = sa^2$. Thus $r(a^n) = r(a) = l(a) = l(a^n)$ for any positive integer $n$. We claim that $a$ is regular in $C(R)$. Note that $a^2 \neq 0$, so there is a positive integer $m$ with $a^{2m} \neq 0$ such that $lr(a^{2m}) = Ra^{2m} \cap X_{2m}$ for some left ideal $X_{2m}$ of $R$ since $R$ is right AGP-injective. Thus $a^{2m-1} \in lr(a^{2m-1}) = lr(a^{2m})$, and so $a^{2m-1} = da^{2m} + x$ for some $d \in R$ and some $x \in X_{2m}$. Then $a^{2m} = ada^{2m} + ax$ and $ax \in Ra^{2m} \cap X_{2m} = 0$. Hence $a^{2m} = ada^{2m}$. Therefore $1 - ad \in l(a^{2m}) = l(a)$, and so $a = ada$. This implies that $C(R)$ is regular by Lemma 2.1.

**Proposition 2.3.** If $R$ is a semiprime right AGP-injective ring, then the center $C(R)$ of $R$ is regular.

**Proof.** For any $0 \neq c \in C(R)$, $Rc \cap l(c) = 0$ since $R$ is semiprime. Therefore, $l(c^m) = l(c) = r(c) = r(c^m)$ for any positive integer $m$. Note that $c^2 \neq 0$ because $Rc \cap l(c) = 0$. As in the proof of Proposition 2.2, $C(R)$ is regular.

Recall that (1) $R$ is called right (resp. left) duo [1] if every right (resp. left) ideal is a two-sided ideal; (2) $R$ is called right quasi-duo [1] if every maximal right ideal is a two-sided ideal; (3) $R$ is said to be weakly right duo (abbreviated WRD) [15] if for any $a \in R$, there exists a positive integer $n$ such that $a^nR$ is a two-sided ideal. Right quasi-duo (resp. WRD) rings are non-trivial generalizations of right duo rings (see [1, 15]). Note that if $R$ is right quasi-duo (resp. WRD) then $R/J$ is reduced (see [17]).

A right $R$-module $M$ is called torsionfree if, for any $0 \neq m \in M$, $mc \neq 0$ for every non-zero-divisor $c$ of $R$. The module $M$ is divisible if $M = Mc$ for each
non-zero-divisor $c$ of $R$.

**Lemma 2.4.** If $R$ is a right AGP-injective ring, then any non-zero-divisor is invertible, thus any right (resp. left) $R$-module is divisible and torsionfree.

**Proof.** Let $a \in R$ be a non-zero-divisor, then $r(a) = l(a) = 0$. Hence $r(a^n) = r(a) = l(a) = l(a^n) = 0$ for any positive integer $n$. Since $R$ is right AGP-injective, there exists a positive integer $n$ with $a^n \neq 0$ such that $R = lr(a^n) = Ra^n \oplus X_{a^n}$ for some left ideal $X_{a^n}$ of $R$. Thus $1 = ra^n + x$ with $r \in R$, $x \in X$, and so $a^n x = a^n - a^n ra^n \in X_{a^n} \cap Ra^n = 0$, which implies that $a^n = a^n ra^n$. If $n = 1$, then $1 = ra = ar$. If $n > 1$, then $1 = (ra^{n-1})a = a(a^{n-1}r)$. It follows that $a$ is invertible, and hence any right (resp. left) $R$-module is divisible and torsionfree.

Let $M$ be a right $R$-module. We write $l_M(X) = \{m \in M \mid ma = 0 \text{ for any } a \in X\}$, where $X$ is a subset of $R$, and write $Z_r(M) = \{m \in M \mid mK = 0 \text{ for some essential right ideal } K \text{ of } R\}$.

**Lemma 2.5.** [20, Proposition 1] Let $M$ be a right $R$-module. Then $M$ is GP-injective if and only if, for any $0 \neq a \in R$, there exists a positive integer $n$ such that $a^n \neq 0$ and $l_M(r(a^n)) = Ma^n$.

**Theorem 2.6.** Let $R$ be a right AGP-injective ring. The following conditions are equivalent:

1. $R$ is strongly regular;
2. $R$ is a right quasi-duo (resp. WRD) ring containing a nonsingular maximal right ideal;
3. $R$ is a right quasi-duo (resp. WRD) ring with $Z(R_R) = 0$;
4. $R$ is a right quasi-duo (resp. WRD) ring whose divisible and torsionfree right $R$-modules are GP-injective;
5. $R$ has a maximal right ideal $M$ such that for any $y \in M$, there exists a central idempotent $c \in M$ and a right regular element $c \in R$ (i.e., $r(c) = 0$) such that $y = ce$;
6. $R$ is a ring with a reduced maximal right ideal;
7. $R$ is a reduced ring.

**Proof.** (1)⇒(2) and (1)⇒(4) are clear.

(2)⇒(3): Let $M$ be a nonsingular maximal right ideal. We claim $Z(R_R) = 0$. In fact, for any $a \in Z(R_R)$, $a \in J \subseteq M$ by [12, Corollary 2.3]. Thus $Z(R_R) = Z(R_R) \cap M = Z_r(M) = 0$.

(3)⇒(7): Assume that the condition (3) holds. Then $R/J$ is reduced. By [12, Corollary 2.3], $J = Z(R_R) = 0$. Thus $R$ is reduced.

(4)⇒(3): Let $0 \neq a \in Z(R_R)$ with $a^2 = 0$. By Lemma 2.4, $aR$ is divisible and torsionfree. Therefore $aR$ is GP-injective. By Lemma 2.5, $l_{aR}(r(a)) = aRa$. Note that $a \in l_{aR}(r(a))$, so $a = aba$ for some $b \in R$. Then $a = 0$ since $ba \in Z(R_R)$, a contradiction. This shows that $Z(R_R)$ is reduced, and so $Z(R_R) = 0$.

(1)⇒(5): By [4, Corollary 4.2], $R$ is unit-regular. Let $M$ be a maximal right
ideal. For any \( y \in M \), there exist unit element \( c, d \in R \) such that \( y = y dy \) and \( d = dc d \). Note that \( y = y^2 d \) and \( d = d^2 c \), thus \( y = y dc \). Let \( e = yd \), then \( e^2 = e \in M \) and \( y = ec \). Since \( R \) is strongly regular, so \( e \) is central.

(5) \( \Rightarrow \) (6): If \( b \in M \) with \( b^2 = 0 \), then \( b = ce \), where \( e \) is a central idempotent in \( M \) and \( c \) is right regular. Then \( r(b) = r(e) = l(e) \) and \( b = be = eb = 0 \) because \( b \in r(b) \). Hence \( M \) is reduced.

(6) \( \Rightarrow \) (7): Let \( M \) be a reduced maximal right ideal. If \( I \) is a nilpotent ideal of \( R \), then \( I \subseteq J \subseteq M \), and thus \( I = 0 \) because \( M \) is reduced, so \( R \) is semiprime. If \( M \) is essential, then there is \( 0 \neq r \in R \) such that \( 0 \neq ar \in M \) for any \( 0 \neq a \in R \), thus \( 0 \neq ara \) and \( 0 \neq a^2 \) since \( M \) is reduced. This proves that \( R \) is reduced. If \( M \) is not essential, then \( R = M \oplus L \). Thus \( M = eR, U = (1 - e)R \), where \( e = e \in R \) and \( U \) is a minimal right ideal of \( R \). Note that \( (1 - e) \in l(eR) = l(M) \subseteq r(M) = eR(e) \) since \( M \) is reduced. So \( eR(1 - e) = 0 \), and hence \( e \) is central since \( R \) is semiprime. Take \( a \in R \) with \( a^2 = 0 \). Then \( a = ec + (1 - e)d \) for some \( c, d \in R \), so \( 0 = a^2 = ec^2 + (1 - e)d^2 \), and therefore \( d(1 - e)d = 0 \). If \( (1 - e)d \neq 0 \), then \( U = (1 - e)dR \) by the minimality of \( U \), which implies \( d(1 - e)d = 0 \). Hence \( (1 - e)d = 0 \) since \( e \) is central, a contradiction. This gives that \( a = ec \in M \). Then \( a = 0 \) since \( M \) is reduced, and so \( R \) is reduced.

(7) \( \Rightarrow \) (1): For any \( 0 \neq a \in R \), \( a^2 \neq 0 \) since \( R \) is reduced. Then there is a positive integer \( n \) with \( a^{2^n} \neq 0 \) such that \( br(a^{2^n}) = Ra^{2^n} \oplus X \) for some \( X \subseteq R \). But \( R \) is reduced, so \( a^{2^n-1} \in br(a^{2^n}) = br(a^{2^n-1}) \) for some \( d \in R \). Since \( R \) is reduced, we obtain \( 1 - ad \in l(a^{2^n}) = l(a) = r(a) \). Thus \( a = a^2 d, R \) is strongly regular.

By the proof “(1) \( \Leftrightarrow \) (3)” of Theorem 2.6, the following result is immediate.

**Corollary 2.7.** A ring \( R \) is regular if and only if \( R \) is a right AGP-injective ring whose divisible and torsionfree right \( R \)-modules are GP-injective.

A ring \( Q \) is called a classical right quotient ring of \( R \) (see [5]) if (a) \( R \subseteq Q \); (b) every non-zero-divisor of \( R \) is invertible in \( Q \); (c) for any \( q \in Q, q = ab^{-1}, a \) and \( b \in R \) and \( b \) is a non-zero-divisor.

**Theorem 2.8.** If \( R \) is a right AGP-injective ring and has classical right quotient ring \( Q \), then the following conditions are equivalent:

1. \( Q \) is strongly regular;
2. \( R \) is reduced.

**Proof.** (1) \( \Rightarrow \) (2): Obvious.

(2) \( \Rightarrow \) (1): For any \( q = ab^{-1} \in Q \) with \( q^2 = 0 \), we see that \( ab^{-1}ab^{-1} = 0 \) and \( ab^{-1}a = 0 \). Since \( Q \) is a classical right quotient ring, there exist \( c, d \in R \) such that \( b^{-1}a = dc^{-1} \), so \( ac = bd \) and \( ade^{-1} = 0 \). Then \( dcd = dac \) and \( ad = 0 \), this implies \( da = 0 \) since \( R \) is reduced. Hence \( dcd = 0, bd = 0 \) since \( R \) is reduced. Thus \( ac = 0, a = 0 \), and hence \( q = ab^{-1} = 0 \). This shows that \( Q \) is reduced. By Theorem 1.6, \( R \) is regular. Take \( q = ab^{-1} \in Q \) with \( a, b \in R \). Then there exists \( r \in R \) such that \( a = ara = ab^{-1}bra = qbra \) since \( R \) is regular. This shows that \( q = ab^{-1} = qbrab^{-1} = qbrq, \) and \( br \in R \subseteq Q \). Hence \( Q \) is strongly regular. \( \blacksquare \)
Let \( e \) be an idempotent element of \( R \). If \( eRe \) is a local ring, then \( e \) is called a local idempotent. It is well known that local idempotents are primitive, but the converse is not true. For the integral ring \( \mathbb{Z} \), 1 is a primitive idempotent, but it is not local idempotent since \( \mathbb{Z} \) is not a local ring.

Recall that a ring \( R \) is called orthogonally finite if \( R \) has no infinite subsets consisting of orthogonal idempotents; \( R \) is called primitively finite if there exist finite orthogonal primitive idempotents \( e_1, e_2, \ldots, e_n \) such that \( 1 = e_1 + e_2 + \cdots + e_n \). It is well known that \( R \) is semiperfect if and only if 1 is a sum of finite orthogonal local idempotents. Thus, every semiperfect ring is primitively finite, but the converse is not true as shown by the following example.

**Example 1.** There exists a primitively finite ring \( R \) which is not semiperfect.

Let \( R = \left\{ \begin{pmatrix} a & c \\ 0 & b \end{pmatrix} \mid a, b, c \in \mathbb{Z} \right\} \). Then \( e = e_{11} + e_{22} \), where

\[
e_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad e_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}
\]

and \( e \) is a unit of \( M_2(\mathbb{Z}) \). Thus \( R \) is primitively finite since \( e_{11} \) and \( e_{22} \) are primitive idempotent elements of \( R \). But \( J = \begin{pmatrix} 0 & \mathbb{Z} \\ 0 & 0 \end{pmatrix} \), and \( R/J \cong \mathbb{Z} \times \mathbb{Z} \), hence \( R \) is not semiperfect.

The following theorem extends the results in [11, Theorem 1.4] and [12, Theorem 2.16].

**Theorem 2.9.** If \( R \) is a primitively finite right AGP-injective ring, then \( R \cong R_1 \times R_2 \), where \( R_1 \) is semisimple and every simple right ideal of \( R_2 \) is nilpotent.

**Proof.** Since \( R \) is primitive finite, there exist orthogonal primitive idempotents \( e_1, \ldots, e_n \), such that \( 1 = e_1 + \cdots + e_n \). We assume that there exists a natural number \( m(1 \leq m \leq n) \) such that \( e_i R \) is simple \( (1 \leq i \leq m) \) and \( e_j R \) is not simple \( (m < j \leq n) \).

Claim (1) \( e_i Re_j = 0, 1 \leq i \leq m, m < j \leq n \).

- If not, then there exist \((0 \neq) a \in e_i Re_j\) and a nonzero right \( R \)-homomorphism

\[
f : e_jR \rightarrow e_iR; \quad e_jr \mapsto ae_ir.
\]

Since \( e_jR \) is simple, so \( f \) is an epimorphism. Since \( e_iR \) is projective, so \( e_jR = \text{Ker} f \oplus T \), where \( T \cong e_iR \). Hence \( \text{Ker} f = 0 \) and \( e_jR \cong e_iR \) since \( e_j \) is primitive, it is a contradiction.

Claim (2) \( e_iRe_j = 0, 1 \leq i \leq m, m < j \leq n \).

- If not, there exist \( 0 \neq b \in e_jRe_i\) and \( R \)-homomorphism

\[
g : e_iR \rightarrow e_jR; \quad e_ir \mapsto be_jr.
\]

Clearly, \( g \) is a nonzero homomorphism, so \( g \) is a monomorphism since \( e_jR \) is simple. Hence \( e_iR \cong g(e_jR) \). Since \( R \) is a right AGP-injective ring, by [12, Proposition 2.13], \( g(e_jR) \) is a direct summand of \( R \). Hence there exists \( u^2 = u \in R \) such that \( g(e_jR) = uR \), thus \( e_jR = e_jR \cap R = e_jR \cap (uR \oplus (1 - u)R) = uR \oplus (e_jR \cap (1 - u)R) \), so \( e_jR \cap (1 - u)R = 0 \) and \( e_jR = uR = g(e_iR) \cong e_iR \) since \( e_j \) is primitive. This is a contradiction.
Claim (3) \( R \cong R_1 \times R_2 \), where \( R_1 \) is semisimple and every simple right ideal of \( R_2 \) is nilpotent.

Let \( e = e_1 + \cdots + e_m \), then \( e^2 = e \in R \) and \( eR(1-e) = (1-e)Re = 0 \) by the above proof. Hence \( e \) and \( 1-e \) are central idempotents of \( R \). \( eR \) and \( (1-e)R \) are two-sided ideals of \( R \). Thus \( R \cong R_1 \times R_2 \), where \( R_1 = eR \) and \( R_2 = (1-e)R \). Obviously, \( R_1 \) is semisimple. We shall prove that every simple right ideal \( K \) of \( R_2 \) is nilpotent.

If not, then there exists a nonzero element \( t \in K \) such that \( tK \neq 0 \), where \( K \) is a simple right ideal of \( R_2 \). Since \( K \) is simple, \( K = tK \). Thus there exists \( u \in K \) such that \( tu = t \), and so \( u^2 - u \in t(t) \cap K \). Hence \( u^2 = u \in K \), \( tK \neq 0 \) and \( K \) is simple. So \( uR_2 = K \). Since \( (1-e)K = K(1-e) \neq 0 \), there exists \( j (m < j \leq n) \) such that \( Ke_j = KRe_j \neq 0 \). Let \( 0 \neq e \in Ke_j = uR_2e_j = uRe_j \), then there exists an \( R \)-homomorphism

\[
h : e_jR \rightarrow uR; e_jr \mapsto e_jr.
\]

Since \( e_j \) is primitive and \( uR = uR_2 = K \) is projective simple, \( e_jR = K \). This contradicts the assumption that \( e_jR \) is not simple. Therefore \( tK = 0 \), and \( K \) is nilpotent.

### 3. Right AGP-Injective Rings with Right Chain Conditions

From [12, Example 1.5(2)], we know that right AGP-injective rings satisfying the a.c.c on right annihilators need not be regular. In this section, we characterize the self-injectivity of right AGP-injective rings.

Recall that a ring \( R \) is \( \pi \)-regular if, for any \( a \in R \), there exists a positive integer \( m \) such that \( a^m = abma \) for some \( b \in R \). Following [2], a ring \( R \) is called generalized \( \pi \)-regular if, for any \( a \in R \), there exists a positive integer \( m \) such that \( a^m = a^mba \) for some \( b \in R \). For convenience, a ring \( R \) is said to be left generalized \( \pi \)-regular if, for any \( a \in R \), there exists a positive integer \( m \) such that \( a^m = aba^m \) for some \( b \in R \).

**Definition 3.1.** A ring \( R \) is said to satisfy the a.c.c on the special right annihilators if, for any \( 0 \neq x \in R \), the chain \( r(x) \subseteq r(x^2) \subseteq \cdots \subseteq r(x^n) \subseteq \cdots \) terminates.

From [21, Theorem 1.5], it is easy to verify that if \( R \) is a right AGP-injective ring satisfying the a.c.c on the special right annihilators, then \( J \) is nilpotent. Now, we have the following result.

**Proposition 3.2.** If \( R \) is a right AGP-injective ring satisfying the a.c.c on the special right annihilators, then \( R \) is left generalized \( \pi \)-regular.

**Proof.** Let \( 0 \neq a \in J \), then there is a positive integer \( n \) such that \( r(a^n) = r(a^{n+1}) \) by hypothesis. If \( a^n = 0 \), we are done. If \( 0 \neq a^n \), then \( 0 \neq a^{n+1} \), and so there is a positive integer \( m \) such that \( 0 \neq a^{m(n+1)} \) and \( lr(a^n) = lr(a^{m(n+1)}) = Ra^{m(n+1)} \oplus X \) with \( X \leq R \). Thus \( a^n = ra^{m(n+1)} + x \) with \( r \in R \) and \( x \in X \).
If $m = 1$, then $a^{n+1} = a ra^{n+1}$. If $m > 1$, then $a^{m(n+1)} = a^{(m-1)(n+1)} ara^{m(n+1)}$. In all cases, $R$ is left generalized $\pi$-regular. 

By Proposition 3.2 and [2, Theorem 2.2], we have the following corollary.

**Corollary 3.3.** If $R$ is a right AGP-injective ring satisfying the a.c.c on the special right annihilators and $N_1 = \{ 0 \neq a \in R \mid a^2 = 0 \}$ is regular (every element of $N_1$ is von Neumann regular), then $R$ is regular.

The following proposition extends [18, Proposition 10].

**Proposition 3.4.** Let $R$ be a right AGP-injective ring satisfying the a.c.c on right annihilators. If $K_1, K_2$ are injective right $R$-modules, $g_1 : M_1 \rightarrow K_1$, $g_2 : M_2 \rightarrow K_2$ right $R$-monomorphisms of right $R$-modules $M_1, M_2$ into $K_1, K_2$ respectively, and $f : M_1 \rightarrow M_2$ a right $R$-homomorphism, then there exists a right $R$-homomorphism $h : K_1 \rightarrow K_2$ such that $hg_1 = g_2 f$. If $g_i(r_{M_i}(J)) = r_{K_i}(J)$, $i = 1, 2$, and $f$ is a monomorphism (resp. isomorphism), then $h$ is a monomorphism (resp. isomorphism).

*Proof.* There exists a right $R$-homomorphism $h : K_1 \rightarrow K_2$ such that $hg_1 = g_2 f$ since $K_2$ is an injective right $R$-module. By [5, Proposition 3.31], $Z(R_R)$ is nilpotent, which shows that $J$ is nilpotent by [12, Corollary 2.3]. The proof of [18, Proposition 10] can be used to prove the result. 

**Corollary 3.5.** If $R$ is a right AGP-injective ring satisfying the a.c.c on right annihilators, $E, Q$ are injective right $R$-modules such that $r_E(J)$ is isomorphic to $r_Q(J)$ (as right $R$-modules), then $E_R \cong Q_R$.

Following [19], a ring $R$ is called right $IF$ if every injective right $R$-module is flat, and $R$ is called right $MI$ if $R$ contains an injective maximal right ideal. In [19], Yue Chi Ming gave an example, in which $R$ is an $MI$-ring but not left self-injective, and proved that if $R$ is a left $MI$-ring, then $R$ is a left $IF$-ring satisfying the a.c.c on right annihilators if and only if $R$ is $QF$. The result remains true if $IF$-rings are replaced by AGP-injective rings as shown by the following theorem.

**Theorem 3.6.** Let $R$ be a right AGP-injective and right $MI$-ring. Then $R$ is right self-injective.

*Proof.* Since $R$ is a right $MI$-ring, there exists $e^2 = e \in R$ such that $R_R = M \oplus L$, where $M = eR$ is an injective maximal right ideal of $R$ and $L = (1 - e)R$ is a minimal right ideal of $R$.

If $LM \neq 0$, we claim that $L_R$ is injective, and hence $R_R = M \oplus L$ also is injective.

In fact, there exists $u \in L$ such that $uM \neq 0$, which implies that $L = uM$. Thus there is a right $R$-epimorphism $\varphi : M \rightarrow L$ with $\varphi(x) = ux$. Since $L = (1 - e)R$ is projective, there exists a right submodule $T$ of $M$ such that
Let $M = \text{Ker}(\varphi) \oplus T$ with $T \cong L$. Hence $L \cong T$ is injective by the injectivity of $M_R$, so $R$ is injective.

If $LM = 0$, then $M = r(L)$ is a two-sided ideal of $R$. Let $f = 1 - e$. Then for any $0 \neq u \in Rf$, $u = uf$ and $r(u) = M = r(f)$ by the maximality of $M$. The fact that $R$ is right AGP-injective implies that there exists a positive integer $n$ with $0 \neq u^n$ such that $Ir(u^n) = Ru^n \oplus X_u^n$ for some left ideal $X_u^n$ of $R$. Note that $M = r(u) = r(u^n)$ by the maximality of $M$, so $Rf = Ir(f) = Ru^n \oplus X_u^n$. Therefore $u^n f = u^n = u^n ru^n$ for some $r \in R$. Thus $(f - ru^n) \in r(u^n) = r(f)$, which implies that $f = fru^n$ and $Rf \subseteq Ru$. This shows that $R(1 - e)$ is a minimal left ideal of $R$. Consequently, $Re$ is a maximal left ideal of $R$. Since $eR = M$ is a two-sided ideal of $R$, we see that $M = eR = Re$ by the maximality of $Re$. Then $R(R(M) = R(1 - e)$ is projective, and so it is flat. Let $L$ be a right ideal of $R$ and $f : L \to (R/M)_R$ a non-zero right $R$-homomorphism. Since $R(R(M)/M)$ is flat, we see that $LM = M \cap L$. If $L \subseteq M$, then $L = M \cap L = LM$, and hence $f(L) = f(LM) = f(L)M = 0$. This contradicts $f \neq 0$. Thus $L \not\subseteq M$, and so $L + M = R$ by the maximality of $M$. This shows that $1 = a + b$ for some $a \in L$ and some $b \in M$. Therefore for every $x \in L$, $x - ax = bx \in L \cap M = LM$, and hence $f(x - ax) \in f(LM) = f(L)M = 0$. This implies that $f(x) = f(ax)x$ for all $x \in L$. Thus $L \cong (R/M)_R$ is injective, and so $R_R = M \oplus L$ is injective.

Corollary 3.7. If $R$ is a right $MI$ ring, then $R$ is QF if and only if $R$ is a right AGP-injective ring satisfying the a.c.c on right annihilators.

Proof. By Theorem 3.6, $R$ is right self-injective. So $R$ is QF by [6, Corollary 24.22].

Corollary 3.8. Let $R$ be a ring with $\text{Soc}(R_R) \not\subseteq J$. The following conditions are equivalent:

1. $R$ is right self-injective;
2. $R$ is right $MI$, right $P$-injective;
3. $R$ is right $MI$, right $AP$-injective;
4. $R$ is right $MI$, right $GP$-injective;
5. $R$ is right $MI$, right $AGP$-injective.

Proof. (2)⇒(3), (2)⇒(4) and (4)⇒(5) are clear.

1)⇒(2). Since $\text{Soc}(R_R) \not\subseteq J$, there exists a minimal right ideal $M$ of $R$ such that $M^2 \neq 0$. Thus there exists $c^2 = c \in R$ such that $M = eR$ by the minimality of $M$. But $R_e = (1 - e)R \oplus eR$ is right self-injective, and so $(1 - e)R$ is an injective maximal right ideal of $R$ by the minimality of $M$.

5)⇒(1). By Theorem 3.6.

3)⇒(1). It follows from the proof of Theorem 3.6.

Recall that a ring $R$ is right uniform if any two nonzero right ideals of $R$ have nonzero intersection, equivalently, if $R \neq 0$ and every nonzero right ideal is essential in $R$. Yue Chi Ming [16] proved that any right Noetherian right uniform right $GP$-injective ring is right Artinian and local. Now we prove that
the result is still true for AGP-injective rings.

**Proposition 3.9.** Let $R$ be a right uniform right AGP-injective ring. Then $R$ is a local ring and $J = Z(R_R)$ is the unique maximal left (and right) ideal of $R$.

**Proof.** If $Z(R_R) = 0$, then $r(a) = 0$ for any $0 \neq a \in R$ since $R$ is right uniform. For any $0 \neq a \in R$, there is a positive integer $n$ such that $0 \neq a^n$ and $lr(a^n) = Ra^n \oplus X_a^n$ for some left ideal $X_a^n$ of $R$ since $R$ is right AGP-injective. Note that $r(a^n) = 0$, and so $R = lr(a^n) = Ra^n \oplus X_a^n$. Hence $1 = ra^n + x$ with $r \in R$, $x \in X_a^n$. This implies $a^n = a^n ra^n$, so $1 - ra^n \in r(a^n) = 0$ and $1 = ra^n$. Thus $a$ is left invertible. Hence $R$ is a division ring. If $Z(R_R) \neq 0$, then any proper left ideal $I$ of $R$ is contained in $Z(R_R)$. If not, there is $0 \neq a \in I \setminus Z(R_R)$. Then $r(a) = 0$. By the preceding proof, there is $c \in R$ such that $ca = 1$, which implies $I = R$. This is a contradiction. By [12, Corollary 2.3], $J = Z(R_R)$ is the unique maximal left (and right) ideal of $R$. ■

The next corollary follows from [21, Corollary 1.2] and Proposition 3.9.

**Corollary 3.10.** Any right Noetherian right uniform right AGP-injective ring is right Artinian and local.

4. Right AGP-Injective Rings and PP Rings

We say that a ring $R$ satisfies left (resp. right) $(\ast)$ if, for any $0 \neq a \in R$, there exists a positive integer $n$ such that $0 \neq a^n$ and $lr(a^n) = Ra^n \oplus X_a^n$ is a projective $R$-module. Clearly, every right PP-ring satisfies right $(\ast)$.

Let us start this section with the following proposition.

**Proposition 4.1.** If $R$ is a ring satisfying right $(\ast)$, then $Z(R_R) = 0$, i.e., $R$ is a right nonsingular ring.

**Proof.** Suppose that $0 \neq a \in Z(R_R)$ with $a^2 = 0$. Then $aR$ is projective, and so the short exact sequence $0 \to r(a) \to R \to aR \to 0$ splits, where $\varphi : R \to aR; r \mapsto ar$. This implies $R = r(a) \oplus L$ for some $L \subseteq R$. But $r(a) \neq R$ since $0 \neq a$, and hence $L \neq 0$, contradicting $a \in Z(R_R)$. Thus $Z(R_R)$ is reduced, $Z(R_R) = 0$. ■

The following theorem extends [13, Theorem 3], [2, Theorem 2.9], [14, Theorem 2.3], and [21, Proposition 2.5].

**Theorem 4.2.** A ring $R$ is von Neumann regular if and only if $R$ is a right AGP-injective ring satisfying right $(\ast)$.

**Proof.** One direction is obvious. Conversely, suppose that $R$ is a right AGP-injective ring satisfying right $(\ast)$. For any $0 \neq a \in R$, there is a positive integer $m$ with $0 \neq a^m$ such that $lr(a^m) = Ra^m \oplus X_{a^m}$ for some left ideal $X_{a^m}$ of $R$. Put $b = a^m$. Then there is a positive integer $n$ such that $0 \neq b^n$ and $b^nR$ is
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projective. Thus the short exact sequence $0 \rightarrow r(b^n) \rightarrow R \rightarrow b^n R \rightarrow 0$ splits. This implies that $r(b^n) = eR$ with $e^2 = e \in R$. Let $f = 1 - e$. Then $br(b^n) = Rf$ and $f^2 = f \in R$, and so $b^n = b^n f$. Since $br(b^n) \subseteq br(b)$, $f = da^m + x$ for some $d \in R$ and some $x \in X_{a^m}$. Thus

$$b^n x = b^n f - b^n da^m = a^{mn} - a^{mn} da^m \in Ra^m \cap X_{a^m} = 0.$$  

This shows that $a^{mn} = a^{mn} da^m$, and so $R$ is generalized $\pi$-regular. If $0 \neq a$ with $a^2 = 0$, the argument above gives that $a$ is a regular element. Thus, by [2, Theorem 2.2], $R$ is a regular ring. $\blacksquare$

Corollary 4.3. [21, Proposition 2.5] A ring $R$ is von Neumann regular if and only if $R$ is right $PP$ and right AGP-injective.

Corollary 4.4. Suppose that $R$ is a right $PP$-ring having a classical right quotient ring $Q$. Then $Q$ is right AGP-injective if and only if $Q$ is regular.

Proof. By [8, Proposition 1], $Q$ is a right $PP$ ring. Thus, by Corollary 3.5, $Q$ is regular. The converse is obvious. $\blacksquare$

Following [6, Definition 1.1], a ring $R$ is called completely right $p$-injective (briefly, right $cp$-injective) if every ring homomorphic image of $R$ is right $P$-injective.

Example 2. (1) Let $R = \mathbb{Z}_q^2$, where $\mathbb{Z}_q$ is the ring of integers modulo $q^2$ and $q$ is a prime number. By [7, Example 1A(1)], $R$ is a commutative $cp$-injective ring but it is not von Neumann regular. By [7, Proposition 1.7], $R$ is strongly $\pi$-regular. Thus $R$ is a GPP ring. Obviously, $cp$-injective rings are AGP-injective. This shows that $R$ is a GPP and AGP-injective ring which is not regular. Thus, by Theorem 4.2, $R$ is a GPP ring which is not a ring satisfying $(\ast)$.

(2) Let $K = \mathbb{Z}_2$ and $R = \left( \begin{array}{cc} K & K \\ 0 & A \end{array} \right)$ with

$$A = \{(a_1, a_2, \ldots, a_n, d, d, \ldots) \mid d, a_1, a_2, \ldots, a_n \in K, n \in \mathbb{N} \}.$$  

Then $\mathbb{Z}_2$ is a left (and right) $A$-module by defining

$$A \times \mathbb{Z}_2 \longrightarrow \mathbb{Z}_2; (a, k) \mapsto ak \equiv dk$$

$$\mathbb{Z}_2 \times A \longrightarrow \mathbb{Z}_2; (k, a) \mapsto ka \equiv kd$$

By [2, Example 1], $R$ is a left $PP$ and right $P$-injective ring, so is a right AGP-injective ring satisfying left $(\ast)$. Note that $R$ is not regular since $J \neq 0$, thus by Corollary 4.3, $R$ is neither left AGP-injective nor right $PP$. By Theorem 4.2, $R$ is not a ring satisfying right $(\ast)$.

Remark 1. Example 2 (1) above shows that GPP rings need not be rings satisfying $(\ast)$, and hence GPP rings need not be $PP$ rings. Example 2 (2) shows that the condition “$R$ is right AGP-injective” cannot be replaced by “$R$ is left AGP-injective” in Theorem 3.2, and hence the notion of AGP-injective rings
is not left-right symmetric. Likewise, the notions of PP rings and rings satisfying $(*)$ are not left-right symmetric. However, we do not know if every ring satisfying right $(*)$ is a right PP ring.

**Proposition 3.6.** If $R$ is a right GPP and right AGP-injective ring, then $R$ is $\pi$-regular.

**Proof.** For any $0 \neq a \in R$, there exists a positive integer $m$ such that $a^m R$ is projective by the assumption. Thus the short exact sequence

$$0 \to r(a^m) \to R \to a^m R \to 0$$

splits, and so there exists $T \cong a^m R$ such that $R_R = r(a^m) \oplus T$. By [11, Proposition 2.13(2)], $a^m R$ is a direct summand of $R_R$. Thus $R$ is $\pi$-regular. ■

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**References**