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Minty Vector Variational Inequality, Efficiency and Proper Efficiency

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Abstract. Minty variational inequalities have proven to characterize minimizers of scalar optimization problems. Similar results have been proved in the vector case, but it is shown that these are not equivalent to the scalar ones. Hence different (and stronger) concepts of solution of a Minty inequality are presented and their relations with efficiency and proper efficiency are investigated.

Since scalarization results arise throughout the research, a closing section is devoted to this problem.

1. Introduction

In the scalar case several results are known which state relations between solutions of a Stampacchia or Minty variational inequality and the underlying minimization problem. To summarize briefly, we shall recall that if $x^* \in K \subseteq \mathbb{R}^n$, K convex and nonempty, is a solution of the primitive minimization problem:

$P(f, K) \qquad \min f(x),\ x \in K$

for some $f : \mathbb{R}^n \to \mathbb{R}$, differentiable on an open set containing K, then x^* solves the following Stampacchia type variational inequality:

$VI(f', K) \qquad \langle f'(x^*), y - x^* \rangle \geq 0, \quad \forall y \in K$

where f' is the gradient of f and $\langle \cdot, \cdot \rangle$ is the inner product on $\mathbb{R}^n$.

Conversely, if $x^* \in K$ solves the following Minty variational inequality:

$MVI(f', K)$ $\qquad\qquad \langle f'(y), y - x^* \rangle \geq 0, \quad \forall y \in K,$

then x^* solves also $P(f, K)$ (see e.g. [11]).

The relations between variational inequalities and optimization are even deeper, as it has recently been shown (see [4]) that if x^* solves $MVI(f', K)$, then f is monotone along rays starting at x^* and, furthermore the existence of solutions of $MVI(f', K)$ can be related to well-posedness of $P(f, K)$.

In the eighties, variational inequalities (of Stampacchia type and later of Minty type as well) have been proposed in a vector valued formulation, in which a matrix valued function $F : \mathbb{R}^n \to \mathbb{R}^{l\times n}$ is involved. When F is the Jacobian matrix of some vector valued function $f : \mathbb{R}^n \to \mathbb{R}^l$, then Stampacchia and Minty vector variational inequalities have been related to the vector minimization problem:

$VP(f, K)$ $\qquad\qquad v - \min_K f(x),\ x \in K$

with $f : \mathbb{R}^n \to \mathbb{R}^l$. As it is known (see [9, 14, 19]) several notions have been introduced to define solutions of $VP(f, K)$. In [6] solutions of vector variational inequalities are linked to weak efficiency and in [2, 7, 21] also to efficiency.

Sec. 2 summarizes these results and underlines some gaps between those concerning Minty vector inequality and Minty scalar inequality. A different approach is presented to fill these gaps and Sec. 3 extends it to the characterization of proper efficient solutions of $VP(f, K)$. Finally Sec. 4 is devoted to some remarks on scalarization of a vector variational inequality.

2. Vector Variational Inequalities and Efficiency

We state first the main results in the scalar case, briefly recalled in the Introduction. Throughout the paper K is a nonempty convex subset of $\mathbb{R}^n$ and we will denote the inner product of two vectors λ and $v \in \mathbb{R}^n$, both with $\langle \lambda, v \rangle$ and with $\lambda^\top v$.

Proposition 1. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be differentiable on an open set containing the subset $K \subseteq \mathbb{R}^n$. If $x^* \in K$ solves $P(f, K)$, then x^* solves $VI(f', K)$. Moreover, if f is convex, the converse holds true.*

Proposition 2. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be differentiable on an open set containing the subset $K \subseteq \mathbb{R}^n$. If $x^* \in K$ solves $MVI(f', K)$, then x^* solves $P(f, K)$. Moreover, if f is convex, the converse holds true.*

Note that, in both propositions, the convexity of f is necessary only to prove one of the implications.

In the following we consider a real vector space $\mathbb{R}^l$ endowed with the order induced by a cone C, which is assumed to be pointed, closed, convex, and with nonempty interior, in order to have the order reflexive, antisymmetric and transitive. For a set $A \subseteq \mathbb{R}^l$ its complement is denoted by A^c, its convex hull by $\operatorname{conv} A$, the spanning cone by $\operatorname{co} A$, the convex spanning cone by $\operatorname{cone} A$, the closure by $\mathrm{cl}\, A$ and the interior by $\mathrm{int}\, A$. Moreover the polar cone of A is the set $A^* := \{\lambda \in \mathbb{R}^l \,|\, \langle \lambda, a\rangle \geq 0,\ \forall a \in A\}$ and the strict polar is denoted by $A^o := \{\lambda \in \mathbb{R}^l \,|\, \langle \lambda, a\rangle > 0,\ \forall a \in A\}$. If C is a convex cone which satisfies the previous assumptions, it is known that $C^0 = \mathrm{int}\, C$. Let M be any of the cones C^c, $C \setminus \{0\}$ and $\mathrm{int}\, C$. The vector optimization problem (see e.g. [19]) corresponding to M, where $f : \mathbb{R}^n \to \mathbb{R}^l$, is written as:

$$VP(f,K) \qquad\qquad v - \min_M f(x),\ x \in K.$$

This amounts to find a point $x^* \in K$ (called the optimal solution), such that there is no $y \in K$ with $f(y) \in f(x^*) - M$. The optimal solutions of the vector problem corresponding to $-C^c$ (respectively, $C\setminus\{0\}$ and $\mathrm{int}\, C$) are called ideal solutions (respectively, efficient solutions and weakly efficient solutions).

The former concepts of solutions have been strenghtened by several definitions of proper efficiency (for a survey and some relations among them see [9]). For any $l \times n$ matrix A and $x \in \mathbb{R}^n$ denote by $\langle A, x\rangle_l$ the vector of l inner products $\langle a^i, x\rangle$, where a^i, $i = 1, \ldots, l$ are the rows of A.

Definition 1. *Let f be differentiable and f' be its Jacobian matrix. A point $x^* \in K$ is properly efficient in the sense of Hurwicz for f over K when:*

$$\mathrm{cl}\,\mathrm{cone}\big(f(K) - f(x^*)\big) \cap \big(-C\big) = \{0\}.$$

In the sequel $PE^{Hu}(f,K)$ denotes the sets properly efficient points in the sense of Hurwicz of f over K.

It is classical the following result which states a geometrical characterization of Hurwicz solutions:

Proposition 3. *Let $f : K \subseteq \mathbb{R}^n \to \mathbb{R}^l$ be given. Then $x^* \in PE^{Hu}(f,K)$ if and only if there exists $\lambda \in \mathrm{int} C^*$ such that $\lambda^\top f(y) - \lambda^\top f(x^*) \geq 0\ \forall y \in K$.*

A useful property, when dealing with both optimization problems and variational inequalities, is convexity. We recall the following basic definitions for vector valued functions (see e.g. [14]):

Definition 2. *The function $f : K \subseteq \mathbb{R}^n \to \mathbb{R}^l$ is said to be C–convex (respectively $\mathrm{int} C$–convex) when:*

$$f(tx + (1-t)y) - \big[tf(x) + (1-t)f(y)\big] \in -C, \quad \forall x, y \in K, \forall t \in [0,1].$$
$$\Big(f(tx + (1-t)y) - \big[tf(x) + (1-t)f(y)\big] \in -\mathrm{int} C, \quad \forall x, y \in K, \forall t \in [0,1]\Big)$$

If f is differentiable, then the previous definitions are equivalent to require that:

$$f(y) - f(x) - \langle \nabla f(x), y - x\rangle_l \in C, \quad \forall x, y \in K.$$
$$\Big(f(y) - f(x) - \langle \nabla f(x), y - x\rangle_l \in \mathrm{int} C, \quad \forall x, y \in K\Big)$$

Definition 3. *Let $F : \mathbb{R}^n \to \mathbb{R}^l n$ be given. We say that F is C–monotone (respectively $intC$–monotone) over K, when:*

$$\langle F(y) - F(x), y - x\rangle_l \in C, \quad \forall x, y \in K$$
$$\big(\langle F(y) - F(x), y - x\rangle_l \in \ intC, \quad \forall x, y \in K\big)$$

It is known the following result (see e.g. [1]).

Proposition 4. *Let f be differentiable and f' denote its Jacobian matrix. Then f is C–convex (respectively $intC$–convex) if and only if f' is C–monotone (respectively $intC$–monotone)*

The vector valued formulations of $VI(F, K)$ and $MVI(F, K)$ have been introduced in [6, 7], respectively. Here we make use of the following sets:

$$\Omega(x) := \big\{u \in \mathbb{R}^l \mid u = \langle F(x), y - x\rangle_l,\ y \in K\big\},$$
$$\Theta(x) := \big\{w \in \mathbb{R}^l \mid w = \langle F(y), y - x\rangle_l,\ y \in K\big\}.$$

Definition 4.

i) *A vector $x^* \in K$ is a solution of a strong vector variational inequality of Stampacchia type when*:

$VVI^s(F, K)$ $$\Omega(x^*) \cap (-C) = 0.$$

ii) *A vector $x^* \in K$ is a solution of a weak vector variational inequality of Stampacchia type when:*

$VVI(F, K)$ $$\Omega(x^*) \cap (-\mathrm{int}\, C) = \emptyset.$$

Definition 5.

i) *A vector $x^* \in K$ is a solution of a strong vector variational inequality of Minty type when:*

$MVVI^s(F, K)$ $$\Theta(x^*) \cap (-C) = 0.$$

ii) *A vector $x^* \in K$ is a solution of a weak vector variational inequality of Minty type when:*

$MVVI(F, K)$ $$\Theta(x^*) \cap (-\mathrm{int}\, C) = \emptyset.$$

Clearly (see [8, 21]) any strong solution is a weak solution, but the converse does not necessarily hold true.

The following results (see [6, 7, 12]) are vector extensions of Propositions 1 and 2.

Proposition 5. *Let $f : \mathbb{R}^n \to \mathbb{R}^l$ be differentiable on an open set containing K.*

i) *If x^* is a weakly efficient solution of $VP(f, K)$, then it solves also $VVI(f', K)$.*

ii) *If f is C–convex and x^* is a solution of $VVI(f', K)$, then it is a weakly efficient solution of $VP(f, K)$.*

Proposition 6. *Let $C = \mathbb{R}^l_+$. If f is C–convex and differentiable on an open set containing K, then $x^* \in K$ is a weakly efficient solution of $VP(f, K)$ if and only if it is a solution of $MVVI(f', K)$.*

Remark 1. One can easily check that when $l = 1$, Proposition 5 reduces to Proposition 1. However in Proposition 6, the assumption of C-convexity is essential both for the necessary and sufficient condition. This implies that Proposition 6 does not collapse onto Proposition 2, since there the convexity assumption was required only to state the necessary condition.

Some refinements of the relations between the solutions of $VVI(f', K)$ and those of $VP(f, K)$ are given in [3]. In this paper we focus on Minty vector variational inequalities.

First we show that Proposition 6 cannot be improved, without changing Definition 5.

Example 1. Let $C = \mathbb{R}^2_+$, $K := \left[-\frac{2}{\pi}, 0\right]$ and consider a function $f : \mathbb{R} \to \mathbb{R}^2$, $f(x) := \begin{bmatrix} f_1(x) \\ f_2(x) \end{bmatrix}$, defined as follows. We set:

$$f_1(x) = \begin{cases} x^2 \sin \dfrac{1}{x} - x^2, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

and observe that $-2x^2 \leq f_1(x) \leq 0$, $\forall x \in K$ and f_1 is differentiable on K; its graph is plotted in Fig. 1. Function f_1 has a countable number of local minimizers and of local maximizers over K. The local maximizers of f_1 are the points $y_k = -\frac{1}{\frac{\pi}{2}+2k\pi}$, $k = 0, 1, \dots$ and $f_1(y_k) = 0$. If we denote by x_k, $k = 0, 1, \dots$ the local minimizers of f over K, we have $y_k < x_k < y_{k+1}$, $\forall k = 0, 1, \dots$.

The function f_2 is defined on K as:

$$f_2(x) = \begin{cases} -\frac{f_1(x_k)}{2}\left[\cos\left(\frac{\pi x}{x_k - y_k} + \frac{\pi(x_k - 2y_k)}{x_k - y_k}\right) - 1\right], & x \in [y_k, x_k) \\ -\frac{f_1(x_{k+1})}{2}\left[\cos\left(\frac{\pi x}{y_{k+1} - x_k} + \frac{\pi(2y_{k+1} - 3x_k)}{y_{k+1} - x_k}\right) - 1\right], & x \in [x_k, y_{k+1}) \\ 0, & x = 0 \end{cases}$$

for $k = 0, 1, \dots$. It is easily seen that also f_2 is differentiable on K and clearly f is not C-convex.

The points $x \in [-\frac{2}{\pi}, x_0]$ are (weakly) efficient, while the other points in K are not efficient. In particular, $x^* = 0$ is an ideal maximal point (i.e. $f(x) - f(x^*) \in \mathbb{R}^2_-$, $\forall x \in K$). Anyway, it is easy to see that any point of K is a solution of $MVVI(f', K)$.

To fill this gap we suggest (as in [8], but there for Stampacchia type inequalities) to strengthen the definition of solution of a Minty type vector variational inequality in the following way:

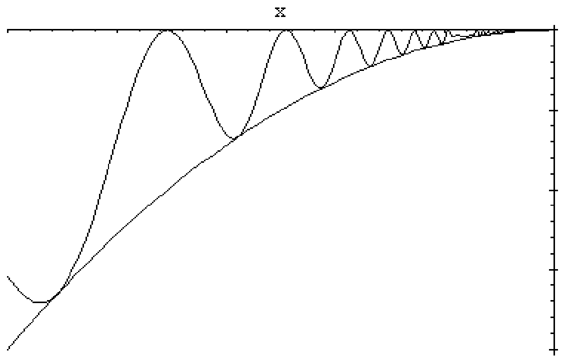

Figure 1. $f_1(x) = -x^2 sin\frac{1}{x} - x^2$.

Definition 6. *A vector $x^* \in K$ is a (weak) solution of a convexified Minty vector variational inequality when:*

$$CMVVI(F,K) \qquad \operatorname{conv}\Theta(x^*) \cap \big(-\operatorname{int} C\big) = \emptyset.$$

Remark 2.
i) Clearly, if $l = 1$ Definition 6 is equivalent to say that x^* solves $MVI(F,K)$.
ii) If $l \geq 2$, it follows from the definitions that, if $x^* \in K$ solves $CMVVI(F,K)$ then it solves also $MVVI(F,K)$. The converse is not always true, as it is shown in the following example.

Example 2. Let $l = 2$, $C = \mathbb{R}^2_+$, $F : \mathbb{R} \to \mathbb{R}^2$, with $F(x) = \begin{bmatrix} 1 \\ 1/(x-1) \end{bmatrix}$ and $K = [-1/2, 1/2]$. It is easy to check that $x^* = 0$ solves $MVVI(F,K)$, since $\Omega(0) \cap (-\operatorname{int}\mathbb{R}^2_+) = \emptyset$. However it is easy to see that conv $\Theta(0) \cap (-\operatorname{int}\mathbb{R}^2_+) \neq \emptyset$.

By means of this stronger variational inequality, the following result holds true (see e.g. [5]).

Theorem 1. *Let $f : \mathbb{R}^n \to \mathbb{R}^l$ be differentiable on an open set containing K and $x^* \in K$ be a solution of $CMVVI(f', K)$. Then x^* is a weakly efficient solution of $VP(f,K)$.*

The converse needs C-convexity of f to hold. This result has been proved in [5], but here we give a shorter proof, based on a scalarization result.

Lemma 1. *A vector $x^* \in K$ solves $CMVVI(F,K)$ if and only if there exists a nonzero vector $\lambda \in C^*$, such that x^* is a solution of the following scalar Minty variational inequality:*

$$MVI(\lambda^\top F, K) \qquad \langle \lambda^\top F(y), y - x^* \rangle \geq 0, \quad \forall y \in K.$$

Proof. Let $x^* \in K$ solve $MVI(\lambda^\top F, K)$ for some nonzero $\lambda \in C^*$. We have

$\langle \lambda, w \rangle < 0$, $\forall w \in -\text{int}\, C$, while $\langle \lambda, w \rangle \geq 0$, $\forall w \in \Theta(x^*)$. It follows easily that:

$$\Theta(x^*) \subseteq \text{ conv } \ \Theta(x^*) \subseteq \{w \in \mathbb{R}^l | \langle \lambda, w \rangle \geq 0\},$$

while:

$$-\text{int}\, C \subseteq \{w \in \mathbb{R}^l | \langle \lambda, w \rangle < 0\}$$

and so conv $\Theta(x^*) \cap -\text{int}\, C = \emptyset$.

Conversely, assume that $x^* \in K$ solves $CMVVI(F, K)$, which means that conv $\Theta(x^*)$ and $-\text{int}\, C$ are two disjoint convex sets. By classical separation arguments the thesis follows easily. ■

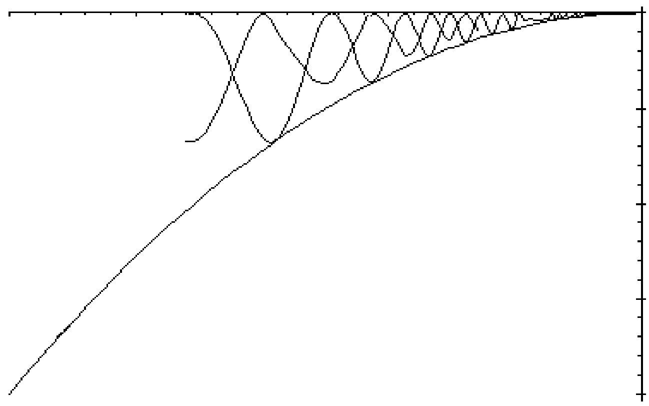

Figure 2. $f_1(x)$ *and* $f_2(x)$

Theorem 2. *Let f be C-convex. If $x^* \in K$ is a weakly efficient solution of $VP(f, K)$, then x^* solves $CMVVI(f', K)$.*

Proof. We claim that there exists a vector $\lambda \in C^*$ such that x^* solves $MVI(\lambda^\top F, K)$.

It is known that, under the C-convexity assumption, any weak efficient point can be written as the solution of a suitable scalarized minimum problem, i.e. there exist $\lambda \in C^*$ such that:

$$\lambda^\top f(x^*) \leq \lambda^\top f(y), \quad \forall y \in K.$$

Hence x^* solves a (scalar) Stampacchia variational inequality defined by the function $(\lambda^\top f)'(x) = \lambda^\top f'(x)$:

$$\langle \lambda^\top f'(x^*), y - x^* \rangle \geq 0, \quad \forall y \in K$$

Since f is C-convex, $\lambda^\top f : K \subseteq \mathbb{R}^n \to \mathbb{R}$ is convex and hence $\lambda^\top f'$ is a monotone map, that is:

$$\langle \lambda^\top f'(y) - \lambda^\top f'(x^*), y - x^* \rangle \geq 0, \text{ i.e.}$$
$$\langle \lambda^\top f'(y), y - x^* \rangle \geq \langle \lambda^\top f'(x^*), y - x^* \rangle \geq 0$$

Finally Lemma [1] applies to get the thesis. ∎

Remark 3. When $l = 1$, Theorems 1 and 2 reduce to Proposition 2, as it should be expected.

We recall that a map $F : \mathbb{R}^n \to \mathbb{R}^{l \times n}$ is hemycontinuous at x^* when its restriction along every ray with origin at x^* is continuous. When this property holds at any point x^*, then we say that F is hemycontinuous.

The following result, proved in [5], allows to extend Proposition 6 to any ordering cone C (pointed, closed, convex and with nonempty interior).

Theorem 3. *Let $F : \mathbb{R}^n \to \mathbb{R}^{l \times n}$ be hemycontinuous and C-monotone. Then, any $x^* \in K$ which solves $MVVI(F, K)$ is a solution of $CMVVI(F, K)$.*

Corollary 1. *Let $f : \mathbb{R}^n \to \mathbb{R}^l$ be a C-convex function whose Jacobian is a hemycontinuous map. Then the conclusions of Proposition* 6 *hold whatever the cone C (closed, pointed, convex and with nonempty interior).*

Remark 4. Observe that the hemycontinuity hypothesis on the Jacobian of f, is not actually additional with respect to Proposition 6 since when f is $\mathbb{R}^l_+$-convex and differentiable, then its Jacobian is necessarily hemycontinuous [18].

3. Proper Efficiency

We now wish to present a solution concept of a Minty vector variational inequality, stronger then $CMVVI(F, K)$, which is a sufficient condition for proper efficiency of the primitive multiobjective problem.

Definition 7. *Let F be a function from $\mathbb{R}^n$ to $\mathbb{R}^l n$. A vector $x^* \in K$ is a proper solution of a convexified Minty vector variational inequality when:*

$$CMVVI^P(F, K) \qquad \operatorname{cl} \operatorname{cone} \Theta(x^*) \cap (-C) = \{0\}.$$

Clearly if x^* solves $CMVVI^P(F, K)$, then x^* solves also $CMVVI(F, K)$, since $\operatorname{conv}\Theta(x^*) \subseteq \operatorname{cl} cone\Theta(x^*)$. Hence it also follows that x^* is a solution of $MVVI(F, K)$. The converse is not always true, as it can be easily seen:

Example 3. Let $l = 2$, $K := [-1, 1]$ and $F(x) := \begin{bmatrix} 1 \\ 2x \end{bmatrix}$. Clearly, $x^* = 0$ satisfies $CMVVI(F, K)$, but, since $\operatorname{cl} \operatorname{cone} \Theta(0) = \{(x, y) \in \mathbb{R}^2 \mid y \geq 0\}$, the same x^* is not a solution of $CMVVI^P(F, K)$.

Remark 5. Note that the function F involved in the previous example can be easily related to the primitive function $f(x) = \begin{bmatrix} x \\ x^2 \end{bmatrix}$. It is classical that $x^* = 0$ is an efficient solution of $VP(f, K)$ but not properly efficient.

The following result links the solutions of $CMVVI^P(F,K)$ to the properly efficient solutions of $VP(F,K)$.

Theorem 4. *Let $f : \mathbb{R}^n \to \mathbb{R}^l$ be differentiable and $f' = F$ be its Jacobian. If x^* solves $CMVVI^P(f',K)$, then $x^* \in PE^{Hu}(f,K)$.*

Proof. Let x^* be a solution of $CMVVI^P(f',K)$. By Taylor's formula for vector valued functions, we have that $\forall y \in K$:

$$f(y) - f(x^*) \in \operatorname{cl}\operatorname{conv}\{\langle f'(ty + (1-t)x^*), y - x^*\rangle_l,\ t \in (0,1)\}.$$

Let $\Phi_y(x^*) := \operatorname{cl}\operatorname{conv}\{\langle f'(ty+(1-t)x^*), y-x^*\rangle_l,\ t \in (0,1)\}$, by Charatheodory Theorem, $\gamma \in \Phi_y(x^*)$ if and only if there exist sequences $\{t_i^k\}_{k\geq 0} \in (0,1)$, $i = 1,\dots,l+1$ and $\{\lambda_i^k\}_{k\geq 0} \in [0,1]$, $i,\dots,l+1$, with $\sum_{i=1}^{l+1}\lambda_i^k = 1$, $\forall k$, such that:

$$\gamma = \lim_{k\to+\infty}\sum_{i=1}^{l+1}\lambda_i^k\langle f'(t_i^k y + (1-t_i^k)x^*), y - x^*\rangle_l.$$

It easily follows that:

$$\begin{aligned}\gamma &= \lim_{k\to+\infty}\sum_{i=1}^{l+1}\frac{\lambda_i^k}{t_i^k}\langle f'(t_i^k y + (1-t_i^k)x^*), t_i^k(y - x^*)\rangle_l \\ &= \lim_{k\to+\infty}\sum_{i=1}^{l+1}\frac{\lambda_i^k}{t_i^k}\langle f'(\xi_i^k), \xi_i^k - x^*\rangle_l,\end{aligned}$$

where $\xi_i^k = t_i^k y + (1-t_i^k)x^* \in K$.

We claim that $\gamma \in \operatorname{clcone}\Theta(x^*)$. In fact, for each i and k we have $\frac{1}{t_i^k}\langle f'(\xi_i^k), \xi_i^k - x^*\rangle_l \in \operatorname{co}\Theta(x^*)$ and hence $\sum_{i=1}^{l+1}\frac{\lambda_i^k}{t_i^k}\langle f'(\xi_i^k), \xi_i^k - x^*\rangle_l \in$ cone $\Theta(x^*)$, $\forall k$, from which the assertion follows. Therefore we have shown:

$$f(y) - f(x^*) \in \operatorname{cl}\operatorname{cone}\Theta(x^*), \quad \forall y \in K.$$

If we denote by $f(K)$ the image of K through the function f it follows that:

$$\operatorname{cl} cone\big\{f(K) - f(x^*)\big\} \subseteq \operatorname{cl}\operatorname{cone}\Theta(x^*)$$

and hence $\operatorname{cl}cone\big\{f(K) - f(x^*)\big\} \cap \big(-C\big) = \big\{0\big\}$. This completes the proof. ■

The converse of the previous theorem is not true without additional assumptions.

Example 4. Let $f : \mathbb{R} \to \mathbb{R}^2$ be defined as $f(x) = \begin{bmatrix} x \\ -xe^x \end{bmatrix}$. It is clear that f is differentiable over $K := [-1,1]$ and $f'(x) = \begin{bmatrix} 1 \\ -e^x(1+x) \end{bmatrix}$. It is easy to show that $x^* = 0$ is Hurwicz properly efficient for f over K, but x^* is not a solution of $CMVVI^P(f', K)$. This is easily seen, since, for $y = -1$ we have $\langle f'(y), y - x^* \rangle_l = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \in -C$ and this point is in $\mathrm{cl\,cone}\,\Theta(x^*)$.

Proposition 7. *Let $f : \mathbb{R}^n \to \mathbb{R}^l n$ be differentiable and C–convex. If $x^* \in PE^{Hu}(f, K)$ then x^* solves $CMVVI^P(f', K)$.*

Proof. By assumption (recall Proposition 3), there exists $\hat{\lambda} \in \mathrm{int}\, C^*$ such that $\hat{\lambda}^\top f(y) - \hat{\lambda}^\top f(x^*) \geq 0$, $\forall y \in K$. Moreover, being f C-convex, it easily follows that $\hat{\lambda}^\top f$ is a real valued convex function and therefore x^* solves $MVI(\hat{\lambda}^\top f', K)$, that is $\langle \hat{\lambda}^\top f'(y), y - x^* \rangle \geq 0$, $\forall y \in K$. By Caratheodory Theorem, each $\theta \in \mathrm{cl\,cone}\,\Theta(x^*)$ can be expressed as:

$$\theta = \lim_{k \to +\infty} \sum_{i=1}^{l+1} \beta_i^k \delta_i^k \langle f'(y_i^k), y_i^k - x^* \rangle,$$

for sequences $\{y_i^k\}_{k \geq 0} \in K$, $\{\delta_i^k\}_{k \geq 0}$, and $\{\beta_i^k\}_{k \geq 0} \in [0, 1]$, $i = 1, \ldots, l+1$, with $\delta_i^k \geq 0$ and $\sum_{i=1}^{l+1} \beta_i^k = 1$, $\forall k$.

Hence, it follows easily that it holds:

$$\widehat{\lambda}^\top \theta \geq 0, \quad \forall \theta \in \mathrm{cl\,cone}\ \Theta(x^*).$$

Now assume, by contradiction, that x^* does not solve $CMVVI^P(f', K)$, that is there exists a vector $\bar{\theta} \in \mathrm{cl\,cone}\ \Theta(x^*) \cap -C \setminus \{0\}$. But this implies that $\forall \lambda \in \mathrm{int}\, C^*$ we have:

$$\lambda^\top \theta < 0$$

which is the absurdo. ∎

Remark 6. It is classical that the assumption of C-convexity of the objective function implies that Hurwicz proper efficiency is equivalent to several other notions of proper efficiency (see e.g. [9]). Therefore the previous Proposition could be stated also with other proper efficient solutions in the hypothesis.

4. Scalarization: Some Remarks

In the previous sections, we have already used some scalarization results (see Lemma 1). When dealing with vector valued problems, *scalarization* is a classical tool, which allows to reduce the original problem to a family of scalar ones.

Several scalarization techniques are known and applied in vector optimization (see e.g. [14, 19]). The most common is linear scalarization, which consists in summing up the l criteria of the vector problem, averaged by nonnegative weights and it has been already used in the previous results. The application of this technique to vector variational inequalities of Stampacchia type is known (see e.g. [3, 13, 16, 22]).

Definition 8. *Let $F : \mathbb{R}^n \to \mathbb{R}^l n$ be given. A vector $x^* \in K$ is a scalarized solution of a Stampacchia type vector variational inequality when there exists $\lambda \in C^*$ (or $\lambda \in int C^*$) such that:*

$$VI(\lambda^\top F, K) \qquad \langle \lambda^\top F(x^*), y - x^* \rangle \geq 0, \quad \forall y \in K$$

Theorem 5. [22] *The following implications hold true:*
i) *if x^* solves $VI(\lambda^\top F, K)$ for some $\lambda \in int C^*$, then it solves also $VVI^s(F, K)$;*
ii) *if x^* solves $VI(\lambda^\top F, K)$ for some $\lambda \in C^*$, then it solves also $VVI(F, K)$;*
iii) *if x^* solves $VVI(F, K)$, then there exists $\lambda \in C^*$ such that x^* is a solution of $VI(\lambda^\top F, K)$.*

In general, implication *i)* is not reversible as we can prove by an example.

Example 5. Let $K := \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \geq 0,\ x_2 \geq 0,\ x_2 \leq -x_1^2 + 2x_1\}$, $F(x_1, x_2) = \begin{bmatrix} x_1 - 2 & x_2 - 1 \\ 0 & x_2 - 2 \end{bmatrix}$ and assume the Pareto order (i.e. $C := \mathbb{R}^2_+$). Therefore:

$$\Omega(x) := \left\{ w \in \mathbb{R}^l \mid w = \begin{bmatrix} (x_1 - 2)(y_1 - x_1) + (x_2 - 1)(y_2 - x_2) \\ (x_2 - 2)(y_2 - x_2) \end{bmatrix},\ y \in K \right\}$$

For $x^* = (1, 1)$ we have $\Omega(x^*) \cap (-C) = \{0\}$.

Let's assume x^* solves $VI(\lambda^\top F, K)$, for some $\lambda^* = \begin{bmatrix} \lambda_1^* \\ \lambda_2^* \end{bmatrix}$, i.e.:

$$\lambda_1^*(1 - y_1) + \lambda_2^*(1 - y_2) \geq 0, \forall y \in K, \tag{1}$$

where $\lambda_i^* > 0$, $i = 1, 2$. If we put $\gamma = \frac{\lambda_1^*}{\lambda_2^*} > 0$ and evaluate (1) along the portion of δK (the boundary of K) defined by the equation $y_2 = -y_1^2 + 2y_1$, we get:

$$y_1^2 - (2 + \gamma)y_1 + 1 + \gamma \geq 0,\ \forall y_1 \in [0, 2].$$

This inequality admits solutions in the intervals $y_1 \leq 1$ and $y_1 \geq 1 + \gamma$. Since $\gamma > 0$, there exists $y_1^* \in (1, 1 + \gamma)$ which does not solve (1) and therefore a contradiction.

Now we focus on the scalarization problem for Minty type vector variational inequalities. We have already presented Lemma 1, which shows that solutions of $CMVVI(F, K)$ coincide with the solutions of scalar Minty variational inequalities. Here we remark that Example 2 shows as well a solution of $MVVI(F, K)$ which does not admit a scalar representation by means of $\lambda \in C^*$ (while points *ii)* and *iii)* of Theorem 5 state a different result for Stampacchia type inequalities). This is another lack of Definition 5which has been filled by $CMMVI(F, K)$.

Also solutions of $CMVVI^P(F, K)$ can be written in terms of scalar Minty variational inequality. The proof of the next lemma follows along the lines of Proposition 7 and is omitted.

Lemma 2. *Let $\lambda \in int C^*$. Any $x^* \in K$ which solves the scalar variational inequality $MVI(\lambda^\top F, K)$ is such that x^* solves $CMVVI^P(F, K)$.*

The converse of the previous Lemma hods true too.

Lemma 3. *Let $x^* \in CMVVI^P(F, K)$. Then there exists $\lambda \in \operatorname{int} C^*$ such that x^* solves $MVI(\lambda^\top F, K)$.*

Proof. The closed convex cone $\text{clcone}\Theta(x^*)$ has a compact base A. Hence we can find a vector $\lambda \in C^*$ such that $\langle \lambda, v\rangle \leq 0$, $\forall v \in -C$ and $\langle \lambda, a\rangle > 0$, $\forall a \in A$. Since $A \cap (-C) = \emptyset$, we can choose $\lambda \in \operatorname{int} C^*$. ∎

Therefore we can present an alternative and quicker proof of Theorem 4 and Proposition 7, as we did for Theorem 1, by means of scalarization:

Theorem 6. *Let $f : \mathbb{R}^n \to \mathbb{R}^l$ be differentiable and $f' = F$ be its Jacobian. If x^* solves $CMVVI^P(f', K)$, then $x^* \in PE^{Hu}(f, K)$. The converse holds true under C-convexity of f.*

Proof. Assume x^* solves $CMVVI^P(f', K)$. Then, by Lemma 2, we know $\exists \lambda \in \operatorname{int} C^*$ such that x^* solves $MVI(\lambda^\top f', K)$. By Proposition 2, the latter means x^* is a minimizer for $\lambda^\top f$ over K, i.e. it is proper efficient in the sense of Hurwicz.

Conversely, convexity is necessary to prove that minimizers of $\lambda^\top f$ over K, for some $\lambda \in \operatorname{int} C^*$, are solutions of $MVI(\lambda^\top f', K)$, and hence of $CMVVI^P(f', K)$. ∎

The following result allows to characterize efficient solutions by means of a scalar Minty variational inequality:

Theorem 7. *Let $f : \mathbb{R}^n \to \mathbb{R}^l$ be differentiable on an open subset containing $K \subseteq \mathbb{R}^n$ and $x^* \in K$ be a solution of the following strict Minty variational inequality (see [7]):*

$$\langle \lambda^\top f'(y), y - x^*\rangle > 0 \quad \forall y \in K,\ y \neq x^*, \tag{2}$$

for some $\lambda \in C^*$. Then x^* is an efficient solution of $VP(f, K)$.

Proof. Since x^* is a solution of (2), then it is the unique minimizer of $\lambda^\top f$ over K (see [5]). By contradiction, assume that x^* is not efficient. Hence there exists a vector $x \in K$ such that:

$$f(x) - f(x^*) \in -\mathbb{C},$$

that is:

$$\lambda^\top \big(f(x) - f(x^*)\big) \leq 0.$$

Hence $\lambda^\top f(x) = \lambda^\top f(x^*)$ which is a contradiction. ∎

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