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On Polynomial Projectors That Preserve Homogeneous Partial Differential Equations

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1. Let $H(\mathbb{C}^n)$ be the space of entire functions on $\mathbb{C}^n$ equipped with its usual compact convergence topology, and $\mathcal{P}_d(\mathbb{C}^n)$ the space of polynomials on $\mathbb{C}^n$ of total degree at most d. A *polynomial projector* of degree d is a continuous linear map Π from $H(\mathbb{C}^n)$ to $\mathcal{P}_d(\mathbb{C}^n)$ for which $\Pi(p) = p$ for every $p \in \mathcal{P}_d(\mathbb{C}^n)$. Such a projector Π is said to *preserve homogeneous partial differential equations (HPDE)* of degree k if for every $f \in H(\mathbb{C}^n)$ and every homogeneous polynomial of degree k,

$$q(z) = \sum_{|\alpha|=k} a_\alpha z^\alpha,$$

we have

$$q(D)f = 0 \Rightarrow q(D)\Pi(f) = 0, \tag{1}$$

where as usual

$$q(D) := \sum_{|\alpha|=k} a_\alpha D^\alpha,$$

$D^\alpha = \partial^{|\alpha|}/\partial z_1^{\alpha_1} \dots \partial z_n^{\alpha_n}$, and $|\alpha| = \sum_{j=1}^n \alpha_j$ denotes the length of the multi-index $\alpha = (\alpha_1, \dots, \alpha_n)$.

In [3] Calvi and Filipsson gave a precise description of the polynomial projectors preserving all HPDE. In particular they show that a polynomial projector preserves all HPDE as soon as it preserves HPDE of degree 1. Then naturally arises the question of the existence of polynomial projectors preserving HPDE of degree k ($k \geq 1$) without preserving HPDE of smaller degree. We proved that such projectors do indeed exist and we extend the basic structure theorem proved in [3] to this more general case. As a consequence we show that a polynomial projector which preserves HPDE of degree k necessarily preserves HPDE of every degree not smaller than k.

We also complete the results of [3] in an other direction. Calvi and Filipsson have used their results to give a new characterization of Kergin interpolation. Namely, they have shown that the interpolation space of a polynomial projector of degree d that preserves HPDE contains no more than — and only Kergin interpolation projectors effectively contains — $d+1$ Dirac (point-evaluation) functionals. Here we give a characterization of Abel-Gontcharoff projectors as the only Birkhoff polynomial projectors that preserve all HPDE.

In [6] (see also [7]) Petersson has settled a convenient formalism (using the concept of pairing of Banach spaces) and extended results of [3] to Banach spaces. Our Theorem 1 is likely to have a similar infinite dimensional counterpart.

2. We recall some definitions and results from [3]. A polynomial projector Π can be completely described by the so called *space of interpolation conditions* $\Im(\Pi) \subset H'(\mathbb{C}^n)$, where $H'(\mathbb{C}^n)$ denotes the space of the continuous linear functionals on $H(\mathbb{C}^n)$ whose elements are usually called analytic functionals. The space $\Im(\Pi)$ is defined as follows : an element $\varphi \in H'(\mathbb{C}^n)$ belongs to $\Im(\Pi)$ if and only if for any $f \in H(\mathbb{C}^n)$ we have

$$\varphi(f) = \varphi(\Pi(f)).$$

Let $\{p_\alpha : |\alpha| \leq d\}$ be a basis of $\mathcal{P}_d(\mathbb{C}^n)$. Then we can represent Π as

$$\Pi(f) = \sum_{|\alpha| \leq d} a_\alpha(f) p_\alpha, f \in H(\mathbb{C}^n), \tag{2}$$

with some $a_\alpha{}'s \in H'(\mathbb{C}^n)$, and $\Im(\Pi)$ is given by

$$\Im(\Pi) = \text{span}\{a_\alpha : |\alpha| \leq d\}.$$

In particular, we may take in (2) $p_\alpha = u_\alpha$ where $u_\alpha(z) := z^\alpha/\alpha!$, $z^\alpha := \prod_{j=1}^n z_j^{\alpha_j}$, $\alpha! := \prod_{j=1}^n \alpha_j!$. Notice that the dimension of $\Im(\Pi)$ is

$$N_d(n) := \binom{n+d}{n},$$

which coincides with the dimension of $\mathcal{P}_d(\mathbb{C}^n)$. Moreover, the restriction of $\Im(\Pi)$ to $\mathcal{P}_d(\mathbb{C}^n)$ is the dual space $\mathcal{P}_d^*(\mathbb{C}^n)$. Conversely, if $\mathbf{I}$ is a subspace of $H'(\mathbb{C}^n)$ of dimension $N_d(n)$ such that the restriction of its elements to $\mathcal{P}_d(\mathbb{C}^n)$ spans $\mathcal{P}_d^*(\mathbb{C}^n)$ then there exists a unique polynomial projector $\wp(\mathbf{I})$ such that $\mathbf{I} = \Im(\wp(\mathbf{I}))$. In that case we say that $\mathbf{I}$ *is an interpolation space for* $\mathcal{P}_d(\mathbb{C}^n)$ and, for $p \in \mathcal{P}_d(\mathbb{C}^n)$, we have

$$\wp(\mathbf{I}, f) = p \Leftrightarrow \varphi(p) = \varphi(f), \ \forall \varphi \in \mathbf{I}.$$

Notice that for every projector Π we have $\wp(\Im(\Pi)) = \Pi$. A function f is called *ridge function* if it is of the form $f(z) = h(a.z)$ with $h \in H(\mathbb{C})$, where

$$y.z := \sum_{j=1}^n y_j.z_j \ \text{ for } \ y, z \in \mathbb{C}^n.$$

Let Π is a polynomial projector preserving HPDE of degree 1. From (1) we can easily see that Π also preserves ridge functions, that is, if $f(z) = h(a.z)$ then there exists a univariate polynomial $p := \Pi_a(h)$ such that

$$\Pi(h(a.\cdot))(z) = \Pi_a(h)(a.z),$$

and Π_a is itself a univariate polynomial projector. If $\varphi \in H'(\mathbb{C}^n)$, for a multi-index α, we define $D^\alpha\varphi$ as the continuous linear functional given by

$$D^\alpha\varphi(f) := \varphi(D^\alpha f).$$

Let $\mu_0, \mu_1, \ldots, \mu_d$ be $d+1$ not necessarily distinct analytic functionals on $H(\mathbb{C}^n)$ such that $\mu_i(1) \neq 0$ for $i = 0, \ldots, d$. Then, it was proved in [3] that

$$\mathbf{I} := \operatorname{span}\{D^\alpha\mu_{|\alpha|} : |\alpha| \le d\} \tag{3}$$

is an interpolation space for $\mathcal{P}_d(\mathbb{C}^n)$. The projector corresponding to space $\mathbf{I}$ in (3) is called D-*Taylor projector*. It was introduced by Calvi [2]. For $\alpha \in \mathbb{Z}_+^n$ and $a \in \mathbb{C}^n$, the analytic functional $D^\alpha[a]$ is defined by

$$D^\alpha[a](f) = D^\alpha f(a), \ \ f \in H(\mathbb{C}^n).$$

It is called *discrete functional*. For $\alpha = 0$, we use the abbreviation: $D^0[a] = [a]$. Typical D-Taylor projector are Abel-Gontcharoff projector when $\mu_i := [a_i]$ in (3). For other natural examples, see [1 - 3, 5].

Theorem A. [3] *Let Π be a polynomial projector of degree d in $H(\mathbb{C}^n)$. Then the following three conditions are equivalent:*
1.Π *preserves all HPDE;*
2. Π *preserves ridge functions;*
3. Π *is a D-Taylor projector.*

This theorem shows that a polynomial projector Π preserving HPDE of degree 1 also preserves all HPDE. Furthermore, the μ_i's constructed in the proof of Theorem A are the only analytic functionals satisfying the identity

$$\Pi(f) = \sum_{|\alpha|\le d} D^\alpha\mu_{|\alpha|}(f)u_\alpha. \tag{4}$$

3. We now extend Theorem A and D-Taylor representation (4) to polynomial projectors preserving HPDE of degree k, $1 \le k \le d$.

Theorem 1. *A polynomial projector Π preserves HPDE of degree k, $1 \le k \le d$, if and only if there are analytic functionals $\mu_k, \mu_{k+1}, \ldots, \mu_d \in H'(\mathbb{C}^n)$ with $\mu_i(1) \neq 0$, $i = k, \ldots, d$, such that Π is represented in the following form*

$$\Pi(f) = \sum_{|\alpha|<k} a_\alpha(f)u_\alpha + \sum_{k\le|\alpha|\le d} D^\alpha\mu_{|\alpha|}(f)u_\alpha, \tag{5}$$

with some a_α's $\in H'(\mathbb{C}^n)$, $|\alpha| < k$.

Observe that (4) is immediately derived from Theorem 1.

Corollary 1. *If the polynomial projector* Π *preserves HPDE of degree* k, $1 \leq k \leq d$, *then* Π *preserves also HPDE of all degree greater than* k.

Corollary 2. *If* $1 < k \leq d$, *there is a polynomial projector which preserves HPDE of degree* k *but not HPDE of all degree smaller than* k.

Corollary 3. *Let* Π *be a polynomial projector preserving HPDE of degree* k, $1 \leq k \leq d$. *Then there are functionals* $\mu_k, \mu_{k+1}, \ldots, \mu_d$ *such that the set*

$$span\{D^\alpha \mu_s : |\alpha| = s, \ s = k, \ldots, d\}$$

is a proper subset of $\Im(\Pi)$. *Moreover, if* Π *is represented as in* (5) *with* $\mu_i(1) = 1, \ i = k, \ldots, d$, *and* $D^\beta \nu \in \Im(\Pi)$ *with* $|\beta| \geq k$, *then we have a relation*

$$\nu = \mu_{|\beta|} + \sum_{j=1}^{d-|\beta|} \sum_{l_1 l_2 \ldots l_j} c_{l_1 l_2 \ldots l_j} \frac{\partial^j \mu_{|\beta|+j}}{\partial z_{l_1} \ldots \partial z_{l_j}},$$

where each l_k *is taken over* $\{1, 2, ..., n\}$.

4. *Birkhoff projector* is called a polynomial projector Π for which $\Im(\Pi)$ is generated by discrete functionals, that is to say, by functionals of the form $D^\alpha[a]$. For results on Birkhoff interpolation we refer to [4]. The following theorem might seem intuitively clear but we failed to find an immediate proof. It is worth noting that this result is typical of the higher dimension. It is indeed not true in dimension 1 in which the concept of polynomial projector preserving all HPDE reduces to a triviality.

Theorem 2. *Let* Π *be a Birkhoff projector of degree* d *on* $\mathbb{C}^n$, $n \geq 2$. *Then* Π *preserves all HPDE if and only if it is an Abel-Gontcharoff projector, that is, there are* $a_0, \ldots, a_d \in \mathbb{C}^n$ *not necessary distinct such that*

$$\Im(\Pi) = \text{span}\{\, D^\alpha[a_s] : |\alpha| = s, \ s = 0, \ldots, d\,\}.$$

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