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On Generalized co-Cohen-Macaulay and co-Buchsbaum Modules Over Commutative Rings

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1. Introduction

The classes of Cohen-Macaulay modules, Buchsbaum modules and generalized Cohen-Macaulay modules play an important role in the category of Noetherian modules. The structure of these modules is known well by means of the theories of the multiplicity, the local cohomology modules, and by the localization method (see [5, 11]).

There is a theory of local homology for Artinian modules [3] which is in some sense dual to the well known theory of local cohomology modules. Let R be a commutative Noetherian ring and A an Artinian R-module. For given ideal I of R, *the i-th local homology module of A with respect to I*, denoted by $H_i^I(A)$, is the module $\varprojlim_t \operatorname{Tor}_i^R(R/I^t; A)$. It should be mentioned that $H_i^I(A) = L_i^I(A)$, where $\Lambda_I(A) = \varprojlim_t A/I^t A$, and L_i^I is the i-th left derived functor of Λ_I. Recall that, in the case R is local with $\mathfrak{m}$ is the unique maximal ideal, A is called *co-Cohen-Macaulay* if N-dim $A =$ Width A, (see [12]), where N-dim A is the Noetherian dimension of A defined in [7, 9], and Width A is the width of A defined as in [8]. Then the class of co-Cohen-Macaulay modules, which also plays an impotant role in the category of Artinian modules, can be characterized by local homology modules as follows: A is co-Cohen-Macaulay if and only if $H_i^{\mathfrak{m}}(A) = 0$, for all $i = 0, \ldots,$ N-dim $A - 1$ (see [3]).

The notions of system of parameters and multiplicity for Artinian modules over commutative rings were introduced in [4]. In that paper, many properties of multiplicity for Artinian modules are shown in some sense dual to that of multiplicity for Noetherian modules. Especially, $e(\underline{x}; A) \leq \ell(0 :_A \underline{x})$, and A is co-Cohen-Macaulay if and only if $e(\underline{x}; A) = \ell(0 :_A \underline{x})$ for all system of parameters $\underline{x}$ of A, where $e(\underline{x}; A)$ is the multiplicity of A with respect to $\underline{x}$. Thus, A is co-Cohen-Macaulay if and only if $I(A) = \sup_{\underline{x}} I(\underline{x}; A) = 0$, where $I(\underline{x}; A) = \ell_R(0 :_A \underline{x}R) - e_R(\underline{x}; A)$ and the supremum takes over all the systems of parameters $\underline{x}$ of A. Therefore, it is natural to ask about the structure of Artinian modules A satisfying $I(A) < \infty$ or $I(\underline{x}; A)$ is a constance not depending on $\underline{x}$.

The aim of this short note is to study structure of modules A satifying $I(A) < \infty$ (resp. $I(\underline{x}; A)$ is a constance) called *generalized co-Cohen-Macaulay* (resp. *co-Buchsbaum*). The structure of these modules will be described in terms of local homology modules and multiplicity. By using $\mathfrak{q}$-weak co-sequence and co-standard system of parameters, some characterizations of these modules are given. The complete proofs of the statements given here can be found in [2].

2. Generalized co-Cohen-Macaulay Modules

In this paper, we always assume that R is a commutative Noetherian ring (not necessarily local) and A is an Artinian $R-$module with N-dim $A = d$. The following notations will be used from now on. We have known by [10] that $\operatorname{Supp} A$ is a finite set, and each element in $\operatorname{Supp} A$ is a maximal ideal of R. Let $\operatorname{Supp} A = \{\mathfrak{m}_1, \ldots, \mathfrak{m}_t\}$. Then $A = A_1 \oplus \ldots \oplus A_t$, where $A_j = \bigcup_{n\geq 0} (0 :_A \mathfrak{m}_j^n)$, for $j = 1, \ldots, t$. Set $\mathfrak{m} = \bigcap_{j=1}^{t} \mathfrak{m}_j$. Note that if R is a local ring then $\mathfrak{m}$ is exactly the unique maximal ideal of R. For a system of parameters $\underline{x} = (x_1, \ldots, x_d)$ of A we set

$$I(\underline{x}; A) = \ell_R(0 :_A \underline{x}R) - e(\underline{x}; A) \text{ and } I(A) = \sup_{\underline{x}} I(\underline{x}; A),$$

where the supremum takes over all systems of parameters $\underline{x}$ of A.

We introduce first the notion of generalized co-Cohen-Macaulay modules.

Definition 2.1. *A is called generalized co-Cohen-Macaulay (gCCM for short) if $I(A) < \infty$.*

To give characterizations of gCCM modules, we introduce the notion of $\mathfrak{q}$-weak co-sequences for Artinian modules, which is in some sense dual to the concept of $\mathfrak{q}$-weak sequences defined by Stückrad and Vogel [11] for Noetherian modules.

Definition 2.2. *Let $\mathfrak{q}$ be an ideal of R such that $Rad(\mathfrak{q}) = \mathfrak{m}$. A sequence* $x_1, \ldots, x_r$ of elements in $\mathfrak{m}$ *is called a $\mathfrak{q}$-weak co-sequence of A if*

$$x_i(0 :_A (x_1, \ldots, x_{i-1})R) \supseteq \mathfrak{q}(0 :_A (x_1, \ldots, x_{i-1})R) \text{ for all } i = 1, \ldots, r.$$

If $x_1, \ldots, x_r$ is an $\mathfrak{m}$-weak co-sequence then it is called a weak co-sequence.

Now we have the following characterizations of gCCM.

Theorem 2.3. *The following statements are equivalent:*
(i) *A is gCCM.*
(ii) $\ell_R(H_i^{\mathfrak{m}}(A)) < \infty$ *for all* $i \leq d-1$.
(iii) *There exists a system of parameters $(x_1, \ldots, x_d)$ of A and an ideal $\mathfrak{q}$ such that Rad $(\mathfrak{q}) = \mathfrak{m}$ and $(x_1^n, \ldots, x_d^n)$ is a $\mathfrak{q}-$weak co-sequence for all positive integers n.*
(iv) *There exists an ideal $\mathfrak{q}$ such that Rad $(\mathfrak{q}) = \mathfrak{m}$ and every system of parameters of A is a $\mathfrak{q}$-weak co-sequence.*
(v) *There exists a positive integer s and a system of parameters $(x_1, \ldots, x_d)$ such that $I(x_1^n, \ldots, x_d^n; A) \leq s$ for all $n \geq 1$.*
When A satisfies one of the above conditions, we have

$$I(A) = \sum_{i=0}^{d-1} \binom{d-1}{i} \ell_R(H_i^{\mathfrak{m}}(A)).$$

To prove Theorem 2.3, we need the following lemma.

Lemma 2.4. *For every system of parameters $\underline{x} = (x_1, \ldots, x_d)$ of A we have*

$$I(\underline{x}; A) \leq \sum_{i=0}^{d-1} \binom{d-1}{i} \ell_R(H_i^{\mathfrak{m}}(A)).$$

In particular, if $\ell_R(H_i^{\mathfrak{m}}(A)) < \infty$ for all $i = 0, \ldots, d-1$ then there exists an ideal $\mathfrak{q}$ such that Rad $(\mathfrak{q}) = \mathfrak{m}$ and the equality holds for every system of parameters $\underline{x}$ of A contained in $\mathfrak{q}$.

The notion of standard system of parameters, which makes an important role in the study generalized Cohen-Macaulay modules, was introduced by Trung [13]. Below we introduce the dual notion for Artinian modules.

Definition 2.5. *An system of parameters $\underline{x} = (x_1, ..., x_d)$ of A is called co-standard if*

$$\ell(0 :_A \underline{x}A) - e(\underline{x}; A) = \ell(0 :_A (x_1^2, ..., x_d^2)R) - e(x_1^2, ..., x_d^2; A).$$

By using co-standard system of parameters, we have the following characterization of gCCM modules.

Theorem 2.6. *A is gCCM if and only if there exists a co-standard system of parameters of A.*

It is natural to ask that whether the generalized co-Cohen-Macaulayness is preserved by localization? The following result gives a positive answer to this question. Note that $A_{\mathfrak{p}} = 0$ for all $\mathfrak{p} \notin \{\mathfrak{m}_1, \ldots, \mathfrak{m}_t\}$ and $A_{\mathfrak{m}_j} \cong A_j$ for $j = 1, \ldots, t$.

Theorem 2.7. *A is gCCM if and only if A_j is gCCM and N-dim $A_j = d$ or N-dim $A_j = 0$ for all $j = 1, \dots, t$.*

The following corollary produces many examples of gCCM modules.

Corollary 2.8. *The following statements are true.*

(i) *If A is gCCM and x is a parameter element of A then $0 :_A x$ is gCCM.*

(ii) *If $B_1, \dots, B_n$ are gCCM R-modules of Noetherian dimension 0 or d then $A = \overset{n}{\underset{i=n}{\oplus}} B_i$ is gCCM.*

(iii) *Let $(R, \mathfrak{m})$ be a local ring.*

(a) *A is a gCCM R-module if and only if it is a gCCM $\widehat{R}$-module, where $\widehat{R}$ is the $\mathfrak{m}$-adic completion of R.*

(b) *If M is generalized Cohen-Macaulay then Hom $(M; E)$ is gCCM, where E is the injective hull of $R/\mathfrak{m}$.*

(c) *If M is generalized Cohen-Macaulay of dimension d then $H^d_{\mathfrak{m}}(M)$ is gCCM.*

3. Co-Buchsbaum Modules

We first introduce the notion of co-Buchsbaum module.

Definition 3.1. *A is called co-Buchsbaum if $I(\underline{x}; A)$ is a constance not depending on system of parameters $\underline{x}$ of A.*

The following result gives a property of local homology modules of co-Buchsbaum modules.

Proposition 3.2. *Suppose that A is co-Buchsbaum. Then $\mathfrak{m}H^{\mathfrak{m}}_i(A) = 0$ for all $i = 0, \dots, d-1$.*

From Proposition 3.2, we can produce many examples of gCCM modules which are not co-Buchsbaum modules. For example, let $R = k[x_1, \dots, x_d]$ be the polynomial ring of d variables over a field k. Let $B = k[x_1^{-1}, \dots, x_d^{-1}]$ be the Artinian $R-$module of inverse polynomials (see [6] for the definition). Let $A = B \oplus R/(x_1^n, x_2, \dots, x_d)R$, where n is an integer such that $n > 1$. Then B is co-Cohen-Macaulay, Supp $A = \{\mathfrak{m}\}$, where $\mathfrak{m} = (x_1, \dots, x_d)R$, the unique homogenuous maximal ideal of R. Then A is gCCM, but A is not co-Buchsbaum since $\mathfrak{m}H^{\mathfrak{m}}_0(A) \neq 0$.

The following results are characterizations of co-Buchsbaum modules in terms of weak co-sequence and co-standard system of parameters.

Theorem 3.3. *The following statements are equivalent:*

(i) *A is co-Buchsbaum;*

(ii) *every system of parameters of A is a weak co-sequence;*

(iii) *every system of parameters of A is co-standard.*

Below, we give the property of co-Buchsbaum modules by passing Artinian modules A_j over local rings $R_{\mathfrak{m}_j}$.

Proposition 3.4. *A is co-Buchsbaum if A_j is co-Buchsbaum and* N-dim $A_j = d$ *or $\mathfrak{m}A_j = 0$ for all $j = 1, \ldots, t$. In particular, A is co-Cohen-Macaulay if and only if A_j is co-Cohen-Macaulay and* N-dim $A_j = d$ *for all $j = 1, \ldots, t$.*

The following corollary gives a lot of examples of co-Buchsbaum modules.

Corollary 3.5. *The following statements are true.*

(i) *If $B_1, \ldots, B_n$ are co-Buchsbaum R-modules such that* N-dim $B_i = d$ *or $\mathfrak{m}B_i = 0$ then $A = \overset{n}{\underset{i=n}{\oplus}} B_i$ is co-Buchsbaum.*

(ii) *Let $(R, \mathfrak{m})$ be a local ring. Then we have*

(a) *A is a co-Buchsbaum R-module if and only if A is a co-Buchsbaum $\widehat{R}$-module.*

(b) *If M is Buchsbaum then* Hom $(M; E)$ *is co-Buchsbaum, where E is the injective hull of $R/\mathfrak{m}$.*

(c) *If M is Buchsbaum of dimension d then $H^d_{\mathfrak{m}}(M)$ is co-Buchsbaum.*

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