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L_p-Convergence of Some Weighted Dependent Sequences of Random Variables

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Abstract. The paper is concerned with the L_p-convergence of some weighted dependent sequences random variables.

1. Introduction

Let $\{X_n, n \geq 1\}$ be a sequence of random variables and let $S_n = X_1 + X_2 + \cdots + X_n$. Pyke and Root (1968) showed that if $\{X_n, n \geq 1\}$ is an independent and identically distributed (i.i.d) sequence of random variables and $E\lfloor |X_1|^p \rfloor < \infty$ for some $0 < p < 2$, then $n^{-1}E\lfloor |S_n - a_n|^p \rfloor \to 0$ as $n \to \infty$ where $a_n = 0$ for $0 < p < 1$ and $a_n = nE[X_1]$ for $1 \leq p < 2$. Considering $\{X_n, n \geq 1\}$ to be dominated in distribution by a random variable X such that $E\lfloor |X|^p \rfloor < \infty$ and taking $a_n = \sum\limits_{k=1}^{n} E[X_k | X_1, X_2, \ldots, X_{k-1}]$, Chatterjee (1969) proved the above result for $1 \leq p < 2$. Chow (1971) strengthened this result by replacing the domination condition by the condition of Uniform Integrability (UI) of $\{|X_n|^p, n \geq 1\}$.

Chandra (1969) introduced a new condition called Cesaro Uniform Integrability (CUI) to establish L_1-convergence in the weak law of large numbers. This is weaker than the usual UI condition and yet was shown to be sufficient enough to derive the result.

Bose and Chandra (1992) proved L_p-convergence $(0 < p < 2)$ for some pair-wise independent and dependent sequences under the condition of Cesaro uniform integrability. In this paper we establish L_p-convergence $(0 < p < 2)$

for some weighted dependent sequences of random variables under more general condition. The different sequences under consideration include martingale difference, *-mixing, mixingale difference and martingale transforms.

2. Definition and Preliminaries

This section is devoted to the background materials which have been used in this paper. Let $\{X_n,\, n \geq 1\}$ be a sequence of random variables defined on a probability space $(\Omega, \Im, P)$.

Definition 1. *Given a sequence of random variables* $\{X_k,\, k \geq 1\}$, *let* $\mathcal{G}_{ij} = \mathcal{B}\{X_k, i \leq k \leq j\}$ *for all* $1 \leq i \leq j < \infty$. $\{X_k,\, k \geq 1\}$ *is said to be* $*$*-mixing if there exist an integer* M *and a function* φ *for which* $\varphi(m) \to 0$ *as* $m \to \infty$ *and* $A \in \mathcal{G}_{1n}$, $\mathcal{B} \in \mathcal{G}_{m+n,m+n}$ *implies*

$$\left|P(A \cap B) - P(A)P(B)\right| \leq \varphi(m)P(A)P(B)$$

for all $m \geq M$ *and all* $n \geq 1$.

Definition 2. *Let* $\{X_k, \Im_k,\, k \geq 1\}$ *be a martingale difference sequence and* ν_k *be* $\Im_{k-1}$*-measurable for each* $k \geq 1$. *Then* $\{T_n,\, n \geq 1\}$ *defined by* $T_n = \sum\limits_{k=1}^{n} \nu_k X_k$ *is called a martingale transform and* $\{\nu_k,\, k \geq 1\}$ *is called the transforming sequence.*

Let $\{X_n, n \geq 1\}$ *be a sequence of random variables on* $(\Omega, \Im, P)$ *such that* $E(|X_n|^p) < \infty$ *for some* $p \geq 1$ *and each* $n \geq 1$ *and* $\{\Im_n : n = 0, \pm 1, \pm 2, \ldots\}$ *be an increasing sequence of sub-sigma fields of* $\Im$.

Definition 3. *The pair* $\{(X_n, n \geq 1),\, \Im_n : n = 0, \pm 1,\, \pm 2, \ldots\}$ *is called an* L_p*-mixingale difference sequence if there exist two sequences of constants* $\{c_n,\, n \geq 1\}$ *and* $\{\psi_m,\, m \geq 0\}$ *such that* $\psi_m \to 0$ *as* $m \to \infty$ *and*
(a) $\|E[X_n|\Im_{n-m}]\|_p \leq c_n \psi_m$,
(b) $\|X_n - E[X_n \,|\, \Im_{n+m}]\|_p \leq c_n \psi_{m+1}$,
where $\|X\|_p = [E|X|^p]^{1/p}$ *for* $p > 0$.

We assume that the following conditions hold.
(A1) Let $(a_{nk},\, k = 1, 2, \ldots, n)$ represent a double sequence of real numbers such that
(i) $a_{nk} \to 0$ as $n \to \infty$ for every k,
(ii) for $M > 0$, a constant, $\sum_{k=1}^{n} |a_{nk}|^r \leq M < \infty$ for all n and $0 < r \leq 1$.
(A2) For $0 < p < 2$,

$$\limsup_{n\to\infty} \sum_{k=1}^{n} |a_{nk}|^p E\left[|X_k|^p I_{[|X_k| > a(n)]}\right] = 0,$$

where $a(n)$ is a function of n such that $a(n) \to \infty$ as $n \to \infty$.

Remark 1. If $\{X_k\}$ is uniformly integrable then it satisfies Condition (A2).

Remark 2. Condition (A2) implies that for $0 < p < 2$,

$$\limsup_{n\to\infty} [a(n)]^p |a_{nk}|^p P[|X_k| > a(n)] = 0.$$

Lemma 1. *Let $\{X_n, n \geq 1\}$ and $\{Y_n, n \geq 1\}$ be two sequences of random variables on $(\Omega, \Im, P)$. Then the following results hold.*

(i) *If $\{|X_n|, n \geq 1\}$ satisfies condition (A2) and $|Y_n| \leq |X_n|$ a.s. then, $\{|Y_n|, n \geq 1\}$ satisfies condition (A2).*

(ii) *If for some $p > 0$, $\{X_n, n \geq 1\}$ and $\{Y_n, n \geq 1\}$ satisfy condition (A2), then so also $\{| X_n + Y_n|^p, n \geq 1\}$.*

(iii) *Let $\{\Im_n, n \geq 1\}$ be a sequence of sub-sigma fields of $\Im$ and $p > 0$. If $\{|X_n|, n \geq 1\}$ satisfies condition (A2), then so also $\{Y_n = E[|X_n|^p|\Im_{n-1}], n > 1\}$.*

Proof.
(i) is trivial. Again

$$E\lfloor |X_k + Y_k|^p I_{[|X_k+Y_k|>a(n)]}\rfloor \leq 2^p \big\{E\lfloor |X_k|^p I_{[|X_k|>a(n)]}\rfloor + E\lfloor |Y_k|^p I_{[|Y_k|>a(n)]}\rfloor\big\}$$

and $|X_k|^p$, $|Y_k|^p$ satisfy (A2) for some $0 < p < 2$ and so (ii) is also trivial. To prove (iii), we have

$$\begin{aligned} E\lfloor |Y_n| I_{[|Y_n|>a(n)]}\rfloor &= E\big[\big|E\big(|X_n|^p|\Im_{n-1}\big)\big| I_{[|Y_n|>a(n)]}\,\big] \\ &= \int\limits_{[|Y_n|>a(n)]} E(|X_n|^p|\Im_{n-1})\, dP \leq \int\limits_{[|X_n|>a(n)]} E(|X_n|^p|\Im_{n-1})\, dP \\ &= \int\limits_{[|X_n|>a(n)]} |X_n|^p dP = E\lfloor |X_n|^p I_{[|X_n|>a(n)]}\rfloor. \end{aligned}$$

Hence if $|X_n|^p$ satisfies (A2), then so also $E\big(|X_n|^p|\Im_{n-1}\big)$.

Lemma 2. *If $f_n : \Re \to \Re^+$ where $0 \leq f_n \leq 1$ for all $n \geq 1$ and $\sup_{n\in N}\big[x f_n(x)\big] \to 0$ as $x \to \infty$ then,*

$$\sup_{n\in N}\Big[\frac{1}{y}\int\limits_0^y x\, f_n(x)\, dx\Big] \to 0 \quad \text{as} \quad y \to \infty.$$

Proof. Put $f^*(x) = \sup_{n\in N}\big[x f_n(x)\big]$. Then

$$\sup_{n\in N}\Big[\frac{1}{y}\int\limits_0^y x\, f_n(x)\, dx\Big] \leq \frac{1}{y}\int\limits_0^y f^*(x) dx \quad \text{for all} \quad y > 0.$$

Thus it is sufficient to show that $\dfrac{1}{y}\displaystyle\int_0^y f^*(x)\, dx \to 0$ as $y \to \infty$.

Since $f^*(x) \to 0$ as $x \to \infty$, for any fixed $\varepsilon > 0$ there exists a $x_0(\varepsilon) > 0$ such that if $x > x_0(\varepsilon)$ and $y > x_0(\varepsilon)$, then $0 \leq f^*(x) < \varepsilon$. Again

$$\frac{1}{y}\int\limits_{0}^{x_0(\varepsilon)} f^*(x)dx \to 0 \quad \text{as} \quad y \to \infty,$$

and

$$\frac{1}{y}\int\limits_{x_0(\varepsilon)}^{y} f^*(x)dx \le \frac{\varepsilon}{y}\int\limits_{x_0(\varepsilon)}^{y} dx < \varepsilon \quad \text{as} \quad y \to \infty.$$

The result follows, since

$$\frac{1}{y}\int\limits_{0}^{y} f^*(x)dx = \Big[\frac{1}{y}\int\limits_{0}^{x_0(\varepsilon)} f^*(x)\,dx + \frac{1}{y}\int\limits_{x_0(\varepsilon)}^{y} f^*(x)\,dx\Big].$$

3. L_p-Convergence of Martingale Difference Sequence

Theorem 1. *Let $0 < p < 1$ and $\{X_n, n \ge 1\}$ be a martingale difference sequence satisfying condition* (A2). *Then $E\lfloor |S_n|^p \rfloor \to 0$ as $n \to \infty$ where $S_n = \sum_{k=1}^{n} a_{nk}X_k$.*

Proof. Let $Y_{nk} = a_{nk}X_k\, I_{(|X_k| \le |a_{nk}|^{-r})}$ and $Z_{nk} = a_{nk}X_k\, I_{(|X_k| > |a_{nk}|^{-r})}$ for $n \ge 1$ and $0 < r < p$. So

$$\begin{aligned} E\big[|S_n|^p\big] &= E\Big[\Big|\sum_{k=1}^{n} a_{nk}X_k\Big|^p\Big] = E\Big[\big|\sum_{k=1}^{n}(Y_{nk}+Z_{nk})\,\big|^p\Big] \\ &\le E\Big[\Big|\sum_{k=1}^{n} Y_{nk}\Big|^p\Big] + E\Big[\Big|\sum_{k=1}^{n} Z_{nk}\Big|^p\Big] \end{aligned} \tag{3.1}$$

Now

$$\begin{aligned} E\Big[\Big|\sum_{k=1}^{n} Y_{nk}\Big|^p\Big] &= E\Big[\Big|\sum_{k=1}^{n} a_{nk}X_k I_{(|X_k| \le |a_{nk}|^{-r})}\Big|^p\Big] \\ &\le \sum_{k=1}^{n} |a_{nk}|^p E\Big[|X_k|^p I_{(|X_k| \le |a_{nk}|^{-r})}\Big] \\ &= p\sum_{k=1}^{n} |a_{nk}|^p \int\limits_{(0<x\le|a_{nk}|^{-r})} x^{p-1} P(\,|X_k| > x)\,dx \\ &\le p\sum_{k=1}^{n} |a_{nk}|^p \int\limits_{(0<x\le|a_{nk}|^{-r})} |a_{nk}|^{r(1-p)} P(|X_k| > x)\,dx \\ &< p\varepsilon \sum_{k=1}^{n} |a_{nk}|^p \quad \text{(by Lemma 2)} \\ &\le p\,\varepsilon\, M \end{aligned} \tag{3.2}$$

and

$$\limsup_{n\to\infty} E\Big[\Big|\sum_{k=1}^{n} Z_{nk}\Big|^p\Big] \le \limsup_{n\to\infty}\sum_{k=1}^{n} E\big[|Z_{nk}|^p\big]$$
$$= \limsup_{n\to\infty}\sum_{k=1}^{n} E\left[|a_{nk}X_k\, I_{(|X_k|>|a_{nk}|^{-r})}|^p\right]$$
$$= 0 \quad \text{(by condition (A2))} \tag{3.3}$$

Hence the result follows from relations (3.2) and (3.3).

Theorem 2. *Let* $\{X_n, n \ge 1\}$ *be a martingale difference sequence satisfying condition* (A2). *Then* $E[|S_n|] \to 0$ *as* $n \to \infty$, *where* $S_n = \sum_{k=1}^{n} a_{nk}X_k$.

Proof. Put $X_{nk} = a_{nk}X_k I_{(|X_k|\le|a_{nk}|^{-(1/2)})}$ and $Z_n = \sum\limits_{k=1}^{n}\big[X_{nk} - E(X_{nk}|\Im_{k-1})\big]$.
Now

$$E(Z_n)^2 = E\Big(\sum_{k=1}^{n}\big[X_{nk} - E(X_{nk}|\Im_{k-1})\big]\Big)^2 = E\Big(\sum_{k=1}^{n}\big[X_{nk} - E(X_{nk}|\Im_{k-1})\big]^2\Big)$$
$$+ 2\sum_{k<j=1}^{n} E\big\{[X_{nk} - E(X_{nk}|\Im_{k-1})][X_{nj} - E(X_{nj}|\Im_{j-1})]\big\}$$
$$= E\Big(\sum_{k=1}^{n}[X_{nk} - E(X_{nk}|\Im_{k-1})]^2\Big)$$
$$+ 2\sum_{k<j=1}^{n} E\{[X_{nk} - E(X_{nk}|\Im_{k-1})][X_{nj} - E(X_{nj}|\Im_{j-1})]|\Im_k\}$$
$$= E\Big(\sum_{k=1}^{n}[X_{nk} - E(X_{nk}|\Im_{k-1})]^2\Big)$$
$$+ 2\sum_{k<j=1}^{n} E\big[\{X_{nk} - E(X_{nk}|\Im_{k-1})\}E\{[X_{nj} - E(X_{nj}|\Im_{j-1})]\,|\,\Im_k\}\big]$$
$$= E\Big(\sum_{k=1}^{n}\big[X_{nk} - E(X_{nk}\,|\,\Im_{k-1})\big]^2\Big)$$
$$+ 2\sum_{k<j=1}^{n} E\big[\{X_{nk} - E(X_{nk}\,|\,\Im_{k-1})\}\;\{E\big[X_{nj}\,|\,\Im_k\big] - E\big[E(X_{nj}\,|\,\Im_{j-1})\,|\,\Im_k\big]\}\big]$$
$$= E\Big(\sum_{k=1}^{n}\big[X_{nk} - E(X_{nk}\,|\,\Im_{k-1})\big]^2\Big)$$
$$+ 2\sum_{k<j=1}^{n} E\big[\{X_{nk} - E(X_{nk}\,|\,\Im_{k-1})\}\;\{E\big[X_{nj}\,|\,\Im_k\big] - E\big[X_{nj}\,|\,\Im_k\big]\}\big]$$
$$= E\Big(\sum_{k=1}^{n}\big[X_{nk} - E(X_{nk}\,|\,\Im_{k-1})\big]^2\Big)$$

$$
\begin{aligned}
&= \sum_{k=1}^{n} \left\{ E[X_{nk}^2] - E\big[E(X_{nk} \mid \Im_{k-1})\big]^2 \right\} \\
&\leq \sum_{k=1}^{n} E[X_{nk}^2] \\
&= \sum_{k=1}^{n} E\big[a_{nk}^2 X_k^2 I_{(|X_k| \leq |a_{nk}|^{-1/2})}\big] \\
&\leq \sum_{k=1}^{n} \Big[|a_{nk}| \frac{1}{|a_{nk}|^{-1/2}} \int_{0<x<|a_{nk}|^{-1/2}} |a_{nk}|^{1/2} x P[|X_k| > x] dx \Big] \\
&< \varepsilon M
\end{aligned}
$$

(by Remark 2 and Lemma 2).
So

$$E[|Z_n|] \leq E^{1/2}[|Z_n|^2] \to 0 \quad \text{as} \quad n \to \infty. \tag{3.4}$$

Again $E(X_k \mid \Im_{k-1}) = 0$ as X_n is a martingale difference sequence. So

$$
\begin{aligned}
& E\Big[X_k I_{\left(|X_k| \leq |a_{nk}|^{-1/2}\right)} \mid \Im_{k-1}\Big] + E\Big[X_k I_{\left(|X_k| > |a_{nk}|^{-1/2}\right)} \mid \Im_{k-1}\Big] = 0 \\
\Rightarrow\ & E\Big[X_k I_{\left(|X_k| \leq |a_{nk}|^{-1/2}\right)} \mid \Im_{k-1}\Big] = -E\Big[X_k I_{\left(|X_k| > |a_{nk}|^{-1/2}\right)} \mid \Im_{k-1}\Big].
\end{aligned} \tag{3.5}
$$

Now

$$
\begin{aligned}
& E\Big|\sum_{k=1}^{n} E\big(X_{nk} \mid \Im_{k-1}\big)\Big| = E\Big|\sum_{k=1}^{n} E\big(a_{nk} X_k I_{\left(|X_k| > |a_{nk}|^{-1/2}\right)} \mid \Im_{k-1}\big)\Big| \\
&\leq \sum_{k=1}^{n} E\Big(|a_{nk}| \big|E\big[X_k I_{\left(|X_k| \leq |a_{nk}|^{-1/2}\right)} \mid \Im_{k-1}\big]\big|\Big) \\
&\leq \sum_{k=1}^{n} E\Big(|a_{nk}| \big|E\big[X_k I_{\left(|X_k| > |a_{nk}|^{-1/2}\right)} \mid \Im_{k-1}\big]\big|\Big) \quad \text{(by Relation (3.5))} \\
&\leq \sum_{k=1}^{n} |a_{nk}| E\big[X_k I_{\left(|X_k| > |a_{nk}|^{-1/2}\right)} \mid \Im_{k-1}\big] \\
&\to 0 \quad \text{as} \quad n \to \infty \qquad \text{(by condition (A2))}.
\end{aligned}
$$

Hence $\sum_{k=1}^{n} E(X_{nk} \mid \Im_{k-1}) \to 0$ in L_1 as $n \to \infty$. Also from relation (3.4), we have $E \mid Z_n| \to 0$ as $n \to \infty$.
i.e.

$$E\Big[\Big|\sum_{k=1}^{n} X_{nk}\Big|\Big] \to 0 \quad \text{as} \quad n \to \infty.$$

i.e.

$$E\Big[\Big|\sum_{k=1}^{n} a_{nk} X_k I_{\left(|X_k| \leq |a_{nk}|^{-1/2}\right)}\Big|\Big] \to 0 \quad \text{as} \quad n \to \infty. \tag{3.6}$$

Thus

$$\begin{aligned} E(\,|\,S_n|\,) &= E\Big[\Big|\sum_{k=1}^{n} a_{nk}X_k\Big|\Big] \\ &= E\Big[\Big|\sum_{k=1}^{n} a_{nk}X_k I_{\left(|X_k|\le|a_{nk}|^{-1/2}\right)} + \sum_{k=1}^{n} a_{nk}X_k I_{\left(|X_k|>|a_{nk}|^{-1/2}\right)}\Big|\Big] \\ &\le E\Big[\Big|\sum_{k=1}^{n} a_{nk}X_k I_{\left(|X_k|\le|a_{nk}|^{-1/2}\right)}\Big|\Big] + \sum_{k=1}^{n} a_{nk}E\Big[|X_k| I_{\left(|X_k|>|a_{nk}|^{-1/2}\right)}\Big] \\ &\quad \to 0 \quad \text{as} \quad n\to\infty \quad \text{(using condition (A2) and realtion (3.6))}. \end{aligned}$$

Theorem 3. *Let* $1 < p < 2$ *and* $\{X_n, n \ge 1\}$ *be a martingale difference sequence satisfying condition* (A2). *Then* $E\lfloor\,|\,S_n|^p\rfloor \to 0$ *as* $n \to \infty$, *where* $S_n = \sum_{k=1}^{n} a_{nk}X_k$.

Proof. Using Burkholder's inequality (cf. [7, Th. 2.10]), we have

$$\begin{aligned} E\big[|S_n|^p\big] &= E\Big[\Big|\sum_{k=1}^{n} a_{nk}X_k\Big|^p\Big] \\ &\le cE\Big[\Big|\sum_{k=1}^{n} a_{nk}^2X_k^2\Big|^{p/2}\Big] \text{ (where } c \text{ denotes a generic constant)} \\ &= cE\Big[\Big|\sum_{k=1}^{n} Y_{nk}^2 + \sum_{k=1}^{n} Z_{nk}^2\Big|^{p/2}\Big] \end{aligned}$$

where

$$Y_{nk} = a_{nk}X_k I_{\left(|X_k|\le|a_{nk}|^{-1/2}\right)} \quad \text{and} \quad Z_{nk} = a_{nk}X_k I_{\left(|X_k|>|a_{nk}|^{-1/2}\right)} \quad \text{for} \quad n \ge 1.$$

So

$$\begin{aligned} E\big[|S_n|^p\big] &\le cE\Big[\Big|\sum_{k=1}^{n} Y_{nk}^2\Big|^{p/2}\Big] + cE\Big[\Big|\sum_{k=1}^{n} Z_{nk}^2\Big|^{p/2}\Big] \\ &\le cE\Big[\Big|\sum_{k=1}^{n} Y_{nk}^2\Big|^{p}\Big] + cE\Big[\Big|\sum_{k=1}^{n} Z_{nk}^2\Big|^{p}\Big]. \end{aligned} \tag{3.7}$$

Now

$$\begin{aligned} E\big[\sum_{k=1}^{n} |\,Y_{nk}|^p\big] &= E\Big[\sum_{k=1}^{n} \big|a_{nk}X_k I_{\left(|X_k|\le|a_{nk}|^{-1/2}\right)}\big|^p\Big] \\ &= \sum_{k=1}^{n} |a_{nk}|^p E\big[|X_k|^p\big] I_{\left(|X_k|\le|a_{nk}|^{-1/2}\right)} \end{aligned}$$

$$= p \sum_{k=1}^{n} |a_{nk}|^p \int_{(0<x\leq|a_{nk}|^{-1/2})} P\big[|X_k| > x\big] dx$$

$$= p \sum_{k=1}^{n} |a_{nk}|^{1/2} \Big[\frac{1}{|a_{nk}|^{-1/2}}\Big] \int_{(0<x\leq|a_{nk}|^{-1/2})} |a_{nk}|^{p-1} x^{p-1} P\big[|X_k| > x\big] dx$$

$$< pM\varepsilon \quad \text{(using Remark 2 and Lemma 2).}$$

Again $E\big[\sum_{k=1}^{n} \mid Z_{nk} \mid^p\big] \to 0$ as $n \to \infty$ by condition (A2).
Hence the result follows from relation (3.7).

Theorem 4. $\{X_k, \Im_k, k \geq 1\}$ *be a martingale difference sequence having transforming sequence* $\{\nu_k, k \geq 1\}$ *with* $\sup \mid \nu_k| < \infty$, *and* X_k *satisfying the condition* (A2). *Then for* $S'_n = \sum_{k=1}^{n} a_{nk}\nu_k X_k$ *and* $0 < p < 2$, $E\big[|S'_n|^p\big] \to 0$ *as* $n \to \infty$.

Proof. Fix $L > 0$. Let

$$t = \begin{cases} \inf\big\{n \geq 1 : \sum_{k=1}^{n} X_k \geq L\big\} \\ \infty, \quad \text{if no such } n \text{ exists} \end{cases}$$

and $R_k = \nu_k X_k I_{(t\geq k)} I_{(|\nu_k|\leq L)}$ for $k \geq 1$. Here $\nu_k I_{(t\geq k)} I_{(|\nu_k|\leq L)}$ is $\Im_{k-1}$-measurable.

Hence

$$E[R_k|\Im_{k-1}] = E\lfloor \nu_k X_k I_{(t\geq k)} I_{(|\nu_k|\leq L)} |\Im_{k-1}\rfloor = \nu_k I_{(t\geq k)} I_{(|\nu_k|\leq L)} E[X_k|\Im_{k-1}] = 0$$

(since X_k is a martingale difference sequence).

So, $\{R_k, k \geq 1\}$ is a martingale difference sequence.

But since $|R_k| = |\nu_k X_k I_{(t\geq k)} I_{(|\nu_k|\leq L)}| \leq L|X_k I_{(t\geq k)}| \leq L|X_k|$ and X_k satisfies condition (A2), $|R_k|$ also satisfies condition (A2) by Lemma 1.
Hence

$$E\Big[\Big|\sum_{k=1}^{n} a_{nk} R_k\Big|^p\Big] \to 0 \quad \text{as} \quad n \to \infty \quad \text{for} \quad 0 < p < 2 \tag{3.8}$$

(by Theorems 1, 2 and 3.), and

$$E\Big[\Big|\sum_{k=1}^{n} a_{nk} X_k\Big|^p\Big] \to 0 \quad \text{as} \quad n \to \infty \tag{3.9}$$

(by Theorems 1, 2 and 3.).

Since $\sup|\nu_k| < \infty$, for $0 < p \leq 1$ we have

$$E\big[|S'_n|^p\big] = E\Big[\Big|\sum_{k=1}^{n} a_{nk}\nu_k X_k\Big|^p\Big]$$

$$\leq E\Big[\Big|\sum_{k=1}^{n} a_{nk}\nu_k X_k I_{(t\geq k)} I_{(|\nu_k|\leq L)}\Big|^p\Big] + E\Big[\Big|\sum_{k=1}^{n} a_{nk}\nu_k X_k I_{(t<k)} I_{(|\nu_k|\leq L)}\Big|^p\Big]$$

$$
\begin{aligned}
&\le E\Big[\Big|\sum_{k=1}^{n} a_{nk}R_k\Big|^p\Big] + L^p E\Big[\Big|\sum_{k=1}^{n} a_{nk}X_k I_{(t<k)}\Big|^p\Big] \\
&\le E\Big[\Big|\sum_{k=1}^{n} a_{nk}R_k\Big|^p\Big] + L^p E\Big[\Big|\sum_{k=1}^{n} a_{nk}X_k\Big|^p\Big] \\
&\quad \to 0 \quad \text{as} \quad n \to \infty \quad \text{(by relations (3.8) and (3.9))}. \qquad (3.10)
\end{aligned}
$$

Now for $1 < p < 2$, we have

$$
\begin{aligned}
E\big[|S_n'|^p\big] &= E\Big[\Big|\sum_{k=1}^{n} a_{nk}\nu_k X_k\Big|^p\Big] \\
&\le \Big\{E^{1/p}\Big[\Big|\sum_{k=1}^{n} a_{nk}\nu_k X_k I_{(t\ge k)} I_{(|\nu_k|\le L)}\Big|^p\Big] \\
&+ E^{1/p}\Big[\Big|\sum_{k=1}^{n} a_{nk}\nu_k X_k I_{(t<k)} I_{(|\nu_k|\le L)}\Big|^p\Big]\Big\}^p \\
&\le \Big\{E^{1/p}\Big[\Big|\sum_{k=1}^{n} a_{nk}R_k\Big|^p\Big] + LE^{1/p}\Big[\Big|\sum_{k=1}^{n} a_{nk}X_k I_{(t<k)}\Big|^p\Big]\Big\}^p \\
&\le \Big\{E^{1/p}\Big[\Big|\sum_{k=1}^{n} a_{nk}R_k\Big|^p\Big] + LE^{1/p}\Big[\Big|\sum_{k=1}^{n} a_{nk}X_k\Big|^p\Big]\Big\}^p \\
&\quad \to 0 \quad \text{as} \quad n \to \infty \quad \text{(by relations (3.8) and (3.9))}. \qquad (3.11)
\end{aligned}
$$

Hence the result follows by combining (3.10) and (3.11).

4. L_p-Convergence of Mixing and Mixingale Difference Sequences

Theorem 5. *Let $\{X_k, k \ge 1\}$ be a $*$-mixing sequence with respect to a function φ and an integer M such that $E[X_k] = 0$ and $E[|X_k|] \le K < \infty$ for each k. Further suppose that $\{X_k, k \ge 1\}$ satisfies condition* (A2) *for some $0 < p < 2$. Then $E[|S_n|^p] \to 0$ as $n \to \infty$ and for $0 < p < 2$, where $S_n = \sum_{k=1}^{n} a_{nk}X_k$.*

Proof. Fix $\varepsilon > 0$. As in the proof of Theorem 1

$$
\big|E[X_{nM_1+k}|X_{(n-1)M_1+k}, X_{(n-2)M_1+k}, \ldots, X_{M_1+k}]\big| \le \varepsilon K \qquad (4.1)
$$

for M_1 sufficiently large .

Now fix $0 \le k \le M_1$. Let $\mathcal{G}_n = \mathcal{B}(X_{nM_1+k}, X_{(n-1)M_1+k}, X_{(n-2)M_1+k}, \ldots, X_{M_1+k})$ for each $n \ge 1$ and $\mathcal{G}_0 = \{\phi, \Omega\}$.

Denote $Z_{nM_1+k} = X_{nM_1+k} - E(X_{nM_1+k}\,|\,\mathcal{G}_{n-1})$. So $\{Z_{nM_1+k}, \mathcal{G}_n, n \ge 2\}$ is a martingale difference sequence.

As $\big\{|X_{nM_1+k}|^p,\ n \ge 1\big\}$ satisfies condition (A2), then so also $\big\{E\big(|X_{nM_1+k}|^p\,|\mathcal{G}_{n-1}\big), n \ge 1\big\}$ by Lemma 1(iii).

But by Lemma 1(ii) $\big\{|Z_{nM_1+k}|^p, n \ge 1\big\}$ satisfies condition (A2) for $0 < p < 2$. Thus by Theorems 1, 2 and 3

$$E\Big|\sum_{n=2}^{N} a_{N,n} Z_{nM_1+k}\Big|^p \to 0 \quad \text{as} \quad N\to\infty \quad \text{for} \quad 0<p<2.$$

i.e.,

$$E\Big[\Big|\sum_{n=2}^{N} a_{N,n} X_{nM_1+k} - \sum_{n=2}^{N} a_{N,n} E\big[X_{nM_1+k}\,|\,\mathcal{G}_{n-1}\big]\Big|^p\Big] \to 0$$

as $N\to\infty$ for $0<p<2$ and $0\le k\le M_1$.

Since $\varepsilon>0$ was arbitrarily chosen, by using relation (4.1),

$$E\Big|\sum_{n=2}^{N} a_{N,n} X_{nM_1+k}\Big|^p \to 0 \quad \text{as} \quad N\to\infty \quad \text{for} \quad 0<p<2$$

and hence the result follows.

Theorem 6. *Let $\{(X_n, n\ge 1), \Im_n : n=0,\pm1,\pm2,\dots\}$ be an L_p-mixingale difference sequence and $\{|X_n|^p, n\ge 1\}$ satisfy condition* (A2) *for $1\le p<2$. Further assume that $\limsup_n |\sum_{k=1}^n a_{nk}c_k| < \infty$, for c_k as in the definition of mixingale difference sequence. Then $E\lfloor |S_n|^p\rfloor \to 0$ as $n\to\infty$ for $1\le p<2$, where $S_n=\sum_{k=1}^n a_{nk}X_k$.*

Proof. For $n\ge 1$ and $i=0,\pm1,\pm2,\dots$, let

$$Y_{ni} = E[X_i|\Im_{n+i}] - E\big[X_i|\Im_{n+i-1}\big].$$

So $E[Y_{ni}|\Im_{n+i-1}]=0$ and hence $\{Y_{ni}, \Im_{n+i}, n\ge 1\}$ is a martingale difference sequence for each i.

Define $S_{ni}=\sum_{k=1}^n a_{nk}Y_{ki}$. So by Theorems 1, 2 and 3

$$E\lfloor |S_{ni}|^p\rfloor \to 0 \quad \text{for} \quad 1\le p<2.$$

Therefore

$$\begin{aligned}
\Big\|\sum_{k=1}^n a_{nk}X_k\Big\|_p &= \Big\|\sum_{k=1}^n a_{nk}\big[X_k - E(X_k\,|\,\Im_{k+m})\big] \\
&+ \sum_{k=1}^n a_{nk}E[X_k\,|\,\Im_{k-m}] + \sum_{i=-m+1}^{m} S_{ni}\Big\|_p \\
&\le \Big\|\sum_{k=1}^n a_{nk}\big[X_k - E(X_k\,|\,\Im_{k+m})\big]\Big\|_p + \Big\|\sum_{k=1}^n a_{nk}E\big[X_k\,|\,\Im_{k-m}\big]\Big\|_p + \Big\|\sum_{i=-m+1}^{m} S_{ni}\Big\|_p \\
&\le \sum_{k=1}^n a_{nk}\big\|\big[X_k - E(X_k\,|\,\Im_{k+m})\big]\big\|_p + \sum_{k=1}^n a_{nk}\big\|E\big[X_k\,|\,\Im_{k-m}\big]\big\|_p + \Big\|\sum_{i=-m+1}^{m} S_{ni}\Big\|_p \\
&\le \sum_{k=1}^n a_{nk}c_k\psi_{m+1} + \sum_{k=1}^n a_{nk}c_k\psi_m + \Big\|\sum_{i=-m+1}^{m} S_{ni}\Big\|_p.
\end{aligned}$$

Now since $\limsup_n \left|\sum_{k=1}^n a_{nk}c_k\right| < \infty$, $\psi_m \to 0$ as $m \to \infty$ and $E\big[|S_{ni}|\big]^p \to 0$ as $n \to \infty$, the result follows.

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