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Degree of C^0-Sufficiency of an Analytic Germ with Respect to a Principal Ideal

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Abstract. We give formulas for the degree of C^0-sufficiency of an analytic germ with respect to a principal ideal.

1. Introduction

Let $\mathbb{C}\{x, y\}$ be the ring of germs of analytic functions of two complex variables, and $f, g \in \mathbb{C}\{x, y\}$. We say that f and g are *topologically equivalent* if there exists a homeomorphism $\phi\colon (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ such that $f \circ \phi = g$.

Let $I \subset \mathbb{C}\{x, y\}$ be an ideal in $\mathbb{C}\{x, y\}$. A germ f is said to be *C^0-sufficient of order r with respect to I*, if for any g such that $f - g \in I^{r+1}$, the germs f and g are topologically equivalent.

We call

$$\mathrm{Suff}_I(f) = \min\{r \mid f \text{ is } C^0\text{-sufficient w.r.t. } I\}$$

the *degree of C^0-sufficiency of f w.r.t. I.*

Problem. *How to compute* $\mathrm{Suff}_I(f)$?

In the case when I is a maximal ideal of $\mathbb{C}\{x, y\}$, this problem has been solved by Kuo and Lu [1]. In that case, the degree of C^0-sufficiency is equal to $[\mathcal{L}] + 1$, where $\mathcal{L}$ is the Lojasiewicz number of f at the origin. Kuo and Lu also gave a formula for $\mathcal{L}$ in terms of Puiseux's expansions of f. In [3] Lê and Weber computed the Lojasiewicz number via the data of the resolution tree of f.

In this paper, we give a formula for $\mathrm{Suff}_I(f)$ in the case when I is a principal ideal, $I = (g)\mathbb{C}\{x, y\}$. We compute $\mathrm{Suff}_I(f)$ in several ways: in terms of the jacobian quotients of the map $\Phi = (f, g)$; in terms of the resolution tree of the germ $fg = 0$ and finally; in terms of intersection multiplicities of germs $f^{-1}(0)$ and $g^{-1}(0)$ with jacobian curve.

To do this, we follow the method of Lê and Weber [3] and we use results of Maugendre on the jacobian quotients [4, 5].

2. Statement of Results

We always suppose that

$$f^{-1}(0) \cap g^{-1}(0) = \{0\}.$$

Let us denote

$$J(f,g) = \det \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}.$$

Let $J(f,g) = q_1^{s_1} \dots q_k^{s_k}$ be a decomposition of $J(f,g)$ into irreducible factors, and $J(f,g)_{red}$ be the product of those q_j which are not factors of fg. Write

$$\Sigma = \{(x,y) \in \mathbb{C}^2 \mid J(f,g)_{\text{red}} = 0\}.$$

We consider the map

$$\Phi \colon (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0), \quad (x,y) \mapsto (u = f(x,y); v = g(x,y)).$$

Let

$$\Delta = \Phi(\Sigma).$$

For any irreducible component δ of Δ, let $u = av^{\frac{m_\delta}{n_\delta}} + \cdots$ be the Puiseux expansion of δ. Following Maugendre [4], the numbers $\frac{m_\delta}{n_\delta}$ will be called *jacobian quotients* of Φ.

Theorem 1. *If a germ $f(x,y)$ has at most an isolated singularity and I is the ideal generated by a germ $g(x,y)$, then*

$$\text{Suff}_I(f) = \max\Big\{ \Big[\frac{m_\delta}{n_\delta}\Big] \mid \delta \in \Delta \Big\}.$$

Let π be the minimal resolution of $fg = 0$. Let $\pi^{-1}(0)$ be the set of exceptional divisors. An exceptional divisor E is said to be "rupture", if E intersects with at least three irreducible components of $\pi^{-1}((f.g)^{-1}(0))$. For every E we denote by $C(E)$ a germ of a curve in $(\mathbb{C}^2, 0)$, whose strict transformation intersects E transversally.

Corollary 1. *Under the conditions of Theorem* 1, *we have*

$$\text{Suff}_I(f) = \max\Big\{ \frac{(f^{-1}(0), C(E))_0}{(g^{-1}(0), C(E))_0} \Big\},$$

where $(\ ,\ )_0$ *is the intersection multiplicity at* 0 *of two curves and the maximum is taken over all the rupture components* E.

Corollary 2. *With the assumptions of Theorem* 1*, we have*

$$\mathrm{Suff}_I(f) = \max \frac{(f^{-1}(0), q_i^{-1}(0))_0}{(g^{-1}(0), q_i^{-1}(0))_0},$$

where the maximum is taken over all irreducible factors of $J_{red}(f, g)$*.*

3. Proofs

Let $h_t(x, y) = f(x, y) - tg(x, y), t \in \mathbb{C}$, and V_t be the germ of $h_t^{-1}(0)$ at the origin. We denote by $B(f, g)$ the set of all the special values of the parametry of the family $V_t : B(f, g)$ consists of all values $t_0 \in \mathbb{C}$, at which the family is not topologically equisingular. First, we are interested in finding the set $B(f, g)$ for given f and g. This problem was solved by Lê and Weber in [3]. Here, it is more convenient for us to characterize $B(f, g)$ using jacobian curve.

Let us denote by $\mathcal{P}$ the set of all Puiseux expansions of the curve $J(f, g)(x, y) = 0$. Each element of $\mathcal{P}$ can be written in the form $(x(s), y(s))$, where $x(s)$ and $y(s)$ are convergent series in s.

Lemma 1.

$$B(f, g) = \Big\{ t_0 \in \mathbb{C} \,|\, \exists (x(s), y(s)) \in \mathcal{P},\, t_0 = \lim_{s \to 0} \frac{f(x(s), y(s))}{g(x(s), y(s))} \Big\}.$$

Proof. Assume that $t_0 = \lim_{s\to 0} \frac{f(x(s),y(s))}{g(x(s),y(s))}$ and $q_1((x(s), y(s)) = 0$. There are two cases:
(a) $h_{t_0}(x(s), y(s)) \equiv 0$; and
(b) Otherwise.

Case (a). We can write $y = y(x)$ as the Puiseux expansions $(x(s), y(s))$ which give $t_0 = \lim_{s\to 0} \frac{f(x,y(x))}{g(x,y(x))}$ and $h_{t_0}(x, y(x)) \equiv 0$. It implies

$$\frac{\partial f}{\partial x}(x, y(x)) + \frac{\partial f}{\partial y}(x, y(x)).\dot{y}(x) - t_0 \frac{\partial g}{\partial x}(x, y(x)) - t_0 \frac{\partial g}{\partial y}(x, y(x)).\dot{y}(x) \equiv 0.$$

Since $(x, y(x)) \in q_1^{-1}(0) \subset J(f, g)^{-1}(0)$,

$$\lambda(x) \text{ grad } f(x, y(x)) = \text{grad } g(x, y(x))$$

for some $\lambda(x) \in \mathbb{C}$.

We get then

$$(1 - t_0\lambda(x)) \Big[\frac{\partial f}{\partial x}(x, y(x)) + \frac{\partial f}{\partial y}(x, y(x)).\dot{y}(x) \Big] \equiv 0.$$

If $(1 - t_0\lambda(x)) \not\equiv 0$ then

$$\frac{\partial f}{\partial x}(x, y(x)) + \frac{\partial f}{\partial y}(x, y(x)).\dot{y}(x) = 0$$

and

$$\frac{\partial g}{\partial x}(x, y(x)) + \frac{\partial g}{\partial y}(x, y(x)).\dot{y}(x) = 0.$$

Therefore $f(x, y(x)) = g(x, y(x)) = 0$, which means that $q_1^{-1}(0) \subset g^{-1}(0) \cap f^{-1}(0)$, what contradicts to $g^{-1}(0) \cap f^{-1}(0) = \{0\}$.

Thus $(1 - t_0\lambda(x)) \equiv 0$, and consequently $\text{grad}h_{t_0}(x, y(x)) \equiv 0$. The curve V_{t_0} has non-isolated singularity at $0 \in \mathbb{C}^2$. Hence, $t_0 \in B(f, g)$.

Case (b). $f(x(s), y(s)) - t_0 g(x(s), y(s)) \not\equiv 0$.

Let U be a sufficiently small ball centered at $0 \in \mathbb{C}^2$ such that $U \cap V_{t_0} \cap J(f, g)^{-1}(0) = \{0\}$. Put $t(s) = \frac{f(x(s),y(s))}{g(x(s),y(s))}$; i.e., $(x(s), y(s)) \in V_{t(s)}$.

Consider the following map

$$\pi_g^{t(s)}: U \cap V_{t(s)} \setminus \{0\} \to \mathbb{C}, \quad (x, y) \mapsto g(x, y).$$

If there exists a sequence of singular points of $V_{t(s)}$, going to $0 \in \mathbb{C}^2$ as $s \to 0$, then it must be that $\mu(V_{t_0}, 0) > \mu(V_{t(s)}, 0)$, i.e., the Milnor number is not constant at t_0. We have then $t_0 \in B(f, g)$. Assume that $V_{t(s)} \setminus \{0\}$ is nonsingular. We see that the map $\pi_g^{t_0}$ is a finite covering without ramification, while $\pi_g^{t(s)}$ is ramified over the value $\pi_g^{t(s)}(x(s), y(s))$. From this fact we can conclude that $\chi(V_{t_0} \setminus \{0\}) > \chi(V_{t(s)} \setminus \{0\})$, where $\chi(.)$ is the Euler characteristics. It implies $t_0 \in B(f, g)$.

We have shown that if $t_0 = \lim_{s\to 0} \frac{f(x(s),y(s))}{g(x(s),y(s))}$ for some $(x(s), y(s)) \in \mathcal{P}$ then $t_0 \in B(f, g)$. Now we are going to prove that the converse is also true.

Assume that $t_0 \neq \lim_{s\to 0} \frac{f(x(s),y(s))}{g(x(s),y(s))}$ for any $(x(s), y(s)) \in \mathcal{P}$. Then there exist a disc $D \subset \mathbb{C}, t_0 \in D$ and a ball $U \subset \mathbb{C}^2$, such that $U \cap J(f, g)^{-1}(0) \cap V_t = \{0\}$ for every $t \in D$. This means that if $(x, y) \in U$, the vectors $\text{grad}f(x, y)$ and $\text{grad}g(x, y)$ are linearly independent. Moreover, if the ball U and the disc D are sufficiently small, the curve V_t intersects with ∂U transversally, and this holds for every $t \in D$. Using these facts, we can construct a vector field whose integral curve $(x(\tau), y(\tau))$ have the following properties

(i) $g(x(\tau), y(\tau)) = \text{const}$;
(ii) $(x(\tau), y(\tau)) \in V_{t(\tau)}$; and
(iii) If $(x(0), y(0)) \in \partial U$ then $(x(\tau), y(\tau)) \in \partial U$.

One can see that the flow of this vector field will give a smooth trivialization of the family $V_t \setminus \{0\}, t \in D$, and consequently, it induces a topologically trivialization of $V_t, t \in D$. Thus $t_0 \notin B(f, g)$. The lemma is proved. ■

Lemma 2. *Under the condition of Theorem* 1*, the set* $B(f, g)$ *is empty if and only if every jacobian quotient of* (f, g) *is less than* 1.

Proof. By an argument similar to that in the proof of Lemma 1, we can show that if $f^{-1}(0)$ has an isolated singularity at $0 \in \mathbb{C}^2$, then $J(f, g)^{-1}(0) \cap f^{-1}(0) = \{0\}$ in a small neighborhood of 0. Let $q_i(x, y)$ be an irreducible factor of $J(f, g)$, and $y = y_i(x)$ its Puiseux expansion. Assume that $q_i(x, y)$ is not a factor of $g(x, y)$.

We denote by $v(f(x,y_i(x))$ and $v(g(x,y_i(x))$ the valuations of corresponding series. By a classical formula on the intersection multiplicities, we have

$$v(f(x,y_i(x)) = (f^{-1}(0) \cap q_i^{-1}(0))_0$$

and

$$v(g(x,y_i(x)) = (g^{-1}(0) \cap q_i^{-1}(0))_0.$$

By [4, Lemma 1.1] the number $\frac{(f^{-1}(0)\cap q_i^{-1}(0))_0}{(g^{-1}(0)\cap q_i^{-1}(0))_0}$ belongs to the set of jacobian quotients. The condition (ii) gives then $\frac{v(f(x,y_i(x))}{v(g(x,y_i(x))} < 1$. It follows from Lemma 1, that every component $q_i(x,y)$, which is not a factor of $g(x,y)$, does not contribute any special value. In particular, we have $0 \notin B(f,g)$. Using again Lemma 1, we can see that if $0 \notin B(f,g)$ then the map $t_0 \in B(f,g) \mapsto \frac{1}{t_0} \in B(g,f)$ is bijective.

Assume now that $q_i(x,y)$ is a factor of $g(x,y)$. Then, as in the proof of Lemma 1, we can show that $B(g,f) = \{0\}$. Hence, by the above bijection, $B(f,g) = \emptyset$, i.e., the conditions (i) - (ii) are sufficient for the family V_t to be equisingular. It follows from Lemma 1 that they are also nessecary for $B(f,g)$ to be empty. ■

Proof of Theorem 1. Let

$$l = \max\Big\{ \Big[\frac{m_\delta}{n_\delta}\Big], \delta \in \Delta \Big\}.$$

(A) Let $k \geq l+1, I = (g(x,y))\mathbb{C}\{x,y\}$, and $r(x,y) \in I^k$. We have to show that $f(x,y)$ and $f(x,y)+r(x,y)$ have the same topologically type. To do this, by [2], it suffices to show that the family

$$V_t = \{f(x,y) - tr(x,y) = 0\}$$

is equisingular for $t \in \mathbb{C}$.

Case 1. $r(x,y) = g(x,y)^k$. A number α is an jacobian quotient of (f,g) if and only if $\frac{\alpha}{k}$ is that of (f,g^k). Thus, if $k > l$ then $1 > \frac{m_\delta}{n_\delta}$ for any component $\delta \in \Delta$ and the equisingularity follows from Lemma 2.

Case 2. $r(x,y) = g^{l+1}(x,y)g_1(x,y)$, where $g_1(x,y)$ and $f(x,y)$ have no common factor.

Case 2.1. Additionally, we suppose that $f.r$ and $f.g^{l+1}$ have the same minimal resolution π. For an exceptional divisor $D \in \pi^{-1}(0)$, let us denote by $v_D(.)$ the multiplicity on D. We see that

$$\frac{v_D(f\circ\pi)}{v_D(r\circ\pi)} = \frac{v_D(f\circ\pi)}{v_D(g^{l+1}\circ\pi) + v_D(g_1\circ\pi)} \leq \frac{v_D(f\circ\pi)}{v_D(g^{l+1}\circ\pi)}.$$

The numbers $\frac{v_D(f\circ\pi)}{v_D(g^{l+1}\circ\pi)}$ and $\frac{v_D(f\circ\pi)}{v_D(r\circ\pi)}$ are called contact quotients (see [3]). According to ([5, Theorem 1.1]), each contact quotient at a rupture vertex belongs to the set of jacobian quotients. From these facts it is easy to see that every jacobian quotient of (f,r) is less than 1, and the equisingularity of the family $f(x,y) - tg(x,y) = 0, t \in \mathbb{C}$, follows from Lemma 2.

Case 2.2. The minimal resolutions of $f.g^{l+1}$ and $f.r$ may be different.

To investigate this case, we do use the result of Maugendre on the behavior of contact quotients on a colored resolution tree of a product of two germs. Let $A_c(f.r)$ be the colored resolution tree of $f.g$ (we refer [5] for this notion). The minimal resolution π' of $f.r$ can be obtained from the minimal resolution π of $f.g^{l+1}$ by making some additional blowing-ups in order to solve the singularities caused by the factor $g_1(x,y)$. According to [5], the part of $A_c(f.r)$, corresponding to these additional blowing-ups must be colored in green color. Maugendre proved that the contact quotients are decreasing if we are following a green path in a geodesic direction [5]. This implies

$$\max_{D'\in\pi'^{-1}(0)} \frac{v_{D'}(f\circ\pi')}{v_{D'}(r\circ\pi')} \le \max_{D\in\pi^{-1}(0)} \frac{v_D(f\circ\pi)}{v_D(g^{l+1}\circ\pi)}.$$

Hence every jacobian quotient of (f,r) is less than 1. The equisingularity of considered family is proved.

Case 3. $r(x,y) = g^{l+1}(x,y)g_1(x,y)$, and f and g_1 have a common factor: $f(x,y) = \varphi(x,y)f_1(x,y), g_1(x,y) = \varphi(x,y)g_2(x,y)$. The equisingularity of the family $f(x,y) - tr(x,y) = 0$ is then equivalent to that of the family

$$f_1(x,y) - tg^{l+1}(x,y)g_2(x,y) = 0.$$

In comparing the resolution π of $f.g^{l+1}$ and π_1 of $f_1.g^{l+1}$ and by an argument similar to that of the Case 2.2, we can show that

$$\max_{D_1\in\pi_1^{-1}(0)} \frac{v_{D_1}(f_1\circ\pi_1)}{v_{D_1}((g^{l+1}g_2)\circ\pi')} \le \max_{D\in\pi^{-1}(0)} \frac{v_D(f\circ\pi)}{v_D(g^{l+1}\circ\pi)} < 1.$$

The equisingularity of the family is then followed from Lemma 2.

(B) If $k \le l$, we take $r(x,y) = g^k(x,y)$. It is clear that there exists $\delta \in \Delta$ such that $\frac{m_\delta}{kn_\delta} \ge 1$. Hence, by Lemma 2, the family $f(x,y) - tg^k(x,y)$ can not be equisingular; and therefore $f(x,y)$ and $f(x,y) - t_0 g^k(x,y)$, for $t_0 \in B(f,g^k)$, are not topologically equivalent.

Proof of Corollary 1. It follows from Theorem 1 and ([5, Theorem 1.1]).

Proof of Corollary 2. It follows from Lemma 1 and Theorem 1.

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