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On Subnormal Subgroups of the Multiplicative Group of a Division Ring

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Abstract. Let D be a centrally finite division ring (i.e. D is a finite dimensional vector space over its center F). We say that a subgroup H of $D^* := GL_1(D)$ is *radical over* F if for any $x \in H$ there exists a positive integer $n(x)$ such that $x^{n(x)} \in F$. In this paper we prove that, if a subnormal subgroup H of D^* is radical over F then H is central. This is a generalization of some Herstein's result.

Let D be a division ring with the center F. In 1978, Herstein [2] conjectured that given a normal subgroup H of D^*, if H is radical over F then H is central. He showed that the conjecture is true if D is centrally finite, but in general it is still open. In this paper, we replace "normal" by "subnormal" in the above conjecture and show that the resulting conjecture is also true in the finite dimensional case. Our main result is the following theorem:

Theorem 1. *Let D be a centrally finite division ring with the center F. If a subnormal subgroup S of D^* is radical over F then S is central.*

To prove this theorem we need some known results. For the convenience we list them here.

Theorem A. [7, p. 440] *Every solvable subnormal subgroup of the multiplicative group of a division ring is central.*

Theorem B. [6, Lemma 3] *Let D be a centrally finite division ring. Then every solvable subgroup of D^* has an abelian normal subgroup of finite index.*

From Theorem 1 in [8] it follows the following:

Theorem C. *Let K be a field of characteristics zero. If G is a subgroup of $GL_n(K)$ then either G contains non-abelian free subgroup or it contains a solvable subgroup of a finite index.*

Theorem D. [4, Th. (9.9'), p. 154] *A subgroup $G \leq GL_n(K)$ over a field K is torsion if and only if it is locally finite.*

Theorem E. (Poincaré) [3, Th. (13.2.2), p. 83] *Let G be a group and H be its subgroup of index m. Then H contains a normal subgroup N of G such that*

$$m | [G : N] \text{ and } [G : N] | m!.$$

Beside listing above theorems, we need also the following result:

Theorem 2. *Let D be a division ring with the center F and S be a subnormal subgroup of D^*. If the center $Z(S)$ of S is the subgroup of a finite index in S then S is central.*

Proof. Writing

$$S = Z(S)x_1 \cup Z(S)x_2 \cup \ldots \cup Z(S)x_n \tag{1}$$

for some $x_1 = 1, x_2, \ldots, x_n \in S$, where $\{x_1, x_2, \ldots, x_n\}$ is a complete system of representative elements of all cosets. Then we have $S = Z(S).H$, where $H := \langle x_1, x_2, \ldots, x_n \rangle$ is the subgroup of S generated by $x_1, x_2, \ldots, x_n$. Since $Z(S)$ is an abelian normal subgroup of S, it is subnormal in D^*. Hence $Z(S) \subseteq F$ by Theorem A. Set

$$L := \{\sum_{i=1}^{n} f_i x_i |\ f_i \in F\}.$$

First, we prove that L is a division subring of D. Clearly L is closed under the additive operation. For the multiplicative operation, note that in view of (1) for arbitrary i and j with $1 \leq i, j \leq n$, there exists some $k \in \{1, \ldots, n\}$ such that $x_i x_j \in Z(S)x_k \subseteq Fx_k$. It follows that L is also closed under the multiplicative operation. Since the set $\{x_1, x_2, \ldots, x_n\}$ contains the identity $1, L$ is a subring of D. Moreover, $F \subseteq L$ and clearly L is a finite dimensional vector space over F. Now, let us consider an arbitrary non-zero element $a \in L$. Since $[L : F] < \infty$, there exists some positive integer m such that the identity 1 can be expressed in the form

$$1 = f_1 a + f_2 a^2 + \ldots + f_m a^m$$

with $f_1, f_2, \ldots, f_m \in F$. Now $1 = a(f_1 + f_2 a + \ldots + f_m a^{m-1}) = (f_1 + f_2 a + \ldots + f_m a^{m-1})a$ shows that the element a is invertible in L. Hence L is a division subring of D. As we have noted above , L is a finite dimensional vector space over F, hence L is a finite dimensional vector space over its center $Z(L)$. Now, since $S = Z(S).H, H$ is normal in S, so H is subnormal in D^* and hence H

is subnormal in L^*. By [**5**, Th. 1] $H \subseteq Z(L)$, hence H is abelian. Now, by Theorem A, $H \subseteq F$ and hence $S \subseteq F$. The proof is complete. ∎

From Corollary (13.24) in [4, p. 225] it follows that, if $[D^* : F^*] < \infty$ then D is a field. Clearly, the result obtained in Theorem 2 is a generalization of the result mentioned above. Note that some other generalization was obtained in [1, Cor. 4].

We need also

Lemma 1. *Let D be a division ring of characteristics $p > 0$. Then every locally finite subgroup of D^* is abelian.*

Proof. Suppose G is a locally finite subgroup of D^* and $x, y \in G$. Then $\langle x, y\rangle$ is the finite subgroup of D^*. By [4, Corollary (13.3), p. 215], $\langle x, y\rangle$ is cyclic. Hence G is abelian. ∎

Now, we are ready to prove Theorem 1.

Proof of Theorem 1. Let F be the center of D. Then the subgroup S can be considered as some linear group of $GL_n(F)$. In fact, for an element $a \in D^*$, the map $L_a : D \to D$, defined by $L_a(x) = ax$, for every $x \in D$, is a F-endomorphism of D. Moreover, L_a is a F-automorphism (with $(L_a)^{-1} = L_{a^{-1}}$). Clearly $a \mapsto L_a$ is a monomorphism from D^* into the multiplicative group of all invertible elements in $End_F(D)$. Since $End_F(D) \simeq M_n(F)$ with $n = dim_F(D)$, D^* can be considered as a subgroup of $GL_n(F)$. Hence the same conclusion holds for every subgroup of D^*, in particular for S. Moreover, it can be supposed that $F^* \subseteq S$. In fact, if it is necessary, S can be replaced by $S_F := F^*S$.

Case 1. char $D = 0$.
By Theorem C either S contains a non-abelian free subgroup or it contains a solvable subgroup of a finite index. Since S is radical over $F, S/F^*$ is a torsion group, so only the last assertion occurs. Thus, suppose G is a solvable subgroup of a finite index in S. By Theorem B there exists in G some abelian normal subgroup T of a finite index. So, T is an abelian subgroup of finite index in S. Hence, by Theorem E, S contains some abelian normal subgroup S_0 of a finite index. Since S is subnormal in D^*, S_0 is subnormal in D^* too. By Theorem A, S_0 is central, hence $S_0 \subseteq Z(S)$. Since $[S : S_0] < \infty$, it follows that $[S : Z(S)] < [S : S_0] < \infty$. Now, by Theorem 2, S is central.

Case 2. char $D = p > 0$.
Put $H := [D^*, D^*] \cap S$. Consider an arbitrary element $x \in H$. Since S is radical over F, there exists a positive integer $n(x)$ such that $a := x^{n(x)} \in F$. Hence $RN_{D/F}(x^{n(x)}) = RN_{D/F}(a) = a^n$, where $n^2 = [D : F]$ and $RN_{D/F}$ denotes the reduced norm of D to F. On the other hand, since $x^{n(x)} \in [D^*, D^*]$, $RN_{D/F}(x^{n(x)})$ $= 1$, so $a^n = 1$ or a is periodic. Hence x is periodic too. Thus, the subgroup H is torsion, hence $[S, S]$ is torsion too. So, by Theorem D, $[S, S]$ is locally finite. Therefore, by Lemma 1, $[S, S]$ is abelian. Hence S is a solvable subnormal subgroup of D^*. Now, by Theorem A, S is central. ∎

Corollary 1. *Let D be a division ring with the center F and S be a subnormal subgroup of D^*. If SF^*/F^* is locally finite then S is central.*

Proof. Let x, y be arbitrary elements in S. Since SF^*/F^* is locally finite, $\langle xF^*, yF^*\rangle$ is finite. Hence $F^*\langle x, y\rangle/F^*$ is finite too. Put $G := F^*\langle x, y\rangle$. Since G/F^* is finite, we can find some elements $u_1 \in F^*, u_2, \dots, u_n \in G$ such that

$$G/F^* = \{u_1F^*, \dots, u_nF^*\}. \tag{2}$$

Putting $K := \{\sum_{i=1}^{n} f_iu_i \mid f_i \in F\}$, we claim that K is centrally finite division subring of D. In fact, K is closed under the additive operation. For the multiplicative operation, note that in view of (2), for arbitrary i and j with $1 \leq i, j \leq n$, there exists some $k \in \{1, \dots, n\}$ such that $u_iu_j \equiv u_k (\operatorname{mod} F^*)$. So, K is also closed under the multiplicative operation. Clearly $F \subseteq K$ and K is a finite dimensional vector space over F. Hence, K is a finite dimensional F-algebra. So, as in the proof of Theorem 2, we can see that every non-zero element of K is invertible. Thus, K is centrally finite division subring of D. Since S is subnormal subgroup in $D^*, S \cap K^*$ is a subnormal in K^*. On the other hand, since SF^*/F^* is locally finite, it is torsion. So S is radical over F and it follows $S \cap K^*$ is radical over $Z(K)$. By Theorem 1, $S \cap K^*$ is central in K. Hence x and y commute with each other. Thus, S is an abelian subgroup of D^*. By Theorem A, S is central. ∎

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