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Convergence to Normal Distribution of Random Sums

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Abstract. Let (X_{nk}) be a double sequence of random variables with zero mean and finite variances and (Z_n) be a sequence of positive integral-valued random variables such that for each n, $Z_n, X_{n1}, X_{n2}, ...$ are independent. In this paper, we give necessary and sufficient conditions for weak convergence of the distribution functions of random sums

$$S_{Z_n} = X_{n1} + X_{n2} + \cdots + X_{nZ_n}$$

to the standard normal distribution function Φ. Moreover, we give a bound of $\sup\limits_{-\infty<x<\infty} |P(S_{Z_n} \leq x) - \Phi(x)|$ and show that it tends to 0 when (S_{Z_n}) converges weakly to Φ.

1. Introduction and Main Results

The convergence of a sequence of distribution functions of random sums was first investigated by Robins [11] in 1948 and has been discussed many times in numerous papers. In this work we investigate the case whose limit distribution function is the standard normal distribution function Φ.

In the case of one array, let (X_n) be a sequence of independent random variables with zero mean (this is not an essential restriction) and finite variances. Let (Z_n) be a sequence of positive integral-valued random variables which are independent of (X_n). Many authors (e.g. [1, 3, 6, 9, 10, 15, 18, 20, 25]) gave conditions for the convergence of the sequence of distribution functions of random sums $X_1 + X_2 + \cdots + X_{Z_n}$ to Φ.

In this work we consider a double array of random variables. Let (X_{nk}) be a double sequence of random variables with zero mean and finite variances σ_{nk}^2. For each n, we assume $Z_n, X_{n1}, X_{n2}, ...$ are independent. In [8, 12, 22], the authors investigated the convergence of the sequence of distribution functions of random sums

$$S_{Z_n} = X_{n1} + X_{n2} + \cdots + X_{nZ_n}$$

in case $X_{n1}, X_{n2}, ...$ are identically distributed for every n. The aim of our investigation the case $X_{n1}, X_{n2}, ...$ are not necessary identically distributed. Before we give the main results we state one of the most important versions of central limit theorem of sums.

Theorem 1.1. ([4, Chap. 12.2]) *Let (k_n) be a sequence of positive integers. Assume that $\lim_{n\to\infty} \sum_{k=1}^{k_n} \sigma_{nk}^2 = 1$. Then*

(i) *the sequence of distribution functions of the sums*

$$S_n = X_{n1} + X_{n2} + \cdots + X_{nk_n}$$

converges weakly to Φ and

(ii) *$(X_{nk}), k = 1, 2, ..., k_n$ is infinitesimal, i.e.*

$$\max_{1\le k\le k_n} P(|X_{nk}| \ge \varepsilon) \to 0$$

for every $\varepsilon > 0$, if and only if $(X_{nk}), k = 1, 2, ..., k_n$, satisfies the Lindeberg condition, i.e.

$$\sum_{k=1}^{k_n} \int_{|x|<\varepsilon} x^2 dF_{nk}(x) \to 1$$

for every $\varepsilon > 0$, where F_{nk} is the distribution function of X_{nk}.

In this work, we will extend Theorem 1.1 to the case of random sums and will find a bound of the estimation. This paper is organized as follows.

In Sec. 2 we give necessary and sufficient conditions for the convergence of sequence of distribution functions of random sums S_{Z_n} to Φ. The following theorems are main results of Sec. 2.

Theorem 1.2. *Let (X_{nk}, Z_n) be such that $\sum_{k=1}^{Z_n} \sigma_{nk}^2 \xrightarrow{p} 1$ and satisfy* **random infinitesimal condition** (RI), *i.e.,*

$$\max_{1\le k\le Z_n} P(|X_{nk}| \ge \varepsilon) \xrightarrow{p} 0$$

for every $\varepsilon > 0$.

Then the sequence of distribution functions of the random sums S_{Z_n} converges weakly to Φ if and only if $K_{Z_n}(u) \xrightarrow{p} K(u)$ for every continuity point u of K, where $K_n(u) = \sum_{k=1}^{n} \int_{-\infty}^{u} x^2 dF_{nk}(x)$ and

$$K(u) = \begin{cases} 0 & \text{for} \quad u < 0 \\ 1 & \text{for} \quad u \geq 0. \end{cases}$$

Theorem 1.3. *Let (X_{nk}, Z_n) be such that $\sum_{k=1}^{Z_n} \sigma_{nk}^2 \xrightarrow{p} 1$ and satisfy* (RI). *Then the sequence of distribution functions of random sums S_{Z_n} converges weakly to Φ if and only if (X_{nk}, Z_n) satisfies* **random Lindeberg condition** (RL), *i.e.,*

$$\sum_{k=1}^{Z_n} \int_{|x|<\varepsilon} x^2 dF_{nk}(x) \xrightarrow{p} 1$$

for every $\varepsilon > 0$.

Theorem 1.4. *Assume that $\sum_{k=1}^{Z_n} \sigma_{nk}^2 \xrightarrow{p} 1$. Then*

(i) *the sequence of distribution functions of random sums S_{Z_n} converges weakly to Φ and*
(ii) *(X_{nk}, Z_n) satisfies* (RI)
if and only if (X_{nk}, Z_n) satisfies (RL).

In Sec. 3, we give a bound of the approximation in Sec. 2. The main theorems of Sec. 3 are the followings.

Theorem 1.5. *If $\sigma_{nk}^2 \leq 1$ for all n and k, then for $\varepsilon > 0$ we have a constant C such that*

$$\sup_{-\infty<x<\infty} |F_n(x) - \Phi(x)| \leq CE(g_n(Z_n, \varepsilon)),$$

where F_n is the distribution function of S_{Z_n} and

$$g_n(j, \varepsilon) = \Big[\frac{1}{3} \max_{1\leq k\leq j} \sigma_{nk}^2 \sum_{k=1}^{j} \sigma_{nk}^2\Big]^{\frac{1}{5}} + \Big[\frac{10\varepsilon}{9} \max\Big(\sum_{k=1}^{j} \sigma_{nk}^2, 1\Big)\Big]^{\frac{1}{4}}$$
$$+ \Big[\sum_{k=1}^{j} \int_{|x|>\varepsilon} x^2 dF_{nk} + \frac{1}{2}\Big|\sum_{k=1}^{j} \sigma_{nk}^2 - 1\Big|\Big]^{\frac{1}{3}}.$$

Theorem 1.6. *Let (X_{nk}, Z_n) be such that $\sum_{k=1}^{Z_n} \sigma_{nk}^2 \xrightarrow{p} 1$ and satisfy* (RI). *If $\Big(\sum_{k=1}^{Z_n} \sigma_{nk}^2\Big)$ is bounded and $\sigma_{nk}^2 \leq 1$ for all n and k, then the sequence of distribution functions of the random sums S_{Z_n} converges weakly to Φ if and only if there exists a sequence of positive real numbers (ε_n) such that $E[g_n(Z_n, \varepsilon_n)] \to 0$.*

The following corollary follows directly from Theorem 1.5 and Hölder inequality.

Corollary 1.7. *Let (X_n) be a sequence of independent identically distributed random variables with zero mean and variance σ^2. Assume that (X_n) and (Z_n) are independent. Then for $\varepsilon > 0$*

$$\sup_{-\infty<x<\infty}\Big|P\Big(\frac{S_{Z_n}}{\sqrt{n}}\le x\Big)-\Phi(x)\Big|\le$$
$$\Big(\frac{\sigma^4}{n^2}E(Z_n)\Big)^{\frac{1}{5}}+\Big(\frac{10\varepsilon}{9}E\Big(\max\Big(\frac{\sigma^2}{n}Z_n,1\Big)\Big)\Big)^{\frac{1}{4}}+\Big(\frac{\sigma^2}{n}E(Z_n)+\frac{1}{2}E\Big[\Big|\frac{\sigma^2}{n}Z_n-1\Big|\Big]\Big)^{\frac{1}{3}}.$$

We also give examples of the convergence in Sec. 4.

2. Convergence Theorems

2.1. Auxiliary Results

In this section we give some auxiliary results for proving the main theorems in Subsec. 2.2.

Proposition 2.1. [13] *For every n, let (a_{nk}), be a nondecreasing sequence of non-negative real numbers and let $a\ge 0$ be fixed. Then $a_{nZ_n}\xrightarrow{p} a$ if and only if $a_{nl_n(q)}\to a$ for all $q\in(0,1)$ where $l_n:(0,1)\to\mathbb{N}$ defined by*

$$l_n(q)=\max\{k\in\mathbb{N}\,|\,P(Z_n<k)\le q\}.$$

In what follows, we let F_n, $F_n^{(q)}$ and F_{nk} be the distribution functions of $S_{Z_n}=X_{n1}+X_{n2}+\cdots+X_{nZ_n}$, $S_n^{(q)}=X_{n1}+X_{n2}+\cdots+X_{nl_n(q)}$ and X_{nk} respectively.

Proposition 2.2. [24] *Let (X_{nk},Z_n) satisfy* (RI). *If $F_n\xrightarrow{w} F$ for some distribution function F, then there exists a subsequence (n') such that for a.e. $q\in(0,1)$, there exist a distribution function $\overline{F}^{(q)}$ and a bounded sequence of real numbers $(a_{n'}^{(q)})$ such that*

$$F_{n'}^{(q)}*E_{a_{n'}^{(q)}}\xrightarrow{w}\overline{F}^{(q)},$$

where E_a stands for the degenerated distribution function with parameter $a\in\mathbb{R}$.

Proposition 2.3. [24] *If for a.e. $q\in(0,1)$, there exists a distribution function $F^{(q)}$ such that $F_n^{(q)}\xrightarrow{w}F^{(q)}$. Then $F_n\xrightarrow{w}F$, where F is the distribution function defined by $F(x)=\int_0^1 F^{(q)}(x)dq$.*

Theorem 2.4. [16] *Let (Y_n) be a sequence of random variables and put $H_n(x)=P(Y_n\le x)$. Suppose $\sup_{n\in\mathbb{N}}E[Y_n^2]<\infty$. If $H_n\xrightarrow{w}H$ for some distribution function H then we have $\lim_{n\to\infty}E(Y_n)=\int_{-\infty}^{\infty}x\,dH(x)<\infty$.*

Theorem 2.5. [5, p. 116] *For some suitably chosen constants A_n the sequence of distributions of the sums*

$$X_{n1} + X_{n2} + \cdots + X_{nk_n} - A_n$$

of independent infinitesimal random variables converges to a limit, it is necessary and sufficient that there exist non-decreasing functions M and N defined on the intervals $(-\infty, 0)$ and $(0, +\infty)$, respectively, such that $M(-\infty) = 0$ and $N(+\infty) = 0$ and a constant $\sigma \geq 0$ such that

(i) $\lim\limits_{n\to\infty} \sum\limits_{k=1}^{k_n} F_{nk}(u) = M(u)$ *for every continuity point u of M;*

(ii) $\lim\limits_{n\to\infty} \sum\limits_{k=1}^{k_n} (F_{nk}(u) - 1) = N(u)$ *for every continuity point u of N;*

(iii) $$\lim_{\varepsilon\to 0^+} \liminf_{n\to\infty} \sum_{k=1}^{k_n} \Big\{ \int\limits_{|x|<\varepsilon} x^2 dF_{nk}(x) - \Big(\int\limits_{|x|<\varepsilon} x dF_{nk}(x) \Big)^2 \Big\}$$
$$= \lim_{\varepsilon\to 0^+} \limsup_{n\to\infty} \sum_{k=1}^{k_n} \Big\{ \int\limits_{|x|<\varepsilon} x^2 dF_{nk}(x) - \Big(\int\limits_{|x|<\varepsilon} x dF_{nk}(x) \Big)^2 \Big\}$$
$$= \sigma^2.$$

The constants A_n may be chosen according to the formula

$$A_n = \sum_{k=1}^{k_n} \int\limits_{|x|<\tau} x dF_{nk}(x) - \gamma(\tau),$$

where $\gamma(\tau)$ is any constant and $-\tau$ and τ are continuity points of M and N, respectively.

If the limit distribution function is Φ, then $M \equiv N \equiv 0$ and $\sigma^2 = 1$.

Theorem 2.6. [5, p. 98] *Let (k_n) be a sequence of positive integers. Assume that $(X_{nj}), j = 1, 2, ..., k_n$, is infinitesimal and $\lim\limits_{n\to\infty} \sum\limits_{k=1}^{k_n} \sigma_{nk}^2 < \infty$. Then the sequence of distribution functions of $X_{n1} + X_{n2} + \cdots + X_{nk_n}$ converges weakly to a limit distribution function if and only if the accompanying distribution function of $X_{n1} + X_{n2} + \cdots + X_{nk_n}$ converges weakly to the same limit distribution function, where the accompanying distribution function of $X_{n1} + X_{n2} + \cdots + X_{nk_n}$ is the distribution function whose logarithm of its characteristic function $\hat{\varphi}_n(t)$ is given by*

$$\ln \hat{\varphi}_n(t) = \sum_{k=1}^{k_n} \int\limits_{-\infty}^{\infty} (e^{itx} - 1) dF_{nk}(x).$$

We also know that the limit distribution function is infinitely divisible [5, p. 73].

Proposition 2.7. [13] *For a.e. $q \in (0, 1)$ let $F^{(q)} = L(a_q, \sigma_q^2, M_q, N_q)$ be an infinitely divisible distribution function with zero mean. Suppose that σ_q^2 and the functions $M_q, |N_q|$ are non-decreasing in q and the integral $F(x) = \int\limits_0^1 F^{(q)}(x)dq$ exists for all $x \in \mathbb{R}$. Then we have $F = \Phi$ if and only if $F^{(q)} = \Phi$ a.e. $q \in (0, 1)$.*

Theorem 2.8. *Let* (X_{nk}, Z_n) *be such that* $\sum_{k=1}^{Z_n} \sigma_{nk}^2 \xrightarrow{p} 1$ *and satisfy* (RI). *Then* $F_n \xrightarrow{w} \Phi$ *if and only if* $F_n^{(q)} \xrightarrow{w} \Phi$ *for every* $q \in (0,1)$.

Proof. ($\Rightarrow$) By Proposition 2.2, there exists a subsequence (n') of (n) such that for a.e. $q \in (0,1)$, we have a distribution function $\overline{F}^{(q)}$ and a bounded sequence $(a_{n'}^{(q)})$ such that

$$F_{n'}^{(q)} * E_{a_{n'}^{(q)}} \xrightarrow{w} \overline{F}^{(q)}. \tag{2.1}$$

Since $\sum_{k=1}^{Z_n} \sigma_{nk}^2 \xrightarrow{p} 1$, by Proposition 2.1, we have $\sum_{k=1}^{l_n(q)} \sigma_{nk}^2 \to 1$ for all $q \in (0,1)$. Then for each $q \in (0,1)$

$$\sup_{n\in\mathbb{N}} E[(S_n^{(q)})^2] = \sup_{n\in\mathbb{N}} \sum_{k=1}^{l_n(q)} \sigma_{nk}^2 < \infty. \tag{2.2}$$

Thus, from (2.2) and the boundedness of $(a_{n'}^{(q)})$, we have

$$\sup_{n'\in\mathbb{N}} E[(S_{n'}^{(q)} + a_{n'}^{(q)})^2] = \sup_{n'\in\mathbb{N}} [E[(S_{n'}^{(q)})^2] + (a_{n'}^{(q)})^2] < \infty \quad \text{a.e.} \quad q \in (0,1).$$

From this fact and (2.1) we can apply Theorem 2.4 to $Y_{n'} = S_{n'}^{(q)} + a_{n'}^{(q)}$ and then

$$\lim_{n'\to\infty} a_{n'}^{(q)} = \lim_{n'\to\infty} (E[S_{n'}^{(q)} + a_{n'}^{(q)}]) = \int_{-\infty}^{\infty} x d\overline{F}^{(q)}(x)$$

for a.e. $q \in (0,1)$. Let $a^{(q)} = \int_{-\infty}^{\infty} x d\overline{F}^{(q)}(x)$. Thus $\lim_{n'\to\infty} a_{n'}^{(q)} = a^{(q)} < \infty$ for a.e. $q \in (0,1)$ and

$$F_{n'}^{(q)} \xrightarrow{w} F^{(q)} \quad \text{a.e.} \quad q \in (0,1),$$

where $F^{(q)} = \overline{F}^{(q)} * E_{-a^{(q)}}$. By Proposition 2.3, $\Phi(x) = \int_0^1 F^{(q)}(x) dq$.

Next, we will show that $F^{(q)}$ is Φ for every $q \in (0,1)$. First we will show that $F^{(q)}$ satisfies all conditions of Proposition 2.7. Applying Theorem 2.4 to $Y_{n'} = S_{n'}^{(q)}$, we have

$$\int_{-\infty}^{\infty} x dF^{(q)}(x) = \lim_{n'\to\infty} E[S_{n'}^{(q)}] = 0$$

for a.e. $q \in (0,1)$. Thus $F^{(q)}$ has zero mean. Since (X_{nk}) satisfies (RI), by Proposition 2.1 we have

$$\lim_{n\to\infty} \sup_{1\le l\le l_n(q)} P(|X_{nl}| \ge \varepsilon) = 0$$

for all $q \in (0,1)$. Applying Theorem 2.6, the sequence of the accompanying distribution functions of $S_{n'}^{(q)}$ converges weakly to $F^{(q)}$ for a.e. $q \in (0,1)$. Let

$F^{(q)} = L(a_q, \sigma_q^2, M_q, N_q)$. From monotonicity of the $l_n(q)$ we can use Theorem 2.5 to show that $\sigma_q^2, M_q, |N_q|$ are non-decreasing in q. Therefore Proposition 2.7 can be applied and it follows that $F^{(q)} = \Phi$ for a.e. $q \in (0,1)$. So $F_{n'}^{(q)} \xrightarrow{w} \Phi$ for a.e. $q \in (0,1)$. Next we will show that $F_{n'}^{(q)} \xrightarrow{w} \Phi$ for all $q \in (0,1)$. Let $q \in (0,1)$ and $A = \{t \in (0,1) | F_{n'}^{(t)} \xrightarrow{w} \Phi\}$. Then there exist q_1 and q_2 in A such that $q_1 < q < q_2$. From Theorem 2.5 and the non-decreasing monotonicity of the $l_{n'}(q)$, we have for $u < 0$,

$$0 = \lim_{n'\to\infty} \sum_{k=1}^{l_{n'}(q_1)} F_{n'k}(u) \leq \lim_{n'\to\infty} \sum_{k=1}^{l_{n'}(q)} F_{n'k}(u) \leq \lim_{n'\to\infty} \sum_{k=1}^{l_{n'}(q_2)} F_{n'k}(u) = 0.$$

Hence $(F_{n'}^{(q)})$ satisfies condition (i) of Theorem 2.5 for $M(u) = 0$. Similarly, we can show that the conditions (ii) and (iii) of Theorem 2.5 hold for $N(u) = 0$ and $\sigma^2 = 1$. Hence, $F_{n'}^{(q)} \xrightarrow{w} \Phi$. By the same argument we can show that every convergent subsequence of $(F_n^{(q)})$ converges weakly to Φ for all q. Thus $(F_n^{(q)})$ converges weakly to Φ for all $q \in (0,1)$.
($\Leftarrow$) Follows directly from Proposition 2.3. ∎

Lemma 2.9. *Let $\overline{K}, K_1, K_2, \ldots$ be bounded, non-decreasing, right-continuous functions from $\mathbb{R}$ into $[0,\infty)$ and vanish at $-\infty$. Assume that the followings hold*

(i)
$$\int_{-\infty}^{\infty} f(t,x)dK_n(x) \to \int_{-\infty}^{\infty} f(t,x)d\overline{K}(x)$$

for every real number t where $f(t,\cdot) : \mathbb{R} \to \mathbb{R}$ defined by

$$f(t,x) = \begin{cases} (e^{itx} - 1 - itx)\frac{1}{x^2} & if\ x \neq 0 \\ -\frac{t^2}{2} & if\ x = 0. \end{cases}$$

(ii) $(K_n(+\infty))$ *is bounded.*
Then $K_n \xrightarrow{w} \overline{K}$.

Proof. By Helley Theorem [23, p. 133] there exist a subsequence (K_{n_k}) of (K_n) and a bounded, non-decreasing, right-continuous function $\overline{\overline{K}}$ such that $K_{n_k} \xrightarrow{w} \overline{\overline{K}}$. Since$f(t,\cdot)$ is bounded for every $t \in \mathbb{R}$,

$$\int_{-\infty}^{\infty} f(t,x)dK_{n_k}(x) \to \int_{-\infty}^{\infty} f(t,x)d\overline{\overline{K}}(x).$$

From this fact and (i) we have

$$\int_{-\infty}^{\infty} f(t,x)d\overline{\overline{K}}(x) = \int_{-\infty}^{\infty} f(t,x)d\overline{K}(x).$$

By the uniqueness of Kolmogorov' formula [8], we have $\overline{\overline{K}} = \overline{K}$. So $K_{n_k} \xrightarrow{w} \overline{K}$. By the same argument we have that every subsequence of (K_n), it contains a subsequence which converges weakly to $\overline{K}$. This implies that (K_n) converges weakly to $\overline{K}$. ■

2.2. Proof of Main Results

In this section we prove our main results of convergent conditions.

Proof of Theorem 1.2.
$(\Rightarrow)$ To prove $K_{Z_n}(u) \xrightarrow{p} K(u)$, by Proposition 2.1 it suffices to show that

$$K_{l_n(q)} \xrightarrow{w} K$$

for every $q \in (0,1)$. By Theorem 2.6, Theorem 2.8 and continuity Theorem, $\ln \hat{\varphi}_{l_n(q)}(t) \to -t^2/2$ for every real number t, where $\hat{\varphi}_{l_n(q)}$ is the characteristic function of the accompanying distribution function of $S_n^{(q)}$. This implies

$$\int_{-\infty}^{\infty} f(t,x)dK_{l_n(q)}(x) \to \int_{-\infty}^{\infty} f(t,x)dK(x).$$

So (i) of Lemma 2.9 is satisfied. Since $\sum_{k=1}^{Z_n} \sigma_{nk}^2 \xrightarrow{p} 1$, by Proposition 2.1, $(K_{l_n(q)}(+\infty))$ is bounded for every q in $(0,1)$. Therefore the condition (ii) of Lemma 2.9 is satisfied. Thus $K_{l_n(q)} \xrightarrow{w} K$.
$(\Leftarrow)$ To prove the sufficient condition, by Proposition 2.3 and Theorem 2.6 it suffices to show that $\hat{\varphi}_{l_n(q)}(t) \to e^{-t^2/2}$ for $q \in (0,1)$ and $t \in \mathbb{R}$.

Let $q \in (0,1)$ and t be any real number. It follows from (i) and Proposition 2.1 that

$$K_{l_n(q)} \xrightarrow{w} K.$$

Since $f(t,\cdot)$ is bounded and continuous,

$$\int_{-\infty}^{\infty} f(t,x)dK_{l_n(q)}(x) \to \int_{-\infty}^{\infty} f(t,x)dK(x),$$

i.e., $\ln \hat{\varphi}_{l_n(q)}(t) \to -t^2/2$ which implies $\hat{\varphi}_{l_n(q)}(t) \to e^{-t^2/2}$. ■

Proof of Theorem 1.3.
To prove the theorem, it suffices to show that (RL) is equivalent to the condition (i) of Theorem 1.2.
$(\Rightarrow)$ Let u be the continuity point of K.
Case 1: $u < 0$.
Since $\sum_{k=1}^{Z_n} \int_{|x| \geq -u} x^2 dF_{nk}(x) = \sum_{k=1}^{Z_n} \sigma_{nk}^2 - \sum_{k=1}^{Z_n} \int_{|x|<-u} x^2 dF_{nk}(x) \xrightarrow{p} 1 - 1 = 0$, we have $\sum_{k=1}^{Z_n} \int_{-\infty}^{u} x^2 dF_{nk}(x) \xrightarrow{p} 0$,i.e., $K_{Z_n}(u) \xrightarrow{p} K(u)$.

Case 2: $u > 0$.

From the fact that $\sum_{k=1}^{Z_n} \int_{|x|\geq u} x^2 dF_{nk}(x) = 0$ we have $\sum_{k=1}^{Z_n} \int_{-\infty}^{-u} x^2 dF_{nk}(x) \xrightarrow{p} 0$. So

$$\sum_{k=1}^{Z_n} \int_{-\infty}^{u} x^2 dF_{nk}(x) = \sum_{k=1}^{Z_n} \int_{-\infty}^{-u} x^2 dF_{nk}(x) + \sum_{k=1}^{Z_n} \int_{|x|<u} x^2 dF_{nk}(x) \xrightarrow{p} 0 + 1 = 1.$$

That is $K_{Z_n}(u) \xrightarrow{p} K(u)$.

($\Leftarrow$) Assume that (i) of Theorem 1.2 holds. Note that

$$\sum_{k=1}^{Z_n} \int_{|x|<\varepsilon} x^2 dF_{nk}(x) = \sum_{k=1}^{Z_n} \int_{-\infty}^{\varepsilon} x^2 dF_{nk}(x) - \sum_{k=1}^{Z_n} \int_{-\infty}^{-\varepsilon} x^2 dF_{nk}(x) \xrightarrow{p} 1 - 0 = 1.$$

Thus (RL) is satisfied. ■

Proof of Theorem 1.4.

($\Rightarrow$) Follows from Theorem 1.3.

($\Leftarrow$) Since $\sum_{k=1}^{Z_n} \sigma_{nk}^2 \xrightarrow{p} 1$ and (X_{nk}, Z_n) satisfies (RL), we have $\sum_{k=1}^{Z_n} \int_{|x|\geq\varepsilon} x^2 dF_{nk}(x) \xrightarrow{p} 0$ for every $\varepsilon > 0$. Hence

$$\begin{aligned} \sup_{1\leq k\leq Z_n} P(|X_{nk}| \geq \varepsilon) &= \sup_{1\leq k\leq Z_n} \int_{|x|\geq\varepsilon} dF_{nk}(x) \\ &\leq \frac{1}{\varepsilon^2} \sum_{k=1}^{Z_n} \int_{|x|\geq\varepsilon} x^2 dF_{nk}(x) \end{aligned}$$

converges in probability to 0, i.e., (X_{nk}, Z_n) satisfies (RI). So (i) follows from Theorem 1.3. ■

3. Error of Estimation

To prove main theorems (Theorem 1.5 and Theorem 1.6), we need the following well-known theorem.

Theorem 3.1. [5 p. 196-197] *Let A, T and $\varepsilon > 0$ be constants, F a nondecreasing function and G a function of bounded variation. If*

1. $F(-\infty) = G(-\infty)$, $F(+\infty) = G(+\infty)$,
2. $\int |F(x) - G(x)|dx < \infty$,
3. $G'(x)$ *exists for all* x *and* $|G'(x)| \leq A$,
4. $\int_{-T}^{T} |\frac{f(t) - g(t)}{t}|dt = \varepsilon$ *where* $f(t) = \int_{\mathbb{R}} e^{itx} dF(x)$ *and* $g(t) = \int_{\mathbb{R}} e^{itx} dG(x)$,

then for every number $a > 1$ there corresponds a positive number $c(a)$ depending only on a such that

$$|F(x) - G(x)| \le a\frac{\varepsilon}{2\pi} + c(a)\frac{A}{T}.$$

Proof of Theorem 1.5.

For each n, let $\operatorname{Im} Z_n = \{k_{nj} | k_{nj} < k_{n(j+1)}\}$, $q_{nj} = \sum_{k=1}^{k_{nj}} P(Z_n = k)$ and $q_{n0} = 0$. Then for $q \in [q_{n(j-1)}, q_{nj})$ we have $l_n(q) = k_{nj}$ and

$$\begin{aligned} F_n(x) &= P(S_{Z_n} \le x) \\ &= \sum_{k_{nj} \in \operatorname{Im} Z_n} P(S_n^{(q)} \le x) P(Z_n = k_{nj}) \\ &= \sum_{k_{nj} \in \operatorname{Im} Z_n} P(S_n^{(q)} \le x)(q_{nj} - q_{n(j-1)}) \\ &= \sum_{k_{nj} \in \operatorname{Im} Z_n} \int\limits_{(q_{n(j-1)}, q_{nj})} F_n^{(q)}(x) dq \\ &= \int\limits_0^1 F_n^{(q)}(x) dq. \end{aligned}$$

Hence

$$|F_n(x) - \Phi(x)| \le \int\limits_0^1 |F_n^{(q)} - \Phi(x)| dq. \tag{3.1}$$

Let $\varphi_{l_n(q)}$ and φ_{nk} be the characteristic functions of $S_n^{(q)}$ and X_{nk}, respectively. Then

$$\begin{aligned} &|\varphi_{l_n(q)}(t) - e^{\frac{t^2}{2}}| \le |\ln \varphi_{l_n(q)}(t) - \frac{t^2}{2}| \\ &\le |\sum_{k=1}^{l_n(q)} (1 - \varphi_{nk}(t)) - \frac{t^2}{2}| + |\ln \varphi_{l_n(q)}(t) - \sum_{k=1}^{l_n(q)} (1 - \varphi_{nk}(t))| \\ &= |\int\limits_{\mathbb{R}} f(t,x) d(K_{l_n(q)}(x) - K(x))| + |\sum_{k=1}^{l_n(q)} \ln \varphi_{nk}(t) - \sum_{k=1}^{l_n(q)} (1 - \varphi_{nk}(t))| \\ &\le A_n + B_n, \end{aligned} \tag{3.2}$$

where

$$A_n = |\int\limits_{\mathbb{R}} f(t,x) d(K_{l_n(q)}(x) - K(x))| \ \text{ and } \ B_n = \sum_{k=1}^{l_n(q)} |\ln \varphi_{nk}(t) - (1 - \varphi_{nk}(t))|.$$

Shapiro [17] showed that

$$A_n \le t^2 \sum_{k=1}^{l_n(q)} \int\limits_{|x|\ge\varepsilon} x^2 dF_{nk}(x) + \frac{t^2}{2}\Big| \sum_{k=1}^{l_n(q)} \sigma_{nk}^2 - 1\Big| + \frac{5}{3}\varepsilon |t|^3 \max\Big(\sum_{k=1}^{l_n(q)} \sigma_{nk}^2, 1\Big). \quad (3.3)$$

Next, we find a bound of B_n. By Taylor formula, we have

$$\varphi_{nk}(t) = 1 + \frac{1}{2}\theta\sigma_{nk}^2 t^2 \quad \text{for some} \quad |\theta| \le 1. \quad (3.4)$$

Let $T_{l_n(q)} = \dfrac{1}{g_n(l_n(q), \varepsilon)}$ and t be such that $|t| < T_{l_n(q)}$. Note that

$$\begin{aligned} |\varphi_{nk}(t) - 1| &= |\theta| \frac{1}{2}\sigma_{nk}^2 t^2 \\ &\le \frac{1}{2}\sigma_{nk}^2 T_{l_n(q)}^2 \\ &\le \frac{\sigma_{nk}^2}{2(g_n(l_n(q),\varepsilon))^2} \\ &\le \frac{4}{5}(\sigma_{nk})^{\frac{2}{5}} \\ &\le \frac{4}{5}. \end{aligned} \quad (3.5)$$

Hence $\ln \varphi_{nk}(t) = \sum\limits_{j=1}^{\infty} \frac{(-1)^{j+1}}{j}(1-\varphi_{nk}(t))^j$ and

$$\begin{aligned} |\ln \varphi_{nk}(t) - (1 - \varphi_{nk}(t))| &\le \sum_{j=2}^{\infty} \frac{|1-\varphi_{nk}(t)|^j}{j} \\ &\le \frac{1}{2}\Big(\frac{|1-\varphi_{nk}(t)|^2}{1-|1-\varphi_{nk}(t)|}\Big) \\ &\le \frac{5}{2}|1-\varphi_{nk}(t)|^2 \qquad \text{(by (3.5))} \\ &\le \frac{5}{8}\sigma_{nk}^4 t^4 \quad \text{(by (3.4))} \\ &\le \frac{5}{8} t^4 [\max_{1\le j\le l_n(q)} \sigma_{nj}^2] \sigma_{nk}^2. \end{aligned}$$

So for $|t| < T_{l_n(q)}$,

$$B_n \le \frac{5}{8} t^4 [\max_{1\le k\le l_n(q)} \sigma_{nk}^2 \sum_{k=1}^{l_n(q)} \sigma_{nk}^2]. \quad (3.6)$$

From (3.2) - (3.6) we have

$$\begin{aligned} |\varphi_{l_n(q)}(t) - e^{\frac{t^2}{2}}| \le \frac{5}{8} t^4 [\max_{1\le k\le l_n(q)} \sigma_{nk}^2 \sum_{k=1}^{l_n(q)} \sigma_{nk}^2] + t^2 \sum_{k=1}^{l_n(q)} \int\limits_{|x|\ge\varepsilon} x^2 dF_{nk}(x) \\ + \frac{t^2}{2}\Big| \sum_{k=1}^{l_n(q)} \sigma_{nk}^2 - 1\Big| + \frac{5}{3}\varepsilon|t|^3 \max\Big(\sum_{k=1}^{l_n(q)} \sigma_{nk}^2, 1\Big) \end{aligned}$$

for $|t| < T_{l_n(q)}$. Therefore

$$
\begin{aligned}
&\int_{-T_{l_n(q)}}^{T_{l_n(q)}} \left|\frac{\varphi_{l_n(q)}(t) - e^{-\frac{t^2}{2}}}{t}\right| dt \\
&\le \frac{5}{4}\Big[\max_{1\le k\le l_n(q)} \sigma_{nk}^2 \sum_{k=1}^{l_n(q)} \sigma_{nk}^2\Big] \int_0^{T_{l_n(q)}} t^3 dt + 2\sum_{k=1}^{l_n(q)} \int_{|x|\ge\varepsilon} x^2 dF_{nk}(x) \int_0^{T_{l_n(q)}} t dt \\
&\quad + \Big|\sum_{k=1}^{l_n(q)} \sigma_{nk}^2 - 1\Big| \int_0^{T_{l_n(q)}} t dt + \frac{10}{3}\varepsilon \max\Big(\sum_{k=1}^{l_n(q)} \sigma_{nk}^2, 1\Big) \int_0^{T_{l_n(q)}} t^2 dt \\
&\le \Big[\frac{1}{3} \max_{1\le k\le l_n(q)} \sigma_{nk}^2 \sum_{k=1}^{l_n(q)} \sigma_{nk}^2\Big] \frac{1}{\Big[\frac{1}{3}\max_{1\le k\le l_n(q)} \sigma_{nk}^2 \sum_{k=1}^{l_n(q)} \sigma_{nk}^2\Big]^{\frac{4}{5}}} \\
&\quad + \Big[\sum_{k=1}^{l_n(q)} \int_{|x|\ge\varepsilon} x^2 dF_{nk}(x) + \frac{1}{2}\Big|\sum_{k=1}^{l_n(q)} \sigma_{nk}^2 - 1\Big|\Big] \\
&\quad\quad \times \frac{1}{\Big[\sum_{k=1}^{l_n(q)} \int_{|x|\ge\varepsilon} x^2 dF_{nk}(x) + \frac{1}{2}\Big|\sum_{k=1}^{l_n(q)} \sigma_{nk}^2 - 1\Big|\Big]^{\frac{2}{3}}} \\
&\quad + \frac{10\varepsilon}{9} \max\Big(\sum_{k=1}^{l_n(q)} \sigma_{nk}^2, 1\Big) \frac{1}{\Big[\frac{10\varepsilon}{9} \max\Big(\sum_{k=1}^{l_n(q)} \sigma_{nk}^2, 1\Big)\varepsilon\Big]^{\frac{3}{4}}} \\
&= g_n(l_n(q), \varepsilon).
\end{aligned}
$$

Now applying Theorem 3.1 we see that for any $a > 1$,

$$
\sup_{-\infty<x<\infty} \left|F_n^{(q)}(x) - \Phi(x)\right| \le \frac{a}{2\pi} g(l_n(q), \varepsilon) + \frac{c(a)}{T_n} = C g_n(l_n(q), \varepsilon), \tag{3.7}
$$

where $C = \dfrac{a}{2\pi} + c(a)$. Then the theorem follows from (3.1), (3.7) and the fact that

$$
\begin{aligned}
\int_0^1 g_n(l_n(q), \varepsilon) dq &= \sum_{k_{nj}\in \mathrm{Im}\, Z_n} \int_{[q_{n(j-1)}, q_{nj})} g_n(l_n(q), \varepsilon) dq \\
&= \sum_{k_{nj}\in \mathrm{Im}\, Z_n} g_n(k_{nj}, \varepsilon)(q_{nj} - q_{n(j-1)}) \\
&= \sum_{k_{nj}\in \mathrm{Im}\, Z_n} g_n(k_{nj}, \varepsilon) P(Z_n = k_{nj}) \\
&= E(g_n(Z_n, \varepsilon)).
\end{aligned}
$$

■

Proof of Theorem 1.6.
($\Rightarrow$) Let (ε_n) be a sequence of positive real numbers such that $0 < \varepsilon_n < 1$ and $\varepsilon_n \to 0$. In order to show $E(g(Z_n, \varepsilon_n)) \to 0$, it suffices to show that $g(Z_n, \varepsilon_n) \xrightarrow{p} 0$. Note that $g_n(Z_n, \varepsilon_n) = C_n + D_n + E_n$ where

$$C_n = \Big[\frac{1}{3} \max_{1 \le k \le Z_n} \sigma_{nk}^2 \sum_{k=1}^{Z_n} \sigma_{nk}^2\Big]^{\frac{1}{5}}$$

$$D_n = \Big[\frac{10\varepsilon_n}{9} \max\Big(\sum_{k=1}^{Z_n} \sigma_{nk}^2, 1\Big)\Big]^{\frac{1}{4}}$$

$$E_n = \Big[\sum_{k=1}^{Z_n} \int\limits_{|x|>\varepsilon} x^2 dF_{nk}(x) + \frac{1}{2}\Big|\sum_{k=1}^{Z_n} \sigma_{nk}^2 - 1\Big|\Big]^{\frac{1}{3}}.$$

By Theorem 1.4, (X_{nk}, Z_n) satisfies (RL). From this fact and the fact that $\sum\limits_{k=1}^{Z_n} \sigma_{nk}^2 \xrightarrow{p} 1$ we have (D_n) and (E_n) converge in probability to 0. To prove $C_n \xrightarrow{p} 0$, it suffices to show $\max\limits_{1 \le k \le Z_n} \sigma_{nk}^2 \xrightarrow{p} 0$. This is true by Theorem 1.2, (RI) and the fact that

$$\begin{aligned}\max_{1 \le k \le Z_n} \sigma_{nk}^2 &= \max_{1 \le k \le Z_n} \int\limits_{-\infty}^{\infty} x^2 dF_{nk}(x) \\ &= \max_{1 \le k \le Z_n} \int\limits_{|x| \le \sqrt{\frac{\varepsilon}{5}}} x^2 dF_{nk}(x) + \max_{1 \le k \le Z_n} \int\limits_{\sqrt{\frac{\varepsilon}{5}} < |x| \le 1} x^2 dF_{nk}(x) \\ &\quad + \max_{1 \le k \le Z_n} \int\limits_{|x|>1} x^2 dF_{nk}(x) \\ &\le \frac{\varepsilon}{5} + \max_{1 \le k \le Z_n} P(|X_{nk}| > \sqrt{\frac{\varepsilon}{5}}) + K_{Z_n}(-1) + K_{Z_n}(+\infty) - K_{Z_n}(1).\end{aligned}$$

for every $\varepsilon > 0$.
($\Leftarrow$) This follows directly from Theorem 1.5. ∎

4. Examples

Example 1. For each n, let Z_n be such that

$$P(Z_n = n) = 1 - \frac{1}{n^2} \text{ and } P(Z_n = n+1) = \frac{1}{n^2}.$$

For each n and k, define X_{nk} as follows.
If $k \neq n+1$, let X_{nk} be defined by

$$P(X_{nk} = \frac{1}{\sqrt{n}}) = P(X_{nk} = -\frac{1}{\sqrt{n}}) = \frac{1}{2}.$$

In case $k = n+1$, let X_{nk} be defined by

$$P(X_{nk} = 2^n) = P(X_{nk} = -2^n) = \frac{1}{2}.$$

Note that $\mu_{nk} = 0$,

$$\sigma_{nk}^2 = \begin{cases} \frac{1}{n} & \text{if } \ k \neq n+1 \\ 2^{2n} & \text{if } \ k = n+1 \end{cases}$$

and

$$l_n(q) = \begin{cases} n & \text{if } \ 0 < q < 1 - \frac{1}{n^2} \\ n+1 & \text{if } \ 1 - \frac{1}{n^2} \leq q < 1 \end{cases}$$

for every $k \in \mathbb{N}$ and $n \geq 2$. It is easy to see that $\sum\limits_{k=1}^{Z_n} \sigma_{nk}^2 \xrightarrow{p} 1$. From Proposition 2.1 and the fact that

$$\begin{aligned} \sum_{k=1}^{l_n(q)} \int\limits_{|x|<\varepsilon} x^2 dF_{nk}(x) &= \sum_{k=1}^{n} \int\limits_{-\varepsilon}^{\varepsilon} x^2 dF_{nk}(x) \\ &= \sum_{k=1}^{n} \Big(\int\limits_{\{-\frac{1}{\sqrt{n}}\}} x^2 dF_{nk}(x) + \int\limits_{\{\frac{1}{\sqrt{n}}\}} x^2 dF_{nk}(x) \Big) \\ &= \sum_{k=1}^{n} \Big(\Big(\frac{1}{2} - 0\Big)\Big(-\frac{1}{\sqrt{n}}\Big)^2 + \Big(1 - \frac{1}{2}\Big)\Big(\frac{1}{\sqrt{n}}\Big)^2 \Big) \\ &= 1, \end{aligned}$$

we have (X_{nk}, Z_n) satisfies (RL). Hence, by Theorem 1.4, the sequence of distribution functions of random sums S_{Z_n} converges weakly to Φ.

Example 2. Let Z_n be such that $P(Z_n = n + j) = 1/2^j$, $j = 1, 2, 3, ...$ and for each $n, k \in \mathbb{N}$, let $X_{nk} = X_k/\sqrt{n}$ where $P(X_k = -1) = P(X_k = 1) = 1/2$. Then

$$\sup_{-\infty<x<\infty} |F_n(x) - \Phi(x)| \leq \frac{C}{n^{\frac{1}{5}}}$$

for some constant C.

5. Remarks

1. Theorem 1.1 is a corollary of Theorem 1.4 by using Proposition 2.1 and the fact that $l_n(q) = k_n$ for every $q \in (0, 1)$.
2. In [13], Bethmann gave the conditions of convergence which are similar to Theorem 1.4 but the assumption "$\sum\limits_{k=1}^{Z_n} \sigma_{nk}^2 \xrightarrow{p} 1$" is changed to "$E(\sum\limits_{k=1}^{Z_n} \sigma_{nk}^2) \to 1$". We note that there exist sequences of random variables which satisfy our assumption but do not satisfy the assumption of Bethmann and conversely.

For examples, (X_{nk}, Z_n) in Example 1 satisfy the condition $\sum_{k=1}^{Z_n} \sigma_{nk}^2 \xrightarrow{p} 1$ but $E\big(\sum_{k=1}^{Z_n} \sigma_{nk}^2\big) \to \infty$. Conversely, if we let $X_{nk} = 0$ for $j \neq n+1, P(X_{n(n+1)} = \sqrt{2}) = P(X_{n(n+1)} = -\sqrt{2}) = 1/2$ and $P(Z_n = n) = P(Z_n = n+1) = 1/2$ we see that $E\big(\sum_{k=1}^{Z_n} \sigma_{nk}^2\big) \to 1$ but $E\big(\sum_{k=1}^{Z_n} \sigma_{nk}^2\big)$ does not converge in probability to 1.

3. There are other authors (eg. [2, 19, 24, 26 - 28]) gave a bound of this estimation. But the assumptions and results are different.

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