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On the Solvability of the First Initial Boundary Value Problem for Schrödinger Systems in Infinite Cylinders

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Abstract. In this paper we prove the existence and uniqueness of generalized solutions of the first initial boundary value problem for strongly Schrödinger systems in infinite cylinders.

1. Introduction

Schrödinger equations and Schrödinger systems play an important role in physics, especially in quantum mechanics. Therefore boundary value problems for Schrödinger equations and Schrödinger systems have been studied by many authors. Till now, these problems in a finite cylinder $Q_T = \Omega \times (0,T)$ were studied nearly completely (see [4, 5]).

In this paper we consider the first initial boundary value problem for Schrödinger systems in an infinite cylinder $Q_\infty = \Omega \times (0,\infty)$. We will prove some results on the existence and uniqueness of generalized solutions of the problem in the space $\overset{\circ}{H}{}^{m,0}_{\gamma}(Q_\infty)$ for $\gamma > 0$ big enough.

The paper is organized as follows. In Sec. 2, we will introduce some notations and functional spaces being used. The main results will be presented in Sec. 3.

2. Notations

Suppose that Ω is a bounded domain in $\mathbb{R}^n$ with its boundary $\partial\Omega$. Let us introduce some notations: $Q_T = \Omega \times (0,T)$, $S_T = \partial\Omega \times (0,T)$, $Q_\infty = \Omega \times (0,\infty)$, $S_\infty = \partial\Omega \times (0,\infty)$, $\Omega_T = Q_\infty \cap \{t = T\}$, $x = (x_1, ..., x_n) \in \Omega$, $u(x,t) =$

$(u_1(x,t),...,u_s(x,t))$ is a vector complex function, $\alpha = (\alpha_1,...,\alpha_n)$ $(\alpha_i \in \mathbb{N})$ is a multi-index, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $D^\alpha = \partial^{|\alpha|}/\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}$, $|D^\alpha u|^2 = \sum_{i=1}^{s} |D^\alpha u_i|^2$, $u_{t^j} = (\partial^j u_1/\partial t^j, ..., \partial^j u_s/\partial t^j)$, $|u_{t^j}|^2 = \sum_{i=1}^{s} \left|\partial^j u_i/\partial t^j\right|^2$, $dx = dx_1 \cdots dx_n$.

In this paper we use some following functional spaces.

• $\overset{\circ}{C}{}^\infty(\Omega)$ - the space of all infinitely differentiable functions which have compact support in Ω.

• $H^l(\Omega)$ - the space of all functions $u(x)$ which have generalized derivatives $D^\alpha u_i$ belonging to $L_2(\Omega)$, $|\alpha| \le l$, $1 \le i \le s$, with the norm

$$\|u\|_{H^l(\Omega)} = \Big(\sum_{|\alpha|=0}^{l} \int_\Omega |D^\alpha u|^2 dx \Big)^{\frac{1}{2}}.$$

• $\overset{\circ}{H}{}^l(\Omega)$ - the closure of $\overset{\circ}{C}{}^\infty(\Omega)$ in the space $H^l(\Omega)$.

• $H^{l,k}(Q_T)$ - the space of all functions $u(x,t)$ such that $D^\alpha u_i \in L_2(Q_T)$, $\dfrac{\partial^j u_i}{\partial t^j} \in L_2(Q_T)$, $|\alpha| \le l$, $1 \le i \le s$, $1 \le j \le k$, with the norm

$$\|u\|_{H^{l,k}(Q_T)} = \Big(\sum_{|\alpha|=0}^{l} \int_{Q_T} |D^\alpha u|^2 dxdt + \sum_{j=1}^{k} \int_{Q_T} |u_{t^j}|^2 dxdt \Big)^{\frac{1}{2}}.$$

In particular

$$\|u\|_{H^{l,0}(Q_T)} = \Big(\sum_{|\alpha|=0}^{l} \int_{Q_T} |D^\alpha u|^2 dxdt \Big)^{\frac{1}{2}}$$

• $\overset{\circ}{H}{}^{l,k}(Q_T)$ - the closure in $H^{l,k}(Q_T)$ of the set of all infinitely differentiable in Q_T functions which vanish near S_T.

• $H^{l,k}_\gamma(Q_\infty)$ - the space of all functions $u(x,t)$ which have generalized derivatives $D^\alpha u_i$, $\dfrac{\partial^j u_i}{\partial t^j}$, $|\alpha| \le l$, $1 \le j \le k$, $1 \le i \le s$, satisfying

$$\|u\|^2_{H^{l,k}_\gamma(Q_\infty)} = \sum_{|\alpha|=0}^{l} \int_{Q_\infty} |D^\alpha u|^2 e^{-2\gamma t} dxdt + \sum_{j=1}^{k} \int_{Q_\infty} |u_{t^j}|^2 e^{-2\gamma t} dxdt < +\infty.$$

In particular

$$\|u\|_{H^{l,0}_\gamma(Q_\infty)} = \Big(\sum_{|\alpha|=0}^{l} \int_{Q_\infty} |D^\alpha u|^2 e^{-2\gamma t} dxdt \Big)^{\frac{1}{2}}.$$

• $\overset{\circ}{H}{}^{l,k}_\gamma(Q_\infty)$ - the closure in $H^{l,k}_\gamma(Q_\infty)$ of the set of all infinitely differentiable in Q_∞ functions which vanish near S_∞.

• $L^\infty(0,\infty; L_2(\Omega))$ - the space of all measurable functions $u : (0,\infty) \longrightarrow L_2(\Omega)$
$t \longmapsto u(x,t)$

satisfying

$$\|u(x,t)\|_{L^\infty(0,\infty;\,L_2(\Omega))} = \operatorname{ess\,sup}_{t>0} \|u(x,t)\|_{L_2(\Omega)} < +\infty.$$

Let us consider the partial differential operator of order $2m$

$$L(x,t,D) = \sum_{|p|,|q|=0}^{m} D^p(a_{pq}(x,t)\,D^q),$$

where a_{pq} are $s \times s$ - matrices, $a_{pq} = (-1)^{|p|+|q|} a^*_{qp}$. Suppose that their elements are measurable bounded in $\overline{Q}_\infty$ functions with complex values and a_{pq} are continuous in $x \in \overline{\Omega}$ uniformly with respect to $t \in [0,+\infty)$, if $|p| = |q| = m$. Moreover, assuming that there exists a positive constant a_0 such that for each $\xi \in \mathbb{R}^n \setminus \{0\}$, $\eta \in \mathbb{C}^s \setminus \{0\}$ and $(x,t) \in \overline{Q}_\infty$, we have

$$\sum_{|p|,|q|=m} a_{pq}(x,t)\xi^p\xi^q\eta\overline{\eta} \geq a_0|\xi|^{2m}|\eta|^2, \tag{2.1}$$

where $\xi^p = \xi_1^{p_1}\cdots\xi_n^{p_n},\ \xi^q = \xi_1^{q_1}\cdots\xi_n^{q_n}$.

Put

$$B(u,u)(t) = \sum_{|p|,|q|=0}^{m} (-1)^{|p|} \int\limits_\Omega a_{pq}D^q u\overline{D^p u}dx,\ u(x,t) \in \overset{\circ}{H}{}^{m,0}_\gamma(Q_\infty).$$

For each fixed $t > 0$, the function $x \mapsto u(x,t)$ belongs to $\overset{\circ}{H}{}^m(\Omega)$. So we have

Lemma 2.1. [3] *There exist two constants μ_0 and $\lambda_0(\mu_0 > 0, \lambda_0 \geq 0)$ such that*

$$(-1)^m B(u,u)(t) \geq \mu_0\|u(x,t)\|^2_{H^m(\Omega)} - \lambda_0\|u(x,t)\|^2_{L_2(\Omega)} \tag{2.2}$$

for all $u(x,t) \in \overset{\circ}{H}{}^{m,0}_\gamma(Q_\infty)$.

Therefore, replacing L by $L_0 = L + (-1)^m\lambda_0 I$ if necessary, one gets

$$(-1)^m B(u,u)(t) \geq \mu_0\|u\|^2_{H^m(\Omega)} \tag{2.3}$$

for all $u(x,t) \in \overset{\circ}{H}{}^{m,0}_\gamma(Q_\infty)$.

Throughout this paper we will assume that the operator $L(x,t,D)$ satisfies (2.3). The inequality (2.3) is a basic tool for proving the existence and uniqueness of solutions to a boundary value problem.

3. Main Results

In this section we study the existence and uniqueness of a generalized solution of the following problem: Find a function $u(x,t)$ such that

$$(-1)^{m-1} i\, L(x,t,D)u - u_t = f(x,t) \text{ in } Q_\infty, \tag{3.1}$$

$$u\big|_{t=0} = 0, \tag{3.2}$$

$$\frac{\partial^j u}{\partial \nu^j}\Big|_{S_\infty} = 0, \quad j = 0, ..., m-1, \tag{3.3}$$

where ν is the outer unit normal to S_∞.

A function $u(x,t)$ is called a generalized solution of the problem (3.1)-(3.3) in the space $\overset{\circ}{H}{}^{m,0}_{\gamma}(Q_\infty)$ if and only if $u(x,t)$ belongs to $\overset{\circ}{H}{}^{m,0}_{\gamma}(Q_\infty)$ and for each $T > 0$ the following equality holds

$$(-1)^{m-1} i \sum_{|p|,|q|=0}^{m} (-1)^{|p|} \int_{Q_T} a_{pq} D^q u \overline{D^p \eta} dxdt + \int_{Q_T} u \overline{\eta}_t dxdt = \int_{Q_T} f \overline{\eta} dxdt \tag{3.4}$$

for all test functions $\eta \in \overset{\circ}{H}{}^{m,1}(Q_T),\ \eta(x,T) = 0.$

First, we prove the uniqueness theorem.

Theorem 3.1. *If*

$$\Big|\frac{\partial a_{pq}}{\partial t}\Big| \le \mu,\ 0 \le |p|,\ |q| \le m,\ \mu = \text{ const } > 0,$$

then the problem (3.1)-(3.3) *has at most one generalized solution in* $\overset{\circ}{H}{}^{m,0}_{\gamma}(Q_\infty)$.

Proof. Suppose that the problem (3.1)-(3.3) has two solutions u^1, u^2 in $\overset{\circ}{H}{}^{m,0}_{\gamma}(Q_\infty)$. For $T > 0$ and $b \in (0,T)$ arbitrary, putting

$$u(x,t) = u^1(x,t) - u^2(x,t)$$

$$\eta(x,t) = \begin{cases} \int_b^t u(x,\tau)d\tau, & 0 \le t \le b \\ 0, & b < t \le T. \end{cases}$$

One can see that $\eta(x,t) \in \overset{\circ}{H}{}^{m,1}(Q_T),\ \eta(x,T) = 0$ and $\eta_t(x,t) = u(x,t),\ 0 \le t \le b$. From definition of generalized solution, one gets

$$(-1)^m \sum_{|p|,|q|=0}^{m} (-1)^{|p|} \int_{Q_b} a_{pq} D^q \eta_t \overline{D^p \eta} dxdt + i \int_{Q_b} |\eta_t|^2 dxdt = 0. \tag{3.5}$$

Adding (3.5) with its complex conjugate and using condition a_{pq}=(-1)$^{|p|+|q|} a^*_{qp}$, we have

$$(-1)^m \sum_{|p|,|q|=0}^{m} (-1)^{|p|} \int_{Q_b} a_{pq} \frac{\partial}{\partial t}(D^q \eta \overline{D^p \eta}) dxdt = 0.$$

Integrating by parts, we obtain

$$(-1)^m \sum_{|p|,|q|=0}^{m} (-1)^{|p|} \int_\Omega a_{pq} D^q \eta \overline{D^p \eta} dx$$
$$= (-1)^m \sum_{|p|,|q|=0}^{m} (-1)^{|p|} \int_{Q_b} \frac{\partial a_{pq}}{\partial t} D^q \eta \overline{D^p \eta} dx dt.$$

Since $\left|\dfrac{\partial a_{pq}}{\partial t}\right|$ are bounded, by using Cauchy's inequality and (2.3), we obtain

$$\|\eta(x,0)\|^2_{H^m(\Omega)} \le C\|\eta(x,t)\|^2_{H^{m,0}(Q_b)}, \tag{3.6}$$

where the constant C depends only on μ, μ_0.

Denoting $v_p(x,t) = \displaystyle\int_t^0 D^p u(x,\tau)d\tau,\ 0 < t < b$, we have

$$D^p \eta(x,t) = \int_b^t D^p u(x,\tau)d\tau = v_p(x,b) - v_p(x,t).$$

Substituting those into (3.6), we obtain

$$\|\eta(x,0)\|^2_{H^m(\Omega)} = \sum_{|p|=0}^{m} \int_\Omega |v_p(x,b)|^2 \le C \sum_{|p|=0}^{m} \int_{Q_b} |D^p \eta|^2 dx dt$$
$$\le C \sum_{|p|=0}^{m} \int_{Q_b} \left[|v_p(x,b)|^2 + |v_p(x,t)|^2\right] dx dt$$
$$= Cb \sum_{|p|=0}^{m} \int_\Omega |v_p(x,b)|^2 dx + C \sum_{|p|=0}^{m} \int_{Q_b} |v_p(x,t)|^2 dx dt.$$

Putting $J(t) = \displaystyle\sum_{|p|=0}^{m} \int_\Omega |v_p(x,t)|^2 dx$, we have

$$(1 - Cb)J(b) \le C \int_0^b J(t)dt.$$

Hence

$$J(b) \le 2C \int_0^b J(t)dt,\ \forall b \in [0, \frac{1}{2C}].$$

By Gronwall-Bellman's inequality, this implies that $J(t) \equiv 0$ on $[0, \frac{1}{2C}]$. So

$$\int_t^0 u(x,\tau)d\tau = 0, \quad \forall t \in [0, \frac{1}{2C}].$$

Therefore $u(x,t) = 0$ a.e. $t \in [0, \frac{1}{2C}]$, i.e., $u^1 \equiv u^2$ a.e. $t \in [0, \frac{1}{2C}]$. By the same argument for two functions u^1, u^2 on $[\frac{1}{2C}, T]$ we can show that $u^1 \equiv u^2$, a.e. $t \in [\frac{1}{2C}, \frac{1}{C}]$. Continuing in this fashion; after finitely by many steps we have $u^1 \equiv u^2$, a.e. $t \in [0,T]$. Since $T > 0$ arbitrary, $u^1 \equiv u^2$ a.e. $t \in [0,\infty)$. This completes the proof. ∎

Denote by m^* the number of multi-indexes which have order not exceeding m, μ_0 is the constant in (2.3). We have

Theorem 3.2. *Let*

(i) $\sup\left\{\left|\frac{\partial a_{pq}}{\partial t}\right| : (x,t)\in\overline{Q}_\infty,\ \ 0\le|p|,\ |q|\le m\right\} = \mu < +\infty.$

(ii) $f, f_t \in L^\infty(0,\infty; L_2(\Omega))$.

Then for every $\gamma > \gamma_0 = \frac{m^*\mu}{2\mu_0}$, *the problem* (3.1)-(3.3) *has a generalized solution* $u(x,t)$ *in the space* $\overset{\circ}{H}{}^{m,0}_{\gamma}(Q_\infty)$ *and the following estimates holds*

$$\|u\|^2_{H^{m,0}_\gamma(Q_\infty)} \le C\left[\|f\|^2_{L^\infty(0,\infty;\,L_2(\Omega))} + \|f_t\|^2_{L^\infty(0,\infty;L_2(\Omega))}\right],$$

where C *is a positive constant independent of* u *and* f.

Proof. Let $\{\varphi_k(x)\}_{k=1}^\infty$ be a basis of $\overset{\circ}{H}{}^m(\Omega)$, which is orthonormal in $L_2(\Omega)$. We find an approximate solution $u^N(x,t)$ in the following form

$$u^N(x,t) = \sum_{k=1}^N C_k^N(t)\varphi_k(x),$$

where $C_k^N(t)$ satisfies

$$(-1)^{m-1} i \sum_{|p|,|q|=0}^m (-1)^{|p|}\int_\Omega a_{pq}D^q u^N \overline{D^p\varphi_l}dx - \int_\Omega u_t^N\overline{\varphi_l}dx = \int_\Omega f\overline{\varphi_l}dx,\ l=1,...,N, \tag{3.7}$$

$$C_k^N(0) = 0. \tag{3.8}$$

Multiplying (3.7) by $\frac{d}{dt}(\overline{C_l^N(t)})$, then taking sum with respect to l from 1 to N, we obtain

$$(-1)^m \sum_{|p|,|q|=0}^m (-1)^{|p|}\int_\Omega a_{pq}D^q u^N\overline{D^p u_t^N}dx - i\int_\Omega u_t^N\overline{u_t^N}dx = i\int_\Omega f\overline{u_t^N}dx. \tag{3.9}$$

Adding (3.9) with its complex conjugate, then integrating with respect to t from 0 to T, we obtain

$$(-1)^m \sum_{|p|,|q|=0}^m (-1)^{|p|}\int_{Q_T} a_{pq}\frac{\partial}{\partial t}(D^q u^N\overline{D^p u^N})dxdt = 2\,\mathrm{Im}\int_{Q_T} f\overline{u_t^N}dxdt.$$

Integrating by parts

$$\begin{aligned}(-1)^m &\sum_{|p|,|q|=0}^m (-1)^{|p|}\int_{\Omega_T} a_{pq}D^q u^N\overline{D^p u^N}dx = \\ &= (-1)^m \sum_{|p|,|q|=0}^m (-1)^{|p|}\int_{Q_T}\frac{\partial a_{pq}}{\partial t}D^q u^N\overline{D^p u^N}dxdt \\ &\quad + 2\,\mathrm{Im}\Big[\int_{\Omega_T} f\overline{u^N}dx - \int_{Q_T} f_t\overline{u^N}dxdt\Big].\end{aligned}$$

Using hypothesis, Cauchy's inequality and (2.3), we obtain

$$\|u^N(x,T)\|^2_{H^m(\Omega)} \le \frac{m^*\mu+\epsilon}{\mu_0-\epsilon}\int_0^T \|u^N(x,t)\|^2_{H^m(\Omega)}dt + \\ + \frac{1}{\epsilon(\mu_0-\epsilon)}\Big[\|f\|^2_{L^\infty(0,\infty;L_2(\Omega))} + T\|f_t\|^2_{L^\infty(0,\infty;L_2(\Omega))}\Big]$$

for $0<\varepsilon<\mu_0$.

Applying Gronwall's Lemma [7], we have

$$\|u^N(x,T)\|^2_{H^m(\Omega)} \le \frac{1}{\epsilon(\mu_0-\epsilon)}\Big[\|f\|^2_{L^\infty(0,\infty;L_2(\Omega))} + T\|f_t\|^2_{L^\infty(0,\infty;L_2(\Omega))}\Big] e^{\frac{(m^*\mu+\epsilon)T}{\mu_0-\epsilon}}. \tag{3.10}$$

For each $\gamma>\gamma_0$ given, choose $\epsilon>0$ such that

$$-2\gamma + \frac{m^*\mu+\varepsilon}{\mu_0-\varepsilon} < 0, \quad \text{i.e.,} \quad \gamma > \frac{m^*\mu+\varepsilon}{2(\mu_0-\varepsilon)}$$

(this always can be done because $\inf\limits_{0<\varepsilon<\mu_0} \dfrac{m^*\mu+\varepsilon}{2(\mu_0-\varepsilon)} = \dfrac{m^*\mu}{2\mu_0} = \gamma_0$).

Multiplying (3.10) with $e^{-2\gamma T}$, then integrating with respect to T from 0 to $+\infty$, we obtain

$$\|u^N\|^2_{H^{m,0}_\gamma(Q_\infty)} \le C\big[\|f\|^2_{L^\infty(0,\infty;L_2(\Omega))} + \|f_t\|^2_{L^\infty(0,\infty;L_2(\Omega))}\big].$$

Since $\{u^N\}$ is uniformly bounded in $\overset{\circ}{H}{}^{m,0}_\gamma(Q_\infty)$, we can choose a subsequence which converges weakly to a function $u(x,t)$ in $\overset{\circ}{H}{}^{m,0}_\gamma(Q_\infty)$. We will prove that $u(x,t)$ is a solution of the problem.

It is obvious that $u(x,t) \in \overset{\circ}{H}{}^{m,0}_\gamma(Q_\infty)$. For $T>0$ arbitrary, since $M = \bigcup\limits_{n=1}^{\infty}\left\{\sum\limits_{k=1}^{n} d_k(t)\varphi_k(x) \,|\, d_k(t)\in H^1(0,T),\ d_k(T)=0\right\}$ is dense in $\overset{\circ}{\widehat{H}}{}^{m,1}(Q_T) = \left\{\eta(x,t)\in \overset{\circ}{H}{}^{m,1}(Q_T) \,|\, \eta(x,T)=0\right\}$, it suffices to show that $u(x,t)$ satisfies (3.4) for all $\eta(x,t)\in M$. Taking $\eta\in M$ arbitrarily, then η can be written in the form $\eta(x,t) = \sum\limits_{l=1}^{k} d_l(t)\varphi_l(x)$, where $d_l(t)\in H^1(0,T)$, $d_l(T)=0$. Multiplying both sides of (3.7) by $d_l(t)$, taking sum with respect to l from 1 to k and integrating with respect to t from 0 to T, we obtain

$$(-1)^{m-1} i \sum_{|p|,|q|=0}^{m} (-1)^{|p|}\int_{Q_T} a_{pq}D^q u^N \overline{D^p\eta}\,dxdt - \int_{Q_T} u^N_t \overline{\eta}\,dxdt = \int_{Q_T} f\overline{\eta}\,dxdt.$$

Since

$$\int_{Q_T} u^N_t\overline{\eta}\,dxdt = \int_\Omega \big(u^N\overline{\eta}\big|_{t=0}^{t=T}\big)dx - \int_{Q_T} u^N\overline{\eta}_t\,dxdt = -\int_{Q_T} u^N\overline{\eta}_t\,dxdt,$$

we have

$$(-1)^{m-1} i \sum_{|p|,|q|=0}^{m} (-1)^{|p|} \int_{Q_T} a_{pq} D^q u^N \overline{D^p \eta} dxdt + \int_{Q_T} u^N \overline{\eta}_t dxdt = \int_{Q_T} f \overline{\eta} dxdt. \tag{3.12}$$

Passing to the limit for a weakly convergent subsequence, we obtain

$$(-1)^{m-1} i \sum_{|p|,|q|=0}^{m} (-1)^{|p|} \int_{Q_T} a_{pq} D^q u \overline{D^p \eta} dxdt + \int_{Q_T} u \overline{\eta}_t dxdt = \int_{Q_T} f \overline{\eta} dxdt.$$

This shows that $u(x,t)$ is a generalized solution of the problem (3.1)-(3.3). Moreover, the weak convergence of $\{u^N\}$ and (3.11) implies that

$$\|u\|^2_{H^{m,0}_\gamma(Q_\infty)} \leq \underline{\lim}_{N\to\infty} \|u^N\|^2_{H^{m,0}_\gamma(Q_\infty)} \leq C \left[\|f\|^2_{L^\infty(0,\infty;L_2(\Omega))} + \|f_t\|^2_{L^\infty(0,\infty;L_2(\Omega))} \right].$$

The theorem is completely proved. ∎

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