Copolyform $\Sigma$-Lifting Modules

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Abstract. A module $M$ is called copolyform if every coessential submodule of $M$ is corational in $M$. It is known that every polyform $\Sigma$-extending module is a direct sum of indecomposable self-injective modules. In this paper we study some properties of copolyform $\Sigma$-lifting modules. We show that a copolyform $\Sigma$-lifting module is a direct sum of indecomposable self-projective modules whose $M$-annihilator submodules are linearly ordered and satisfy ACC. We also prove that every copolyform $\Sigma$-lifting module $M$ is non-$M$-cosingular module. Consequently, for $M$ finitely generated, $\operatorname{End}(M)$ is a left and right serial artinian hereditary ring. We then consider $\Sigma$-lifting injective modules in terms of $\mathcal{Z}_M(N)$ and show that for any indecomposable direct summand $N$ of $M$, $\operatorname{Hom}(\mathcal{Z}_M(N), \mathcal{Z}_M(M))$ is a uniserial $S$-module of finite length where $S = \operatorname{End}(\mathcal{Z}_M(M))$.

1. Preliminaries

Let $R$ be an associative ring with identity. All modules we consider are unitary right $R$-modules. Suppose $M$ is an $R$-module. A submodule $A$ of $M$ is said to be a small submodule of $M$ (denoted by $A \ll M$) if for any $B \subseteq M$, $A + B = M$ implies $B = M$. A module $M$ is called a hollow module if every proper submodule of $M$ is small in $M$. A module is called a local module if it has a unique maximal submodule containing all its proper submodules. It is easy to see that a module is a local module if and only if it is a cyclic hollow module.

For $A \subseteq B \subseteq M$, $A$ is said to be a coessential submodule of $B$ in $M$ (denoted by $A \llce B$ in $M$) if $B/A \ll M/A$. In this case we also say $B$ is a coessential extension of $A$ in $M$. Instead of coessential extension the term lying above was used in Wisbauer [13]. For example, if $R = \mathbb{Z} = M$ and $A = 2\mathbb{Z}$, then for any $k \in \mathbb{N}$, $2^k\mathbb{Z}$ is a coessential submodule of $A$ in $M$. 
A is said to be coclosed in \( M \) (denoted by \( A \preceq M \)) if \( A \) has no proper coessential submodule in \( M \). The following lemma regarding hollow submodules of a module has been proved by Inoue \[6, \text{Proposition 6}\].

**Lemma 1.1.** \[6\] Let \( M \) be an \( R \)-module and \( N \) a hollow submodule of \( M \). Then \( N \ll M \) or \( N \preceq M \).

We recall the definition of an amply supplemented module. If \( N \) and \( L \) are submodules of the module \( M \), then \( N \) is called a *supplement* (resp. *weak supplement*) of \( L \), if \( N + L = M \) and \( N \cap L \ll N \) (resp. \( N \cap L \ll M \)). \( M \) is called *supplemented* (resp. *weakly supplemented*) if each of its submodules has a supplement (resp. weak supplement) in \( M \). A module \( M \) is called *\( \Sigma \)-weakly supplemented*, if any direct sum of copies of \( M \) is weakly supplemented. \( M \) is called *amply supplemented*, if for all submodules \( N \) and \( L \) of \( M \) with \( N + L = M \), \( N \) contains a supplement of \( L \) in \( M \).

A module \( M \) is called *lifting* if every submodule \( A \subseteq M \) contains a direct summand \( B \) of \( M \) such that \( A/B \ll M/B \). \( M \) is said to be (finitely) \( \Sigma \)-lifting if every (finite) direct sum of copies of \( M \) is lifting. The following lemma has been proved by Mohamed and Müller [9, 4.8].

**Lemma 1.2.** \[9\] A module is lifting if and only if it is amply supplemented and its coclosed submodules are direct summand.

Suppose \( M \) is an \( R \)-module. We recall the definitions of \( M \)-projective, self-projective and almost \( M \)-projective modules. An \( R \)-module \( N \) is called *\( M \)-projective* if for every epimorphism \( f : M \rightarrow K \) and every homomorphism \( g : N \rightarrow M \) with \( fh = g \), \( N \) is called *self-projective* (resp. *projective*) if it is \( N \)-projective (resp. \( L \)-projective for any \( R \)-module \( L \)). \( N \) is called almost \( M \)-projective, if for every epimorphism \( f : M \rightarrow K \) and every homomorphism \( g : N \rightarrow K \), either there exists \( h : N \rightarrow M \) with \( fh = g \) or there exists a nonzero direct summand \( M_1 \) of \( M \) and \( h : M_1 \rightarrow N \) with \( gh = f|_{M_1} \).

A family \( \{M_i\}_{i \in I} \) of modules is called (locally) *semi-\( T \)-nilpotent*, if for any countable family \( \{f_n : M_{i_n} \rightarrow M_{i_{n+1}}\}_N \) of non-isomorphisms with \( i_n \in I \) all distinct, (and for any element \( x \in M_{i_k} \)), there exists \( k \in \mathbb{N} \) (\( k \) depending on \( x \)) such that \( f_k \ldots f_1 = 0 \) (\( f_k \ldots f_1(x) = 0 \)). It is obvious that if each \( M_i \) is a local module, then the family \( \{M_i\}_I \) of modules is locally semi-\( T \)-nilpotent if and only if it is semi-\( T \)-nilpotent.

Regarding lifting modules with local endomorphism ring the following results have been proved by Baba and Harada [2].

**Theorem 1.3.** \[2, \text{Theorem 1}\] Let \( \{M_i\}_{i=1}^n \) be a set of hollow modules with local endomorphism ring. Then the following are equivalent.

1. \( \bigoplus_{i=1}^n M_i \) is lifting;
2. \( M_i \) is almost \( M_j \)-projective for any \( i \neq j \);
3. for any subset \( J \) of \( I = \{1, 2, \ldots, n\} \), \( \bigoplus_{i \in J} M_i \) is almost \( \bigoplus_{i \in I \setminus J} M_i \)-projective.
Lemma 1.4. [2, Lemma 3] Let \( \{M_i\}_I \) be a family of modules with local endomorphism ring. If \( \bigoplus_{i \in I} M_i \) is lifting, then \( \{M_i\}_I \) is locally semi-T-nilpotent.

Theorem 1.5. [2, Theorem 2] Let \( \{M_i\}_I \) be a family of local modules with local endomorphism ring. Then the following are equivalent.

1. \( \bigoplus_{i \in I} M_i \) is lifting;
2. \( M_i \) is almost \( M_j \)-projective for any \( i \neq j \) and \( \{M_i\}_I \) is locally semi-T-nilpotent;
3. for any subset \( J \) of \( I \), \( \bigoplus_{i \in J} M_i \) is almost \( \bigoplus_{i \in I \setminus J} M_i \)-projective and \( \{M_i\}_I \) is locally semi-T-nilpotent.

Lemma 1.6. [2, Lemma 4] Let \( M \) be a hollow module with local endomorphism ring. If any direct sum of copies of \( M \) is lifting, then \( M \) is cyclic.

2. \( \Sigma \)-Lifting Modules

In this section we prove that a \( \Sigma \)-lifting module with local endomorphism ring is self-projective. Suppose \( M \) is a \( \Sigma \)-lifting module such that any indecomposable direct summand of \( M \) has local endomorphism ring. Let \( N \) be any indecomposable direct summand of \( M \) and \( A = \{ \text{Ker} f : f : N \to M, \text{Im} f \ll M \} \). We prove that then \( N \) has ACC on \( A \) and if \( M \) has only finitely many non-isomorphic indecomposable direct summands, then \( N \) has DCC on \( A \). Suppose \( M \) is a direct sum of modules with local endomorphism rings and is \( \Sigma \)-lifting. Let \( S = \text{End}(M) \) and \( N \) is any indecomposable direct summand of \( M \). Then as a right \( S \)-module \( A = \text{Hom}(M, N) \) has a waist \( B = \{ f : M \to N | \text{Im} f \ll N \} \) and \( A/B \) is a uniserial module.

Proposition 2.1. Let \( M \) be a nonzero \( \Sigma \)-lifting, indecomposable \( R \)-module with \( \text{End}(M) \) local. Then \( M \) is local and self-projective.

Proof. By Lemma 1.6, \( M \) is a local module. We claim that any surjective map from \( M \) to \( M \) is an isomorphism. Suppose not. Let \( f : M \to M \) be surjective map which is not 1-1.

By 1.4, the family \( F = \{ M_n \}, n \in \mathbb{N} \), where \( M_n = M \), for all \( n \in \mathbb{N} \), is locally semi-T-nilpotent. Since \( M \) is a local module the family \( F \) is semi-T-nilpotent. Consider \( f_n = f : M_n \to M_{n+1} \), for all \( n \in \mathbb{N} \). Since \( \{M_i\}_I \) is semi-T-nilpotent, there exists a positive number \( k \) such that \( f^k : M_1 \to M_k = M \) is a zero epimorphism, which is a contradiction.

As \( M \) is lifting and every nonzero epimorphism \( M \to M \) is 1-1, \( M \) is self-projective [12, Lemma 2.3].

Lemma 2.2. Let \( M \) be an indecomposable self-projective lifting \( R \)-module. If \( A \) and \( B \) are fully invariant submodules of \( M \) such that \( M/A \oplus M/B \) is lifting, then either \( A \subseteq B \) or \( B \subseteq A \).

Proof. We first show that if \( Y \) is a fully invariant submodule of a self-projective module \( X \) and if \( \phi : X/Y \to X/Z \) is an onto map, then \( Y \subseteq Z \). We can lift
Y. Talebi and N. Vanaja

φη′ to a map φ′ : X → X such that φη′φ′ = η where η′ : X → X/Y and η : X → X/Z are the natural maps.

Since φ′(Y) ⊆ Y, we get η(Y) = φ′φ′(Y) = 0. Hence Y ⊆ Z.

Since A and B are fully invariant submodules of a self-projective module M, M/A and M/B are self-projective [14, Proposition 2.1]. As M/A and M/B are hollow self-projective, End(M/A) and End(M/B) are local rings. Suppose f : M/A → M/(A+B) and g : M/B → M/(A+B) are the natural maps. Since M/A⊕M/B is lifting, M/A is almost M/B-projective Theorem 1.3. Also M/B is indecomposable. Hence we can get either a map h : M/A → M/B or a map h′ : M/B → M/A, such that gh = f or fh′ = g.

As f and g are small epimorphisms, the maps h and h′ (if they exist) will be epimorphisms. Hence there exists either an epimorphism h : M/A → M/B or an epimorphism h′ : M/B → M/A. Therefore either A ⊆ B or B ⊆ A.

Proposition 2.3. Suppose M is a Σ-lifting module such that the endomorphism ring of every indecomposable direct summand of M is a local ring. Suppose that N is an indecomposable direct summand of M, K = \{ f : N → M | Im f ≪ M \} and A = \{ Ker f | f ∈ K \}. Then

1. A is linearly ordered by inclusion;
2. N has ACC on A;
3. N has DCC on A, if M has only finitely many non-isomorphic indecomposable direct summands.

Proof. Suppose that L is an indecomposable direct summand of M. Since L is Σ-lifting with local endomorphism ring, then by Proposition 2.1, L is a self-projective local module.

1. Suppose that f ∈ K. As Im f is hollow, then by Lemma 1.1, Im f is coclosed in M. Since M is lifting, Im f is an indecomposable direct summand of M Lemma 1.2. Hence Im f is local and self-projective. Since Ker f ≪ N and N/Ker f is self-projective, Ker f is fully invariant in N [14, Proposition 2.2].

Suppose that f, g ∈ K; then Ker f and Ker g are fully invariant in N. N/Ker f ⊕ N/Ker g (as it is isomorphic to a direct summand of M ⊕ M) is a lifting module. By Lemma 2.2, either Ker f ⊆ Ker g or Ker g ⊆ Ker f. Hence A is linearly ordered by inclusion.

2. Now suppose that there exists a strictly ascending chain

X_1 ≪ X_2 ≪ \cdots ≪ X_i ≪ X_{i+1} ≪ \cdots

of elements in A. Then there exists f_i : N → M such that Ker f_i = X_i and Im f_i ≪ M for every i ∈ N. For each i ∈ N, N/X_i is isomorphic to a direct
summand of \( M \) and hence is a \( \Sigma \)-lifting, self-projective, local module with local endomorphism ring.

As \( M \) is \( \Sigma \)-lifting we have \( \bigoplus_{i=1}^{\infty} N/X_i \) is a lifting module with \( \text{End}(N/X_i) \) local, for all \( i \in \mathbb{N} \). Thus \( \{N/X_i\}_{i \in \mathbb{N}} \) is locally semi-\( T \)-nilpotent (1.4) and hence semi-\( T \)-nilpotent (as each \( N/X_i \) is local). By considering the natural maps \( \eta_i : N/X_i \to N/X_{i+1} \) for all \( i \in \mathbb{N} \), we get a contradiction. Hence \( N \) satisfies ACC on \( A \).

(3) Suppose that 
\[
Y_1 \supseteq Y_2 \supseteq \cdots \supseteq Y_i \supseteq Y_{i+1} \supseteq \cdots
\]
is a strictly descending chain of elements in \( A \). Each \( N/Y_i \) is isomorphic to some indecomposable direct summand of \( M \). As there are only finitely many non-isomorphic indecomposable direct summands of \( M \), we get \( N/Y_\ell \cong N/Y_k \) for some \( k \) and \( \ell \). Suppose \( k < \ell \). Then \( Y_\ell \nsubseteq Y_k \). Since \( N/Y_\ell \cong N/Y_k \) is self-projective, the natural map \( f : N/Y_\ell \to N/Y_k \) splits. Therefore \( Y_k/Y_\ell \) is a nonzero proper direct summand of the hollow module \( N/Y_\ell \), which is a contradiction. \( \blacksquare \)

Recall that a submodule \( B \) of a module \( A \) is called a \textit{waist} if for every submodule \( C \) of \( A \) either \( B \subseteq C \) or \( C \subseteq B \) holds.

**Theorem 2.4.** Let \( M = \bigoplus_i M_i \) be a \( \Sigma \)-lifting module, where each \( M_i \) has a local endomorphism ring. Suppose that \( N \) is a nonzero indecomposable direct summand of \( M \), \( A = \text{Hom}(M,N) \) and \( B = \{ f \in A \mid \text{Im} \ f \ll N \} \). Then \( B \) is a waist of \( A \) such that \( A/B \) is a uniserial right \( S \)-module, where \( S = \text{End}(M) \).

**Proof.** Since each \( M_i \) is indecomposable \( \Sigma \)-lifting, by Proposition 2.1, \( M_i \) is local and self-projective. We note that any nonzero \( f \in A \) such that \( f \notin B \) is an epimorphism. To prove that \( B \) is a waist of \( A \), it is enough to prove that for any onto map \( f \in A \) and \( g \in B \), \( gS \subseteq fS \).

Now consider an epimorphism \( f : M \to N \) and \( g : M \to N \) with \( \text{Im} \ g \ll N \). There exists \( i_0 \in I \) such that \( f_{i_0} : M_{i_0} \to N \) is onto (for \( N \) is a local module), where \( f_{i_0} \) is the restriction of \( f \) to \( M_{i_0} \).

For every \( i \in I \), consider \( g_i : M_i \to N \), the restriction of \( g \) to \( M_i \). As \( M_i \oplus M_{i_0} \) is lifting \( M_i \) and \( M_{i_0} \) are relatively almost-projective modules. Hence for every \( i \in I \), there exists \( \phi_i : M_i \to M_{i_0} \) such that \( f_{i_0} \phi_i = g_i \).

Define \( \phi : M \to M \) by \( \phi_{M_i} = \phi_i \), for every \( i \in I \). It is obvious that \( f\phi = g \).

Hence \( gS \subseteq fS \). Therefore \( B \) is a waist of \( A \).

To prove that \( A/B \) is a uniserial right \( S \)-module, it is enough to show that whenever \( f, g : M \to N \) are surjective maps, either \( fS \subseteq gS \) or \( gS \subseteq fS \).

For every \( i \in I \) put \( [i] = \{ j \in I \mid M_i \cong M_j \} \) and \( \mathcal{F} = \{ [i] \mid i \in I \} \). Define an order on \( \mathcal{F} \) by \( [i] \leq [j] \) if and only if there exists a surjective map \( M_i \to M_j \). We claim that \( (\mathcal{F}, \leq) \) is a partially ordered set. Assume that \( [i] \leq [j] \) and \( [j] \leq [i] \) for some \( i, j \in I \). Then there exist surjective maps \( \theta : M_i \to M_j \) and \( \psi : M_j \to M_i \). Since \( M_i \) is self-projective, the map \( \psi \theta : M_i \to M_i \) splits. Therefore \( M_i \cong M_j \) and hence \( [i] = [j] \). Now it is easy to see that \( (\mathcal{F}, \leq) \) is a partially ordered set.
As each $M_i$ is local, by Lemma 1.4 the family $\{M_i\}_I$ is semi-$T$-nilpotent. Therefore every nonempty subset $G$ of $F$ has a maximal element.

Suppose that

\[ I_f = \{i \in I \mid f_i = f\mid_{M_i} \text{ is onto}\} \quad \text{and} \quad I_g = \{j \in I \mid g_j = g\mid_{M_j} \text{ is onto}\}. \]

Since $N$ is local, $I_f \neq \emptyset$ and $I_g \neq \emptyset$. Define

\[ \tilde{T}_f := \{[i]i \in I_f\} \quad \text{and} \quad \tilde{T}_g := \{[i]i \in I_g\}. \]

Suppose $[i_0]$ is a maximal element of $\tilde{T}_f \cup \tilde{T}_g$ and that $i_o \in I_f$. We claim that $gS \subseteq fS$.

Let $f_{i_0} : M_{i_0} \to N$ be the restriction of $f$ to $M_{i_0}$ and $g_i : M_i \to N$ be the restriction of $g$ to $M_i$, for every $i \in I$. As before $M_i$ and $M_{i_0}$ are relatively almost-projective modules for every $i \in I$. Hence if $g_i$ is not onto, there exists $\phi_i : M_i \to M_{i_0}$ such that $f_{i_0}\phi_i = g_i$. If $g_i$ is onto, then either there exists a surjective map $\phi_i : M_i \to M_{i_0}$ such that $f_{i_0}\phi_i = g_i$ or a surjective map $\psi_i : M_{i_0} \to M_i$ such that $g_i\psi_i = f_{i_0}$.

By the choice of $i_0$, the existence of the surjective map $\psi_i$ from $M_{i_0}$ to $M_i$ will imply that $M_i \simeq M_{i_0}$. Since $M_i$ is self-projective, the map $\psi_i$ is an isomorphism. Hence we always get a map $\phi_i : M_i \to M_{i_0}$ such that $f_{i_0}\phi_i = g_i$.

Define $\phi : M \to M$ by $\phi|_{M_i} = \phi_i$, for every $i \in I$. It is obvious that $f\phi = g$. Hence $gS \subseteq fS$.

3. Copolyform $\Sigma$-Lifting Modules

Clark and Wisbauer [3] have proved that a polyform $\Sigma$-extending module $M$ is a direct sum of self-injective modules. In this section dually we show that every copolyform $\Sigma$-lifting module is a direct sum of self-projective modules whose $M$-annihilator submodules are linearly ordered.

Suppose $M$ is an $R$-module. By $\sigma[M]$ we mean the full subcategory of Mod-$R$ whose objects are submodules of $M$-generated modules. The injective hull of $N \in \sigma[M]$ is denoted by $\tilde{N}$. $N \in \sigma[M]$ is said to be an $M$-small module if $N$ is small in $\tilde{N}$. It is easy to see that $N$ is an $M$-small module if and only if there exists a module $L \in \sigma[M]$ such that $N \ll L$. We define $\mathcal{Z}_M(N)$, as follows:

\[ \mathcal{Z}_M(N) = \text{Re}(N, S) = \bigcap \{\text{Ker}(g) \mid g \in \text{Hom}(N, L), L \in S\}, \]

where $S$ denotes the class of all $M$-small modules. We call $N$ an $M$-cosingular (non-$M$-cosingular) module if $\mathcal{Z}_M(N) = 0$ ($\mathcal{Z}_M(N) = N$). It is easy to see that a module $N \in \sigma[M]$ is non-$M$-small if and only if every nonzero factor module of $N$ is non-$M$-small.

Corational extension and copolyform module which are dual concepts of rational extension and polyform module are defined and studied in [11]. We give the definitions.

Suppose that $A \subseteq B \subseteq M$. We say that $A$ is a coessential submodule of $B$ in $M$ (denoted by $A \leftrightarrow B$ in $M$), if $B/A \ll M/A$. We call $A$ a corational submodule of $B$ in $M$, if $\text{Hom}(M/A, B/X) = 0$, for any submodule $X$ such that
A \subseteq X \subseteq B. We denote this by \( A \overset{cr}{\subseteq} B \) in \( M \). In this case we also say that \( B \) is a corational extension of \( A \) in \( M \).

We call a module \( M \) a copolyform module if \( A \overset{cr}{\subseteq} B \) in \( M \) implies \( A \overset{cr}{\subseteq} B \) in \( M \). Equivalently a module \( M \) is a copolyform module if whenever \( B/A \ll M/A \), \( \hom(M/A, B/X) = 0 \), for \( A \subseteq X \subseteq B \). A module \( M \) is \( \Sigma \)-copolyform, if any direct sum of copies of \( M \) is copolyform.

Suppose \( M \) is an \( R \)-module and \( A \subseteq M \). Consider the set \( \mathcal{A} \) of all coessential submodules of \( A \) in \( M \). Minimal elements of \( \mathcal{A} \) under set inclusion, if they exist, are called coclosures of \( A \) in \( M \). If \( M \) is amply supplemented, then coclosures of every submodule of \( A \) in \( M \) exist. A module \( M \) is called a unique coclosure module (denoted by UCC module), if every submodule of \( M \) has a unique coclosure in \( M \) [5]. We call a module \( M \) a \( \Sigma \)-UCC module, if any direct sum of copies of \( M \) is a UCC module.

Suppose that \( M \) is an amply supplemented module. If \( M \) is copolyform, then \( M \) is a UCC module [5, 4.2]. The converse is not true. For example, consider \( \mathbb{Z}/8\mathbb{Z} \) as a \( \mathbb{Z} \)-module. But if \( M \oplus M \) is UCC, then \( M \) is copolyform [5, 4.6]. The following lemma is trivial.

**Lemma 3.1.** Suppose that \( M \) is a \( \Sigma \)-amply supplemented module. Then \( M \) is a \( \Sigma \)-UCC module if and only if it is a \( \Sigma \)-copolyform module.

It is known that if \( M \) is a polyform module then \( M \) is non-\( M \)-singular. We do not know whether the dual is true, but it has been proved that if \( M \) is \( \Sigma \)-copolyform and \( \Sigma \)-weakly supplemented, then \( M \) is non-\( M \)-cosingular [11, 2.11]. We prove below that if \( M \) is copolyform and \( \Sigma \)-lifting, then \( M \) is non-\( M \)-cosingular.

**Proposition 3.2.** Let \( M \) be a copolyform \( \Sigma \)-lifting module. Define \( N := \oplus_{i \in I} M_i \), where \( M_i = M \) for every \( i \in I \). If \( X \) is a direct summand of \( N \) and \( f : N \rightarrow X \), then \( f(A) \overset{cc}{\subseteq} X \) whenever \( A \overset{cc}{\subseteq} N \).

**Proof.** Suppose \( A \overset{cc}{\subseteq} N \). Since \( N \) is a lifting module, \( A \) is a direct summand of \( N \) Lemma 1.2 and hence \( N = A \oplus B \). Consider the homomorphism \( g : N \rightarrow X \) such that \( g = f \) on \( A \) and \( g = 0 \) on \( B \). Then \( g(A) = f(A) = g(N) \). Moreover, since \( X \) is a lifting module, there exists a direct summand \( Y \subseteq g(A) \) such that \( X = Y \oplus Z \) and \( g(A) = Y \oplus (g(A) \cap Z) \) with \( (g(A) \cap Z) \ll Z \). This gives a map \( h := pg : N \rightarrow Z \), where \( p : X \rightarrow Z \) is the projection map along \( Y \). Now \( \im h = pg(N) = g(A) \cap Z \ll Z \).

Let \( \pi_i : N \rightarrow M_i \) and \( q_i : M_i \rightarrow N \) be the natural projection and inclusion maps for any \( i \in I \). Let \( p_i = \pi_i|Z \); then we get a homomorphism \( h_{ij} = p_iq_j : M_j \rightarrow M_i \), for each \( i, j \in I \).
Y. Talebi and N. Vanaja

\begin{equation}
M_i \xi M_j = 0 \quad [11, 2.3] \quad \therefore h = 0. \quad [\therefore f(A) \subseteq X, \text{ and hence } f(A) \subseteq X.]
\end{equation}

**Proposition 3.3** Let \( M \) be a copolyform \( \Sigma \)-lifting \( R \)-module. Then \( M \) is a non-\( M \)-cosingular module.

**Proof.** We know that a \( \Sigma \)-copolyform \( \Sigma \)-weakly supplemented module \( M \) is non-

\( M \)-cosingular \([11, 2.11]\). We want to prove that a copolyform \( \Sigma \)-lifting module is non-

\( M \)-cosingular. As \( M \) is \( \Sigma \)-lifting, \( M \) is \( \Sigma \)-amply supplemented. Hence it

is enough to prove that \( M \) is \( \Sigma \)-copolyform. By Lemma 3.1 it is enough to show that \( M \) is \( \Sigma \)-UCC.

Suppose \( I \) is any indexing set and \( N = \bigoplus_{i \in I} M_i \), where \( M_i = M \) for every

\( i \in I \). We want to show that \( N \) is a UCC module. For this we prove that, for all epimorphism \( f : N \to N/K \), \( A \subseteq N \) implies \( f(A) \subseteq N/K \) \([5, 3.16]\).

Suppose that \( f : N \to N/K \) is an epimorphism and \( A \subseteq N \). As \( N \) is a lifting module, \( N = L \oplus L' \), where \( L \subseteq K \) and \( K = L \oplus (K \cap L') \) with \( (K \cap L') \ll L' \). We

have an isomorphism \( \phi : N/K \to L'/((L' \cap K)) \). Also \( \phi f = \eta p \), where \( p : N \to L' \)

is the projection along \( L \), and \( \eta : L' \to L'/((L' \cap K)) \) the natural map.

\[ \begin{array}{ccc}
N & \xrightarrow{f} & N/K \\
\downarrow \mu & & \downarrow \phi \\
L & \xrightarrow{\eta} & L'/((L' \cap K))
\end{array} \]

Our aim is to prove that if \( A \subseteq N \), then \( f(A) \subseteq N/K \). It is enough to show that \( f(A) = \eta p(A) \subseteq L'/((L' \cap K)) \), as \( \phi \) is an isomorphism. Now since \( \text{Ker} \eta \ll L' \), \( \eta(B) \subseteq L'/((L' \cap K)) \), whenever \( B \subseteq L' \) \([5, 2.6]\). By Proposition 3.2, \( p(A) \subseteq L' \). Therefore \( \eta p(A) \subseteq L'/((L' \cap K)) \) and hence \( f(A) \subseteq N/K \). Thus \( N \)

is UCC module.

We recall the definition of \( M \)-annihilator submodules. Let \( M \) be an \( R \)-module. For an \( R \)-module \( N \) and any subset \( X \subseteq \text{Hom}(N, M) \), We put

\begin{equation}
q_i : M_i \to Z \\
\phi : Z \to M_i
\end{equation}
\[ \text{Ker}(X) = \bigcap \{ \text{Ker} g \mid g \in X \} \].

Any submodule of \( \text{Ker}(X) \) for some such \( X \) is called an \( M \)-annihilator submodule of \( N \) and we denote the set of \( M \)-annihilator submodules by \( K(N,M) \).

**Proposition 3.4.** Let \( M \) be a copolyform \( \Sigma \)-lifting module, and \( N \) an indecomposable direct summand of \( M \). If \( K(N,M) = \{ \text{Ker}(I) \mid I \subseteq \text{Hom}(N,M) \} \) is the set of all \( M \)-annihilator submodules of \( N \). Then

1. \( \text{End}(N) \) is a division ring and \( N \) is local, self-projective;
2. \( K(N,M) \) is linearly ordered by inclusion and \( N \) has ACC on \( K(N,M) \);
3. \( N \) has DCC on \( K(N,M) \), if \( M \) has only finitely many non-isomorphic indecomposable direct summand submodules.

**Proof.**

1. Let \( N \) be an indecomposable direct summand of \( M \). Then \( N \) is lifting and hence a hollow module. Suppose that \( f \) is a homomorphism. As \( N \) is non-M-cosingular, \( f \) is an epimorphism by Lemma 1.1. Let \( L = \bigoplus_{i \in \mathbb{N}} N_i \), for every \( i \in \mathbb{N} \), \( N_i = N \). Then \( L \) is a UCC lifting module, and hence the sum of any family of coclosed submodules of \( L \) is coclosed in \( L \) [5, 3.16 (3)]. As \( L \) is also lifting, any locally direct submodule of \( L \) is a direct summand of \( L \) Lemma 1.2. Now consider \( f_i = f : N_i \rightarrow N_{i+1} \) for every \( i \in \mathbb{N} \). Then for every family \( \{ f_i : N_i \rightarrow N_{i+1} \} \), there exists \( r \in \mathbb{N} \) and a nonzero map \( h_r : N_{r+1} \rightarrow N_r \) such that \( f_{r-1} \cdots f_1 = h_r f_r \cdots f_1 \) [13, 43.3]. For any \( i \in \mathbb{N} \), \( f_i \) is onto, and \( h_r f_r \) is the identity map on \( N_r \). Hence \( f_r \) is 1-1. Therefore \( \text{End}(N) \) is a division ring. Now by Proposition 2.1, \( N \) is local and self-projective.

2. By Proposition 3.3, \( M \) is non-M-cosingular. Since every direct summand \( N \) of \( M \) is non-M-cosingular, so if \( f \in \text{Hom}(N,M) \) is a nonzero map then \( \text{Im} f \not\subseteq M \). Therefore if \( \mathcal{A} = \{ \text{Ker} f \mid \text{Im} f \not\subseteq M \} \) then \( \mathcal{A} \cup N = K(N,M) \) and hence \( K(N,M) \) is linearly ordered by Proposition 2.3. Suppose that \( \text{Ker}(I_1) \not\supseteq \text{Ker}(I_2) \), where \( I_1, I_2 \subseteq \text{Hom}(N,M) \). Then there exists \( f \in I_1 \), such that \( \text{Ker} f \not\subseteq \text{Ker} g \), for every \( g \in I_2 \); for if not, then for every \( f_0 \in I_1 \), there exists \( g_0 \in I_2 \), such that \( \text{Ker} f_0 \not\supseteq \text{Ker} g_0 \), and hence \( \text{Ker} f_0 \not\supseteq \text{Ker}(I_2) \). Therefore \( \text{Ker}(I_1) \not\supseteq \text{Ker}(I_2) \) which is a contradiction.

Consider
\[ \text{Ker}(I_1) \not\supseteq \text{Ker}(I_2) \not\supseteq \cdots \not\supseteq \text{Ker}(I_r) \not\supseteq \text{Ker}(I_{r+1}) \not\supseteq \cdots \]
a strictly ascending chain of \( M \)-annihilator submodules. Fix \( r \in \mathbb{N} \). As \( \text{Ker}(I_r) \not\subseteq \text{Ker}(I_{r+1}) \), there exists \( f_r \in I_r \) such that for every \( g \in I_{r+1} \), \( \text{Ker} f_r \not\supseteq \text{Ker} g \). Hence we get a strictly increasing chain
\[ \text{Ker} f_1 \not\supseteq \text{Ker} f_2 \not\supseteq \cdots \not\supseteq \text{Ker} f_r \not\supseteq \text{Ker} f_{r+1} \not\supseteq \cdots \]
which is a contradiction Proposition 2.3. So \( N \) satisfies ACC on \( K(N,M) \).

3. Suppose that
\[ \text{Ker}(I_1) \supseteq \text{Ker}(I_2) \supseteq \cdots \supseteq \text{Ker}(I_r) \supseteq \text{Ker}(I_{r+1}) \supseteq \cdots \]
is a strictly descending chain of \( M \)-annihilator submodules of \( N \). Then for each \( r+1 \in \mathbb{N} \), there exists \( f_{r+1} \in I_{r+1} \) such that for every \( f_r \in I_r \), \( \text{Ker} f_{r+1} \not\subseteq \text{Ker} f_r \). So we get a strictly descending chain

Ker \( f \).
$\text{Ker } f_1 \supsetneq \text{Ker } f_2 \supsetneq \cdots \supsetneq \text{Ker } f_r \supsetneq \text{Ker } f_{r+1} \supsetneq \cdots$,

which is a contradiction Proposition 2.3.

The following Lemma regarding copolyform modules has been proved in [11, 2.3].

**Lemma 3.5.** Suppose that $M$ is an amply supplemented module. $M$ is a copolyform module if and only if for any nonzero map $f : M \to M/X$, $\text{Im } f \not\subseteq M/X$.

**Lemma 3.6.** Suppose $M$ is an $R$-module and $f : P \to M$ a projective cover of $M$. Then the following are equivalent.

1. $P$ is copolyform;
2. $M$ is copolyform and $0 \xrightarrow{cr} K$ in $P$, where $K$ is the kernel of $f$.

**Proof.**

1. $\Rightarrow$ (2). As $P$ is copolyform, $M$ is copolyform [11, 2.2]. Since $K \ll P$, $0 \xrightarrow{cr} K$ in $P$. Now $P$ is copolyform implies that $0 \xrightarrow{cr} K$ in $P$.

2. $\Rightarrow$ (1). Suppose that $A \xrightarrow{cr} B$ in $P$. It is easy to see that $(A + K)/K \xrightarrow{cr} (B + K)/K$ in $M$. As $M$ is copolyform, $(A + K)/K \xrightarrow{cr} (B + K)/K$ in $M$. Therefore by [11, 1.1(5)] $(A + K) \xrightarrow{cr} (B + K)$ in $P$. Since $0 \xrightarrow{cr} K$ in $P$ and $A \xrightarrow{cr} A$ in $P$, $A \xrightarrow{cr} (A + K)$ in $P$ [11, 1.1(4)]. Now $(A + K) \xrightarrow{cr} (B + K)$ in $P$ implies $A \xrightarrow{cr} (B + K)$ [11, 1.1(2)]. Again by [11, 1.1(2)] $A \xrightarrow{cr} B$ in $P$. Thus $P$ is a copolyform module.

It has been proved in [4] that, $M$ is a polyform module if and only if $\text{End}(\hat{M})$ is a regular ring. We prove the dual of that result when $M$ is a semiperfect module.

**Theorem 3.7.** Let $M$ be a semiperfect module and $f : P \to M$ be the projective cover. Then the following statements are equivalent.

1. $M$ is copolyform and $0 \xrightarrow{cr} \text{Ker } f$ in $P$;
2. $P$ is copolyform;
3. $\text{End}(P)$ is regular.

**Proof.**

1. $\Leftrightarrow$ (2) follows from Lemma 3.6.

2. $\Rightarrow$ (3). Let $P$ be a copolyform module and $S = \text{End}(P)$. As $P$ is a projective module, $f \in \text{Rad } S$ implies that $\text{Im } f \ll P$. Since $P$ is copolyform $\text{Rad } S = 0$. As $M$ is semiperfect $P$ is semiperfect [8, 5.6]. Then $S$ is $f$-semiperfect [13, 42.12] and therefore $S/\text{Rad } S$ is a regular ring [13, 42.11]. Now $\text{Rad } S = 0$ implies that $S$ is a regular ring.

3. $\Rightarrow$ (2). Let $g : P \to P$ be a homomorphism with $\text{Im } g \ll P$. Since $\text{End}(P)$ is regular, $\text{Im } g$ is a direct summand of $P$ [13, 37.7]. Hence $\text{Im } g = 0$ or $g = 0$.

Now by Lemma 3.5 $P$ is copolyform.

Recall that a ring $R$ is a left $PP$-ring (principal projective) if every cyclic left ideal of $R$ is projective. A ring $R$ is a hereditary (semihereditary) ring if
every left (finitely generated) ideal is projective.

Harmaneci has communicated the following lemma. We give here a proof for the sake of completeness.

**Lemma 3.8.** Let $M$ be a copolyform module and $S = \text{End}(M)$.

1. If $M$ is lifting, then $S$ is a left and right PP-ring.
2. If $M$ is finitely $\Sigma$-lifting, then $S$ is left and right semihereditary.

**Proof.** (1) Let $f \in S$. Since $M$ is lifting and copolyform, $\text{Im} f \xhookrightarrow{\text{cc}} M$ [11, 2.3]. Therefore $\text{Im} f$ is a direct summand of $M$ Lemma 1.2. Hence by [13, 39.11] $S$ is right PP-ring. Now we show that $Sf$ is projective for every $f \in S$. As above $f(M)$ is a direct summand of $M$ and hence $f(M) = e(M)$ for some idempotent $e \in S$. It is enough to prove that the onto map $\phi : S \to Sf$ defined by $\phi(s) = sf$, where $s \in S$ splits. We have $S(1 - e) \subseteq \text{Ker} \phi$. Let $g \in \text{Ker} \phi$. Then $\phi(g) = gf = 0$ and so $gf(M) = ge(M) = 0$. This implies $ge = 0$. Hence $g(1 - e) = g \in S(1 - e)$. Thus $\text{Ker} \phi = S(1 - e)$. Therefore $S$ is a left PP-ring.

(2) Since $M$ is copolyform and finitely $\Sigma$-lifting, $M^n$ is also copolyform and lifting [11, 2.9]. Therefore by (1), for every $n \in \mathbb{N}$, $\text{End}(M^n) \simeq S^n \times S^n$ is a left and right PP-ring. Hence $S$ is left and right semihereditary [13, 39.13].

**Theorem 3.9.** Let $M$ be a copolyform $\Sigma$-lifting module and $S = \text{End}(M)$. Suppose that $N$ is any indecomposable direct summand of $M$. Then we have the following.

1. $M = \bigoplus_i M_i$ where each $M_i$ is self-projective, local and $\text{End}(M_i)$ a division ring;
2. $\text{Hom}(N, M)$ is a uniserial, artinian left $S$-module;
3. $\text{Hom}(N, M)$ is a uniserial left $S$-module of finite length, if $M$ has only finitely many non-isomorphic indecomposable direct summands;
4. $\text{Hom}(M, N)$ is a uniserial right $S$-module;
5. if $M$ is finitely generated, then $S$ is left and right serial, right artinian and left and right hereditary ring;
6. if $M$ is finitely generated, then $M$ has a projective cover $P$ in $\sigma[M]$ and $\text{End}(P)$ is a semisimple ring.

**Proof.** (1). Since in a UCC lifting module $M$, every local direct summand of $M$ is a direct summand [5], $M$ is a direct sum of indecomposable modules [9, 2.17]. Now (1) follows from Proposition 3.4.

(2) By Proposition 3.3, $M$ is non-$M$-cosingular. Hence $N$ is non-$M$-cosingular. Thus for every nonzero $f \in \text{Hom}(N, M)$, $\text{Im} f$ is a hollow module which is not small in $M$. Therefore by Lemma 1.1 $\text{Im} f \xhookrightarrow{\text{cc}} M$ and hence a direct summand of $M$. As $\text{End}(M_i)$ is local, $\text{Im} f \simeq M_i$, for some $i \in I$ [1].

Suppose $f, g : N \to M$ with $\text{Ker} f \subseteq \text{Ker} g$. We claim that $Sg \subseteq Sf$. For there exists an onto map $\psi : \text{Im} f \to \text{Im} g$ such that $\psi f = g$.

Since $\text{Im} f$ is a direct summand of $M$, we can extend $\psi$ to $\phi : M \to M$ such that $\phi f = g$. Therefore $Sg \subseteq Sf$. 

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**Copolyform $\Sigma$-Lifting Modules**

59
Consider any two nonzero \( f, g \in \text{Hom}(N, M) \). Then \( \text{Im} f \cong N / \text{Ker} f \cong M_i \) and \( \text{Im} g \cong N / \text{Ker} g \cong M_j \) for some \( i, j \in I \). By Proposition 2.3 (1), either \( \text{Ker} f \subseteq \text{Ker} g \) or \( \text{Ker} g \subseteq \text{Ker} f \). Hence either \( Sg \subseteq Sf \) or \( Sf \subseteq Sg \). Therefore \( \text{Hom}(N, M) \) is a uniserial left \( S \)-module.

Let \( I_1 \supsetneq I_2 \supsetneq \cdots \supsetneq I_n \supsetneq \cdots \) be a strictly descending chain of \( S \)-submodules of \( \text{Hom}(N, M) \). For each \( n \in \mathbb{N} \) choose \( f_n : N \to M \) such that \( f_n \in I_n \) and \( f_n \notin I_{n+1} \). We have either \( Sf_n \subseteq Sf_{n+1} \) or \( Sf_{n+1} \subseteq Sf_n \) and \( f_n \notin I_{n+1} \) implies that \( Sf_{n+1} \not\subseteq Sf_n \). This along with either \( \text{Ker} f_{n+1} \subseteq \text{Ker} f_n \) or \( \text{Ker} f_n \subseteq \text{Ker} f_{n+1} \) (2.3 (1)) gives us \( \text{Ker} f_n \not\subseteq \text{Ker} f_{n+1} \). Thus we get a strictly ascending chain

\[
\text{Ker} f_1 \not\subseteq \text{Ker} f_2 \not\subseteq \cdots \not\subseteq \text{Ker} f_n \not\subseteq \text{Ker} f_{n+1} \not\subseteq \cdots,
\]

which contradicts Proposition 2.3 (3). Thus \( \text{Hom}(N, M) \) is a uniserial left \( S \)-module of finite length.

(3) By (2) it is enough to prove that \( \text{Hom}(N, M) \) satisfies ACC on \( S \)-submodules. Suppose that

\[
I_1 \not\subseteq I_2 \not\subseteq \cdots \not\subseteq I_n \not\subseteq \cdots
\]

is a strictly ascending chain of \( S \)-submodules of \( \text{Hom}(N, M) \). For each \( n \) there exists \( f_{n+1} \in I_{n+1} \) such that \( f_{n+1} \notin I_n \). As in the proof of (2) we get a strictly descending chain

\[
\text{Ker} f_2 \not\supsetneq \text{Ker} f_3 \not\supsetneq \cdots \not\supsetneq \text{Ker} f_n \not\supsetneq \text{Ker} f_{n+1} \not\supsetneq \cdots,
\]

which contradicts Proposition 2.3 (3). Thus \( \text{Hom}(N, M) \) is a uniserial left \( S \)-module of finite length.

(4) Since for every \( i \in I \), \( M_i \) is hollow and \( N \) is a local module, any nonzero map \( M_i \to N \) is an epimorphism. Hence any nonzero \( f \in \text{Hom}(M, N) \) is an epimorphism. Now (4) follows from Theorem 2.4.

(5) Since \( M \) is finitely generated \( M = \bigoplus_{i=1}^{k} M_i \), where each \( M_i \) is a local module with local endomorphism ring.

Since \( S = \bigoplus_{i=1}^{k} \text{Hom}(M_i, M_i) \), \( S \) is a right serial ring by (4).

Also \( S = \bigoplus_{i=1}^{k} \text{Hom}(M_i, M) \) and for each \( i = 1, \ldots, k \), \( \text{Hom}(M_i, M) \) is left artinian and left uniserial (by (2) and (3)) imply that \( S \) is a left serial and left artinian ring.

We know that every left and right serial, left artinian ring is a right artinian ring [13, 55, 16]. Thus \( S \) is a left and right artinian serial ring. By Lemma 3.8, \( S \) is left and right semihereditary and hence \( S \) is a left and right hereditary ring.

(6) As \( M \) is finitely generated \( M = \bigoplus_{i=1}^{k} M_i \), where each \( M_i \) is a local module with local endomorphism ring. Define \( I := \{1, 2, \ldots, k\} \).

For every \( i \in I \) put \( [i] = \{j \in I | M_i \cong M_j\} \) and \( \mathcal{F} = \{[i] | i \in I\} \). Define an order on \( \mathcal{F} \) by \( [i] \preceq [j] \) if and only if there exists an onto map \( M_i \to M_j \). Then \( (\mathcal{F}, \preceq) \) is a partially ordered set (see the proof of Theorem 2.4). Suppose that

\[
J = \{j \in I | [j] \text{ is a minimal elements of } \mathcal{F}\}.
\]

Let \( N = \bigoplus_{j \in J} M_j \). For \( k, \ell \in J \) any epimorphism from \( M_k \) to \( M_\ell \) is an isomorphism and \( N \) is lifting, \( N \) is self-projective [12, Lemma 2.3].
For any \( i \in I \), there exists a \( j \in J \) such that \([j] \leq [i]\) and hence there exists an epimorphism from \( M_j \) to \( M_i \). Thus \( \sigma[M] = \mathbb{N} \). Since \( N \) is finitely generated and self-projective, \( N \) is projective in \( \mathbb{N} \) and hence in \( \sigma[M] \).

Given \( i \in I \), there exists a \( j \in J \) such that there exists an epimorphism from \( M_j \) to \( M_i \) and hence \( M_j \) is a projective cover of \( M_i \). Thus \( M \) has a projective cover \( P \) which is a direct summand of \( N^{(k)} \).

Since \( P \) is finitely generated and weakly supplemented, \( P/\text{Rad}(P) \) is a semisimple module and hence \( \text{End}(P/\text{Rad}(P)) \) is a semisimple ring. By [11, 2.9] \( N^{(k)} \) is copolyform and hence \( P \) is copolyform. Therefore \( \text{Rad}(\text{End}(P)) = 0 \). By [13, 22.2] \( \text{End}(P)/\text{Rad}(\text{End}(P)) \cong \text{End}(P/\text{Rad}(P)) \) and hence \( \text{End}(P) \) is a semisimple ring.

4. Endomorphism Rings of \( \mathcal{Z}_M(M) \), When \( M \) is \( \Sigma \)-Lifting and Injective

In this section we show that if \( M \) is a \( \Sigma \)-lifting injective module and \( N \) is an indecomposable direct summand of \( M \), then \( \text{Hom}(\mathcal{Z}_M(N), \mathcal{Z}_M(M)) \) is a uniserial \( \mathcal{S} \)-module of finite length, where \( \mathcal{S} = \text{End}(\mathcal{Z}_M(M)) \).

**Proposition 4.1.** Let \( M \) be a \( \Sigma \)-lifting module which is injective in \( \sigma[M] \). Then \( M = \oplus_i M_i \), where each \( M_i \) is local, self-projective and indecomposable. Also we have the following.

1. For every \( k \in I \), \( \mathcal{A} = \{ \text{Ker}(J) \mid J \subseteq \text{Hom}(\mathcal{Z}_M(M_k), M) \} \) is linearly ordered by set inclusion;
2. the family \( \{ \mathcal{Z}_M(M_i) \}_{i} \) is semi-T-nilpotent.

**Proof.** Since \( M \) is a lifting and injective module, \( M = \oplus_i M_i \) where each \( M_i \) is indecomposable [10, 2.4, 2.5]. As \( M_i \) is injective and indecomposable, \( \text{End}(M_i) \) is local. Therefore by Proposition 2.1 each \( M_i \) is local and self-projective.

1. Suppose that \( 0 \neq I_1 \) and \( 0 \neq I_2 \subseteq \text{Hom}(\mathcal{Z}_M(M_k), M) \) and \( \text{Ker}(I_1) \not\subseteq \text{Ker}(I_2) \). Then for any nonzero \( f \in I_1 \), there exists a nonzero \( g \in I_2 \) such that \( \text{Ker} f \not\subseteq \text{Ker} g \); for if not, then for every \( g \in I_2 \), \( \text{Ker} f \subseteq \text{Ker} g \) implies \( \text{Ker} f \subseteq \text{Ker}(I_2) \) and hence \( \text{Ker}(I_1) \not\subseteq \text{Ker}(I_2) \) which is a contradiction.

As \( M \) is injective in \( \sigma[M] \), homomorphisms \( f, g \) can be extended to \( \overline{f}, \overline{g} : M_k \to M \), respectively.

Then \( \text{Ker} f = \mathcal{Z}_M(M_k) \cap \text{Ker} \overline{f} \) and \( \text{Ker} g = \mathcal{Z}_M(M_k) \cap \text{Ker} \overline{g} \). If \( \text{Im} \overline{f} \) is small in \( M \), then \( \text{Im} \overline{f} \) is an \( M \)-small module and hence is \( M \)-cosingular. Thus \( M_k/\text{Ker} \overline{f} \cong \text{Im} \overline{f} \) is \( M \)-cosingular and hence \( \mathcal{Z}_M(M_k) \subseteq \text{Ker} \overline{f} \). Therefore \( \text{Ker} f = \mathcal{Z}_M(M_k) \) which is contradiction. Hence \( \text{Im} \overline{f} \not\subseteq M \). Similarly \( \text{Im} \overline{g} \not\subseteq M \).

Since \( M_k \) is an indecomposable direct summand of \( M \) with local endomorphism ring, by Proposition 2.3 (1) either \( \text{Ker} \overline{f} \subseteq \text{Ker} \overline{g} \) or \( \text{Ker} \overline{g} \subseteq \text{Ker} \overline{f} \). As \( \text{Ker} \overline{f} \not\subseteq \text{Ker} \overline{g} \), so \( \text{Ker} \overline{g} \subseteq \text{Ker} \overline{f} \) and hence \( \text{Ker} g \subseteq \text{Ker} f \). Thus \( \text{Ker}(I_2) \subseteq \text{Ker} f \). Since \( f \) is any nonzero element of \( I_1 \), \( \text{Ker}(I_2) \subseteq \text{Ker}(I_1) \).

2. Suppose that \( f : \mathcal{Z}_M(M_i) \to \mathcal{Z}_M(M_j) \) is a nonzero non-isomorphism. As \( M \) is injective in \( \sigma[M] \), \( M_j \) is \( M_i \)-injective and hence \( f \) can be extended to an homomorphism \( \overline{f} : M_i \to M_j \). We claim that \( \overline{f} \) is also a non-isomorphism.
If $\overline{f}$ is not onto, then $\text{Im} \overline{f}$ is an $M$-small module and hence is $M$-cosingular. Thus $M_i/Ker \overline{f} \simeq \text{Im} \overline{f}$ is $M$-cosingular. Therefore $\text{Ker} \overline{f} \supseteq \underline{Z}_M(M_i)$ and hence $f = 0$, a contradiction. Thus $\overline{f}$ is an surjective map. Suppose $\text{Ker} \overline{f} = 0$. Then $\overline{f}$ is an isomorphism. Hence $M_i \simeq M_j$ and so $\underline{Z}_M(M_i) \simeq \underline{Z}_M(M_j)$ and $\text{Ker} f = 0$. $\underline{Z}_M(M_i)$ is fully invariant in $M_i$ and hence is self-injective. Thus $\text{Im} f \simeq \underline{Z}_M(M_i)$ is $\underline{Z}_M(M_j)$-injective implies that $\text{Im} f$ is a direct summand of $\underline{Z}_M(M_j)$. As $\underline{Z}_M(M_j)$ is uniform $\text{Im} f = \underline{Z}_M(M_j)$. Therefore $f$ is an isomorphism, a contradiction. Thus $\text{Ker} \overline{f} \neq 0$.

By Theorem 1.5 the family $\{M_i\}_I$ is semi-$T$-nilpotent. Since any nonzero non-isomorphism from $\underline{Z}_M(M_i) \to \underline{Z}_M(M_j)$ can be extended to a non-isomorphism from $M_i \to M_j$, the family $\{\underline{Z}_M(M_i)\}_I$ is also semi-$T$-nilpotent.  

**Proposition 4.2.** Let $M$ be a $\Sigma$-lifting module which is injective in $\sigma[M]$. Then $M = \mathcal{O}_1 M$ where each $M_i$ is a local and self-projective module. If $\Sigma = \text{End}(\underline{Z}_M(M))$, then we have

1. for any $k \in I$, Hom$(\underline{Z}_M(M_k), \underline{Z}_M(M))$ is a uniserial, artinian left $\Sigma$-module;
2. if $\{\underline{Z}_M(M_i)\}_I$ contains only a finite number of non-isomorphic modules, then each Hom$(\underline{Z}_M(M_k), \underline{Z}_M(M))$ is a uniserial $\Sigma$-module of finite length;
3. Hom$(\underline{Z}_M(M), \underline{Z}_M(M_k))$ is a uniserial right $\Sigma$-module.

**Proof.** The first assertion follows from Proposition 4.1. (1) Let $f, g$ be two nonzero homomorphisms in Hom$(\underline{Z}_M(M_k), \underline{Z}_M(M))$. By the injectivity of $M$ they can be extended to $\overline{f}, \overline{g}: M_k \to M$. As in the proof of Proposition 4.1(1), $\text{Im} \overline{f}$ and $\text{Im} \overline{g}$ are not small in $M$. By Proposition 2.3(1) either Ker $\overline{f} \subseteq \text{Ker} \overline{g}$ or Ker $\overline{g} \subseteq \text{Ker} \overline{f}$.

Suppose Ker $\overline{f} \subseteq \text{Ker} \overline{g}$. We claim that $S\overline{f} \subseteq S\overline{g}$, where $S = \text{End}(M)$. For, there exists a surjective map $\psi: \text{Im} \overline{f} \to \text{Im} \overline{g}$ such that $\psi \overline{f} = \overline{g}$. Since $\text{Im} \overline{f}$ is a direct summand of $M$, we can extend $\psi$ to $\phi: M \to M$ such that $\phi \overline{f} = \overline{g}$. Therefore $S\overline{f} \subseteq S\overline{g}$. Hence either $S\overline{f} \subseteq S\overline{g}$ or $S\overline{g} \subseteq S\overline{f}$.

Suppose that $S\overline{f} \subseteq S\overline{g}$. As any homomorphism $\underline{Z}_M(M) \to \underline{Z}_M(M)$ can be extended to a homomorphism from $M \to M$, and the restriction of any map from $M \to M$ to $\underline{Z}_M(M)$ can be considered as a map from $\underline{Z}_M(M) \to \underline{Z}_M(M)$, we get so $S\overline{f} \subseteq S\overline{g}$.

Similarly if $S\overline{g} \subseteq S\overline{f}$, then $S\overline{g} \subseteq S\overline{f}$. Hence Hom$(\underline{Z}_M(M_k), \underline{Z}_M(M))$ is a uniserial left $\Sigma$-module.

We have to prove that Hom$(\underline{Z}_M(M_k), \underline{Z}_M(M))$ is an artinian left $\Sigma$-module. Let

\[
I_1 \supsetneq I_2 \supsetneq \cdots \supsetneq I_r \supsetneq \cdots
\]

be a strictly descending chain of $\Sigma$-submodules of Hom$(\underline{Z}_M(M_k), \underline{Z}_M(M))$.

For any $r \in \mathbb{N}$, there exists $f_r \in I_r$ such that $f_r \notin I_{r+1}$. We have either $Sf_r \subseteq Sf_{r+1}$ or $Sf_{r+1} \subseteq Sf_r$. Since $f_r \notin I_{r+1}$ we get $Sf_r \supsetneq Sf_{r+1}$. Hence $Sf_r \supsetneq Sf_{r+1}$.

As $f_{r+1} \in Sf_r$, there exists $\pi \in S$ such that $f_{r+1} = \pi f_r$. Therefore Ker $f_r \subseteq$ Ker $f_{r+1}$ which implies that Ker $f_r \cap \underline{Z}_M(M_k) \subseteq$ Ker $f_{r+1} \cap \underline{Z}_M(M_k)$, where $f_r$ and $f_{r+1}$ are extensions of $f_r$ and $f_{r+1}$ respectively from $M_k$ to $M$.
Suppose \( \text{Ker} \mathcal{f}_r = \text{Ker} \mathcal{f}_{r+1} \). As \( f_r \) and \( f_{r+1} \) are nonzero maps, the images of the maps \( \mathcal{f}_r \) and \( \mathcal{f}_{r+1} \) are not small in \( M \) and hence are isomorphic to direct summands of \( M \). Hence we can define a map \( \phi : M \rightarrow M \) such that \( \mathcal{f}_r = \phi \mathcal{f}_{r+1} \). The restriction to \( \mathcal{Z}_M(M_k) \) gives us \( f_r = \phi|_{\mathcal{Z}_M(M_k)} f_{r+1} \). Since \( \phi|_{\mathcal{Z}_M(M_k)} \) can be considered as an element of \( \mathcal{S} \), we get \( \mathcal{f}_r \subseteq \mathcal{S} f_{r+1} \), a contradiction. Hence \( \text{Ker} \mathcal{f}_r \subseteq \text{Ker} \mathcal{f}_{r+1} \).

Hence we get a strictly ascending chain

\[
\text{Ker} \mathcal{f}_1 \subseteq \text{Ker} \mathcal{f}_2 \subseteq \cdots \subseteq \text{Ker} \mathcal{f}_r \subseteq \cdots.
\]

For all \( r \in \mathbb{N} \), \( \text{Im} \mathcal{f}_r \) is not small in \( M \). Hence by Proposition 2.3 (2) the above chain becomes stationary after finitely many steps and hence this is also true for the chain

\[
I_1 \supsetneq I_2 \supsetneq \cdots \supsetneq I_r \supsetneq \cdots.
\]

Therefore \( \text{Hom}(\mathcal{Z}_M(M_k), \mathcal{Z}_M(M)) \) is an artinian left \( \mathcal{S} \)-module.

(2) By (1) it is enough to prove that \( \text{Hom}(\mathcal{Z}_M(M_k), \mathcal{Z}_M(M)) \) satisfies ACC on \( \mathcal{S} \)-submodules. Suppose that

\[
I_1 \subsetneq I_2 \subsetneq \cdots \subsetneq I_r \subsetneq \cdots
\]

is a strictly ascending chain of \( \mathcal{S} \)-submodules of \( \text{Hom}(\mathcal{Z}_M(M_k), \mathcal{Z}_M(M)) \).

As in the proof of (1) we get homomorphisms \( \mathcal{f}_r : M_k \rightarrow M \) such that \( \text{Im} \mathcal{f}_r \not\cong M \) and

\[
\text{Ker} \mathcal{f}_1 \subsetneq \text{Ker} \mathcal{f}_2 \subsetneq \cdots \subsetneq \text{Ker} \mathcal{f}_r \subsetneq \cdots.
\]

By Proposition 2.3 (3) this chain stops. Therefore \( \text{Hom}(\mathcal{Z}_M(M_k), \mathcal{Z}_M(M)) \) is a uniserial \( \mathcal{S} \)-module of finite length.

(3) Any two nonzero maps \( \phi, \psi \in \text{Hom}(\mathcal{Z}_M(M), \mathcal{Z}_M(M_k)) \) can be extended to nonzero maps \( \overline{\phi}, \overline{\psi} \in \text{Hom}(M, M_k) \). Since \( \text{Im} \overline{\phi} \) and \( \text{Im} \overline{\psi} \) are not small in \( M_k \) (as in the proof of Proposition 4.1 (1)), \( \overline{\phi}, \overline{\psi} \) are surjective maps. By Theorem 2.4, either \( \overline{\phi} S \subseteq \overline{\psi} S \) or \( \overline{\psi} S \subseteq \overline{\phi} S \) where \( S = \text{End}(M) \). Suppose that \( \overline{\phi} S \subseteq \overline{\psi} S \).

As any homomorphism \( \mathcal{Z}_M(M) \rightarrow \mathcal{Z}_M(M) \) can be extended to a homomorphism from \( M \rightarrow M \), and the restriction of any map from \( M \rightarrow M \) to \( \mathcal{Z}_M(M) \) can be considered as a map from \( \mathcal{Z}_M(M) \rightarrow \mathcal{Z}_M(M) \), \( \overline{\phi} S \subseteq \overline{\psi} S \). Hence \( \text{Hom}(\mathcal{Z}_M(M), \mathcal{Z}_M(M_k)) \) is a uniserial right \( \mathcal{S} \)-module.

References