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Copolyform Σ-Lifting Modules

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Abstract. A module M is called copolyform if every coessential submodule of M is corational in M. It is known that every polyform Σ-extending module is a direct sum of indecomposable self-injective modules. In this paper we study some properties of copolyform Σ-lifting modules. We show that a copolyform Σ-lifting module is a direct sum of indecomposable self-projective modules whose M-annihilator submodules are linearly ordered and satisfy ACC. We also prove that every copolyform Σ-lifting module M is non-M-cosingular module. Consequently, for M finitely generated, $\text{End}(M)$ is a left and right serial artinian hereditary ring. We then consider Σ-lifting injective modules in terms of $\overline{Z}_M()$ and show that for any indecomposable direct summand N of M, $\text{Hom}(\overline{Z}_M(N), \overline{Z}_M(M))$ is a uniserial $\overline{S}$-module of finite length where $\overline{S} = \text{End}(\overline{Z}_M(M))$.

1. Preliminaries

Let R be an associative ring with identity. All modules we consider are unitary right R-modules. Suppose M is an R-module. A submodule A of M is said to be a *small submodule* of M (denoted by $A \ll M$) if for any $B \subseteq M$, $A + B = M$ implies $B = M$. A module M is called a *hollow* module if every proper submodule of M is small in M. A module is called a *local* module if it has a unique maximal submodule containing all its proper submodules. It is easy to see that a module is a local module if and only if it is a cyclic hollow module.

For $A \subseteq B \subseteq M$, A is said to be a *coessential submodule of* B *in* M (denoted by $A \overset{ce}{\hookrightarrow} B$ in M) if $B/A \ll M/A$. In this case we also say B is a *coessential extension* of A in M. Instead of coessential extension the term *lying above* was used in Wisbauer [13]. For example, if $R = \mathbb{Z} = M$ and $A = 2\mathbb{Z}$, then for any $k \in \mathbb{N}$, $2^k\mathbb{Z}$ is a coessential submodule of A in M.

A is said to be *coclosed in* M (denoted by $A \overset{cc}{\hookrightarrow} M$) if A has no proper coessential submodule in M. The following lemma regarding hollow submodules of a module has been proved by Inoue [6, Proposition 6].

Lemma 1.1. [6] *Let M be an R-module and N a hollow submodule of M. Then $N \ll M$ or $N \overset{cc}{\hookrightarrow} M$.*

We recall the definition of an amply supplemented module. If N and L are submodules of the module M, then N is called a *supplement* (resp. *weak supplement*) of L, if $N + L = M$ and $N \cap L \ll N$ (resp. $N \cap L \ll M$). M is called *supplemented* (resp. *weakly supplemented*) if each of its submodules has a supplement (resp. weak supplement) in M. A module M is called Σ-*weakly supplemented*, if any direct sum of copies of M is weakly supplemented. M is called *amply supplemented*, if for all submodules N and L of M with $N+L = M$, N contains a supplement of L in M.

A module M is called *lifting* if every submodule $A \subseteq M$ contains a direct summand B of M such that $A/B \ll M/B$. M is said to be (finitely) Σ-lifting if every (finite) direct sum of copies of M is lifting. The following lemma has been proved by Mohamed and Müller [9, 4.8].

Lemma 1.2. [9] *A module is lifting if and only if it is amply supplemented and its coclosed submodules are direct summand.*

Suppose M is an R-module. We recall the definitions of M-projective, self-projective and almost M-projective modules. An R-module N is called M-*projective* if for every epimorphism $f : M \to K$ and every homomorphism $g : N \to K$, there exists $h : N \to M$ with $fh = g$. N is called *self-projective* (resp. *projective*) if it is N-projective (L-projective for any R-module L). N is called almost M-projective, if for every epimorphism $f : M \to K$ and every homomorphism $g : N \to K$, either there exists $h : N \to M$ with $fh = g$ or there exists a nonzero direct summand M_1 of M and $\overline{h} : M_1 \to N$ with $g\overline{h} = f|_{M_1}$.

A family $\{M_i\}_I$ of modules is called (*locally*) *semi-T-nilpotent*, if for any countable family $\{f_n : M_{i_n} \to M_{i_{n+1}}\}_{\mathbb{N}}$ of non-isomorphisms with $i_n \in I$ all distinct, (and for any element $x \in M_{i_1}$), there exists $k \in \mathbb{N}$ (k depending on x) such that $f_k \dots f_1 = 0$ ($f_k \dots f_1(x) = 0$). It is obvious that if each M_i is a local module, then the family $\{M_i\}_I$ of modules is locally semi-T-nilpotent if and only if it is semi-T-nilpotent.

Regarding lifting modules with local endomorphism ring the following results have been proved by Baba and Harada [2].

Theorem 1.3. [2, Theorem 1] *Let $\{M_i\}_{i=1}^n$ be a set of hollow modules with local endomorphism ring. Then the following are equivalent.*

(1) *$\oplus_{i=1}^n M_i$ is lifting;*

(2) *M_i is almost M_j-projective for any $i \neq j$;*

(3) *for any subset J of $I = \{1, 2, \dots, n\}$, $\oplus_{i \in J} M_i$ is almost $\oplus_{i \in I \setminus J} M_i$-projective.*

Lemma 1.4. [2, Lemma 3] *Let $\{M_i\}_I$ be a family of modules with local endomorphism ring. If $\oplus_{i\in I}M_i$ is lifting, then $\{M_i\}_I$ is locally semi-T-nilpotent.*

Theorem 1.5. [2, Theorem 2] *Let $\{M_i\}_I$ be a family of local modules with local endomorphism ring. Then the following are equivalent.*
(1) *$\oplus_{i\in I}M_i$ is lifting;*
(2) *M_i is almost M_j-projective for any $i \neq j$ and $\{M_i\}_I$ is locally semi-T-nilpotent;*
(3) *for any subset J of I, $\oplus_{i\in J}M_i$ is almost $\oplus_{i\in I\setminus J}M_i$-projective and $\{M_i\}_I$ is locally semi-T-nilpotent.*

Lemma 1.6. [2, Lemma 4] *Let M be a hollow module with local endomorphism ring. If any direct sum of copies of M is lifting, then M is cyclic.*

2. Σ-Lifting Modules

In this section we prove that a Σ-lifting module with local endomorphism ring is self-projective. Suppose M is a Σ-lifting module such that any indecomposable direct summand of M has local endomorphism ring. Let N be any indecomposable direct summand of M and $\mathcal{A} = \{\mathrm{Ker} f \mid f : N \to M, \mathrm{Im}\, f \not\ll M\}$. We prove that then N has ACC on $\mathcal{A}$ and if M has only finitely many non-isomorphic indecomposable direct summands, then N has DCC on $\mathcal{A}$. Suppose M is a direct sum of modules with local endomorphism rings and is Σ-lifting. Let $S = \mathrm{End}(M)$ and N is any indecomposable direct summand of M. Then as a right S-module $A = \mathrm{Hom}(M, N)$ has a waist $B = \{f : M \to N \mid \mathrm{Im}\, f \ll N\}$ and A/B is a uniserial module.

Proposition 2.1. *Let M be a nonzero Σ-lifting, indecomposable R-module with* $\mathrm{End}(M)$ *local. Then M is local and self-projective.*

Proof. By Lemma 1.6, M is a local module. We claim that any surjective map from M to M is an isomorphism. Suppose not. Let $f : M \to M$ be surjective map which is not 1-1.

By 1.4, the family $\mathcal{F} = \{M_n\}, n \in \mathbb{N}$, where $M_n = M$, for all $n \in \mathbb{N}$, is locally semi-T-nilpotent. Since M is a local module the family $\mathcal{F}$ is semi-T-nilpotent.

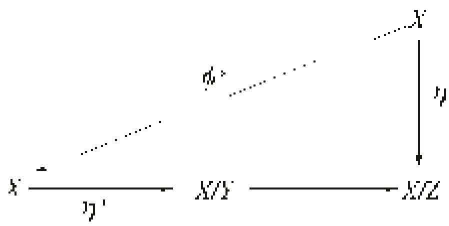
Consider $f_n = f : M_n \to M_{n+1}$, for all $n \in \mathbb{N}$. Since $\{M_i\}_I$ is semi-T-nilpotent, there exists a positive number k such that $f^k : M_1 = M \to M_k = M$ is a zero epimorphism, which is a contradiction.

As $M \oplus M$ is lifting and every nonzero epimorphism $M \to M$ is 1-1, M is self-projective [12, Lemma 2.3].

Lemma 2.2. *Let M be an indecomposable self-projective lifting R-module. If A and B are fully invariant submodules of M such that $M/A \oplus M/B$ is lifting, then either $A \subseteq B$ or $B \subseteq A$.*

Proof. We first show that if Y is a fully invariant submodule of a self-projective module X and if $\phi : X/Y \to X/Z$ is an onto map, then $Y \subseteq Z$. We can lift

$\phi\eta'$ to a map $\phi' : X \to X$ such that $\phi\eta'\phi' = \eta$ where $\eta' : X \to X/Y$ and $\eta : X \to X/Z$ are the natural maps.



Since $\phi'(Y) \subseteq Y$, we get $\eta(Y) = \phi\eta'\phi'(Y) = 0$. Hence $Y \subseteq Z$.

Since A and B are fully invariant submodules of a self-projective module M, M/A and M/B are self-projective [14, Proposition 2.1]. As M/A and M/B are hollow self-projective, End(M/A) and End(M/B) are local rings. Suppose $f : M/A \to M/(A+B)$ and $g : M/B \to M/(A+B)$ are the natural maps. Since $M/A \oplus M/B$ is lifting, M/A is almost M/B-projective Theorem 1.3. Also M/B is indecomposable. Hence we can get either a map $h : M/A \to M/B$ or a map $h' : M/B \to M/A$, such that $gh = f$ or $fh' = g$.

As f and g are small epimorphisms, the maps h and h' (if they exist) will be epimorphisms. Hence there exists either an epimorphism $h : M/A \to M/B$ or an epimorphism $h' : M/B \to M/A$. Therefore either $A \subseteq B$ or $B \subseteq A$. ■

Proposition 2.3. *Suppose M is a Σ-lifting module such that the endomorphism ring of every indecomposable direct summand of M is a local ring. Suppose that N is an indecomposable direct summand of M, $\mathcal{K} = \{f : N \to M \mid \mathrm{Im}\, f \not\ll M\}$ and $\mathcal{A} = \{\mathrm{Ker} f \mid f \in \mathcal{K}\}$. Then*

(1) *$\mathcal{A}$ is linearly ordered by inclusion;*

(2) *N has ACC on $\mathcal{A}$;*

(3) *N has DCC on $\mathcal{A}$, if M has only finitely many non-isomorphic indecomposable direct summands.*

Proof. Suppose that L is an indecomposable direct summand of M. Since L is Σ-lifting with local endomorphism ring, then by Proposition 2.1, L is a self-projective local module.

(1) Suppose that $f \in \mathcal{K}$. As $\mathrm{Im}\, f$ is hollow, them by Lemma 1.1, $\mathrm{Im}\, f$ is coclosed in M. Since M is lifting, $\mathrm{Im}\, f$ is an indecomposable direct summand of M Lemma 1.2. Hence $\mathrm{Im}\, f$ is local and self-projective. Since $\mathrm{Ker} f \ll N$ and $N/\mathrm{Ker} f$ is self-projective, $\mathrm{Ker} f$ is fully invariant in N [14, Proposition 2.2].

Suppose that $f, g \in \mathcal{K}$; then $\mathrm{Ker} f$ and $\mathrm{Ker} g$ are fully invariant in N. $N/\mathrm{Ker} f \oplus N/\mathrm{Ker} g$ (as it is isomorphic to a direct summand of $M \oplus M$) is a lifting module. By Lemma 2.2, either $\mathrm{Ker} f \subseteq \mathrm{Ker} g$ or $\mathrm{Ker} g \subseteq \mathrm{Ker} f$. Hence $\mathcal{A}$ is linearly ordered by inclusion.

(2) Now suppose that there exists a strictly ascending chain

$$X_1 \subsetneqq X_2 \subsetneqq \cdots \subsetneqq X_i \subsetneqq X_{i+1} \subsetneqq \cdots$$

of elements in $\mathcal{A}$. Then there exists $f_i : N \to M$ such that $\mathrm{Ker} f_i = X_i$ and $\mathrm{Im}\, f_i \not\ll M$ for every $i \in \mathbb{N}$. For each $i \in \mathbb{N}$, N/X_i is isomorphic to a direct summand of M and hence is a Σ-lifting, self-projective, local module with local endomorphism ring.

As M is Σ-lifting we have $\oplus_{i=1}^{\infty} N/X_i$ is a lifting module with $\mathrm{End}(N/X_i)$ local, for all $i \in \mathbb{N}$. Thus $\{N/X_i\}_{\mathbb{N}}$ is locally semi-T-nilpotent (1.4) and hence semi-T-nilpotent (as each N/X_i is local). By considering the natural maps $\eta_i : N/X_i \to N/X_{i+1}$ for all $i \in \mathbb{N}$, we get a contradiction. Hence N satisfies ACC on $\mathcal{A}$.

(3) Suppose that

$$Y_1 \supsetneqq Y_2 \supsetneqq \cdots \supsetneqq Y_i \supsetneqq Y_{i+1} \supsetneqq \cdots$$

is a strictly descending chain of elements in $\mathcal{A}$. Each N/Y_i is isomorphic to some indecomposable direct summand of M. As there are only finitely many non-isomorphic indecomposable direct summands of M, we get $N/Y_\ell \simeq N/Y_k$ for some k and ℓ. Suppose $k < \ell$. Then $Y_\ell \subsetneqq Y_k$. Since $N/Y_\ell \simeq N/Y_k$ is self-projective, the natural map $f : N/Y_\ell \to N/Y_k$ splits. Therefore Y_k/Y_ℓ is a nonzero proper direct summand of the hollow module N/Y_ℓ, which is a contradiction. ■

Recall that a submodule B of a module A is called a *waist* if for every submodule C of A either $B \subseteq C$ or $C \subseteq B$ holds.

Theorem 2.4. *Let $M = \oplus_I M_i$ be a Σ-lifting module, where each M_i has a local endomorphism ring. Suppose that N is a nonzero indecomposable direct summand of M, $A = \mathrm{Hom}(M, N)$ and $B = \{f \in A \,|\, \mathrm{Im}\, f \ll N\}$. Then B is a waist of A such that A/B is a uniserial right S-module, where $S = \mathrm{End}(M)$.*

Proof. Since each M_i is indecomposable Σ-lifting, by Proposition 2.1, M_i is local and self-projective. We note that any nonzero $f \in A$ such that $f \notin B$ is an epimorphism. To prove that B is a waist of A, it is enough to prove that for any onto map $f \in A$ and $g \in B$, $gS \subseteq fS$.

Now consider an epimorphism $f : M \to N$ and $g : M \to N$ with $\mathrm{Im}\, g \ll N$. There exists $i_0 \in I$ such that $f_{i_0} : M_{i_o} \to N$ is onto (for N is a local module), where f_{i_0} is the restriction of f to M_{i_0}.

For every $i \in I$, consider $g_i : M_i \to N$, the restriction of g to M_i. As $M_i \oplus M_{i_0}$ is lifting M_i and M_{i_0} are relatively almost-projective modules. Hence for every $i \in I$, there exists $\phi_i : M_i \to M_{i_0}$ such that $f_{i_0}\phi_i = g_i$.

Define $\phi : M \to M$ by $\phi|_{M_i} = \phi_i$, for every $i \in I$. It is obvious that $f\phi = g$. Hence $gS \subseteq fS$. Therefore B is a waist of A.

To prove that A/B is a uniserial right S-module, it is enough to show that whenever $f, g : M \to N$ are surjective maps, either $fS \subseteq gS$ or $gS \subseteq fS$.

For every $i \in I$ put $[i] = \{j \in I | M_i \simeq M_j\}$ and $\mathcal{F} = \{[i] | i \in I\}$. Define an order on $\mathcal{F}$ by $[i] \leq [j]$ if and only if there exists a surjective map $M_i \to M_j$. We claim that $(\mathcal{F}, \leq)$ is a partially ordered set. Assume that $[i] \leq [j]$ and $[j] \leq [i]$ for some $i, j \in I$. Then there exist surjective maps $\theta : M_i \to M_j$ and $\psi : M_j \to M_i$. Since M_i is self-projective, the map $\psi\theta : M_i \to M_i$ splits. Therefore $M_i \simeq M_j$ and hence $[i] = [j]$. Now it is easy to see that $(\mathcal{F}, \leq)$ is a partially ordered set.

As each M_i is local, by Lemma 1.4 the family $\{M_i\}_I$ is semi-T-nilpotent. Therefore every nonempty subset $\mathcal{G}$ of $\mathcal{F}$ has a maximal element.

Suppose that

$$I_f = \{i \in I \,|\, f_i = f|_{M_i} \text{ is onto}\} \quad \text{and} \quad I_g = \{j \in I | g_j = g|_{M_j} \text{ is onto}\}.$$

Since N is local, $I_f \neq \emptyset$ and $I_g \neq \emptyset$. Define

$$\overline{I}_f := \{[i] | i \in I_f\} \text{ and } \overline{I}_g := \{[i] | i \in I_g\}.$$

Suppose $[i_0]$ is a maximal element of $\overline{I}_f \cup \overline{I}_g$ and that $i_o \in I_f$. We claim that $gS \subseteq fS$.

Let $f_{i_0} : M_{i_0} \to N$ be the restriction of f to M_{i_0} and $g_i : M_i \to N$ be the restriction of g to M_i, for every $i \in I$. As before M_i and M_{i_0} are relatively almost-projective modules for every $i \in I$. Hence if g_i is not onto, there exists $\phi_i : M_i \to M_{i_0}$ such that $f_{i_0}\phi_i = g_i$. If g_i is onto, then either there exists a surjective map $\phi_i : M_i \to M_{i_0}$ such that $f_{i_0}\phi_i = g_i$ or a surjective map $\psi_i : M_{i_0} \to M_i$ such that $g_i\psi_i = f_{i_0}$.

By the choice of i_o, the existence of the surjective map ψ_i from M_{i_0} to M_i will imply that $M_i \simeq M_{i_0}$. Since M_i is self-projective, the map ψ_i is an isomorphism. Hence we always get a map $\phi_i : M_i \to M_{i_0}$ such that $f_{i_0}\phi_i = g_i$.

Define $\phi : M \to M$ by $\phi|_{M_i} = \phi_i$, for every $i \in I$. It is obvious that $f\phi = g$. Hence $gS \subseteq fS$. ■

3. Copolyform Σ-Lifting Modules

Clark and Wisbauer [3] have proved that a polyform Σ-extending module M is a direct sum of self-injective modules. In this section dually we show that every copolyform Σ-lifting module is a direct sum of self-projective modules whose M-annihilator submodules are linearly ordered.

Suppose M is an R-module. By $\sigma[M]$ we mean the full subcategory of Mod-R whose objects are submodules of M-generated modules. The injective hull of $N \in \sigma[M]$ is denoted by $\widehat{N}$. $N \in \sigma[M]$ is said to be an *M-small* module if N is small in $\widehat{N}$. It is easy to see that N is an M-small module if and only if there exists a module $L \in \sigma[M]$ such that $N \ll L$. We define $\overline{Z}_M(N)$, as follows:

$$\overline{Z}_M(N) = \text{Re}(N, \mathcal{S}) = \bigcap \big\{\text{Ker}(g) | g \in \text{Hom}(N, L), L \in \mathcal{S}\big\},$$

where $\mathcal{S}$ denotes the class of all M-small modules. We call N an *M-cosingular* (*non-M-cosingular*) module if $\overline{Z}_M(N) = 0$ ($\overline{Z}_M(N) = N$). It is easy to see that a module $N \in \sigma[M]$ is non-M-small if and only if every nonzero factor module of N is non-M-small.

Corational extension and copolyform module which are dual concepts of rational extension and polyform module are defined and studied in [11]. We give the definitions.

Suppose that $A \subseteq B \subseteq M$. We say that A *is a coessential submodule of B in M*(denoted by $A \overset{ce}{\hookrightarrow} B$ in M), if $B/A \ll M/A$. We call A *a corational submodule of B in M*, if $\text{Hom}(M/A, B/X) = 0$, for any submodule X such that

$A \subseteq X \subseteq B$. We denote this by $A \overset{cr}{\hookrightarrow} B$ *in* M. In this case we also say that B is a *corational extension* of A in M.

We call a module M a *copolyform module* if $A \overset{ce}{\hookrightarrow} B$ in M implies $A \overset{cr}{\hookrightarrow} B$ in M. Equivalently a module M is a copolyform module if whenever $B/A \ll M/A$, $\mathrm{Hom}(M/A, B/X) = 0$, for $A \subseteq X \subseteq B$. A module M is Σ-*copolyform*, if any direct sum of copies of M is copolyform.

Suppose M is an R-module and $A \subseteq M$. Consider the set $\mathcal{A}$ of all coessential submodules of A in M. Minimal elements of $\mathcal{A}$ under set inclusion, if they exists, are called *coclosures of* A *in* M. If M is amply supplemented, then coclosures of every submodule of A in M exist. A module M is called a *unique coclosure module* (denoted by UCC module), if every submodule of M has a unique coclosure in M [5]. We call a module M a Σ-*UCC module*, if any direct sum of copies of M is a UCC module.

Suppose that M is an amply supplemented module. If M is copolyform, then M is a UCC module [5, 4.2]. The converse is not true. For example, consider $\mathbb{Z}/8\mathbb{Z}$ as a $\mathbb{Z}$-module. But if $M \oplus M$ is UCC, then M is copolyform [5, 4.6]. The following lemma is trivial.

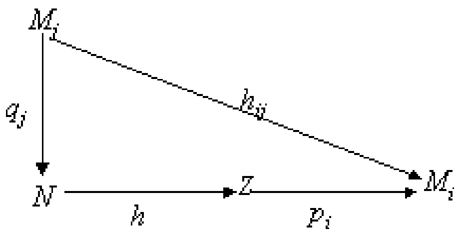
Lemma 3.1. *Suppose that M is a Σ-amply supplemented module. Then M is a Σ-UCC module if and only if it is a Σ-copolyform module.*

It is known that if M is a polyform module then M is non-M-singular. We do not know whether the dual is true; but it has been proved that if M is Σ-copolyform and Σ-weakly supplemented, then M is non-M-cosingular [11, 2.11]. We prove below that if M is copolyform and Σ-lifting, then M is non-M-cosingular.

Proposition 3.2. *Let M be a copolyform Σ-lifting module. Define $N := \oplus_{i \in I} M_i$, where $M_i = M$ for every $i \in I$. If X is a direct summand of N and $f : N \to X$, then $f(A) \overset{cc}{\hookrightarrow} X$ whenever $A \overset{cc}{\hookrightarrow} N$.*

Proof. Suppose $A \overset{cc}{\hookrightarrow} N$. Since N is a lifting module, A is a direct summand of N Lemma 1.2 and hence $N = A \oplus B$. Consider the homomorphism $g : N \to X$ such that $g = f$ on A and $g = 0$ on B. Then $g(A) = f(A) = g(N)$. Moreover, since X is a lifting module, there exists a direct summand $Y \subseteq g(A)$ such that $X = Y \oplus Z$ and $g(A) = Y \oplus (g(A) \cap Z)$ with $(g(A) \cap Z) \ll Z$. This gives a map $h := pg : N \to Z$, where $p : X \to Z$ is the projection map along Y. Now $\mathrm{Im}\, h = pg(N) = g(A) \cap Z \ll Z$.

Let $\pi_i : N \to M_i$ and $q_i : M_i \to N$ be the natural projection and inclusion maps for any $i \in I$. Let $p_i = \pi_i|_Z$; then we get a homomorphism $h_{ij} = p_i h q_j : M_j \to M_i$, for each $i, j \in I$.



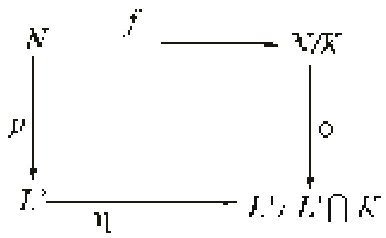
Since M_i is copolyform for all $i, j \in I$ $h_{ij} = 0$ [11, 2.3]. Hence $h = 0$. Therefore $g(A) \cap Z = 0$. Thus $f(A) = g(A)$ is a direct summand of X, and hence $f(A) \overset{cc}{\hookrightarrow} X$. ∎

Proposition 3.3 *Let M be a copolyform Σ-lifting R-module. Then M is a non-M-cosingular module.*

Proof. We know that a Σ-copolyform Σ-weakly supplemented module M is non-M-cosingular [11, 2.11]. We want to prove that a copolyform Σ-lifting module is non-M-cosingular. As M is Σ-lifting, M is Σ-amply supplemented. Hence it is enough to prove that M is Σ-copolyform. By Lemma 3.1 it is enough to show that M is Σ-UCC.

Suppose I is any indexing set and $N = \oplus_{i\in I} M_i$, where $M_i = M$ for every $i \in I$. We want to show that N is a UCC module. For this we prove that, for all epimorphism $f : N \to N/K$, $A \overset{cc}{\hookrightarrow} N$ implies $f(A) \overset{cc}{\hookrightarrow} N/K$ [5, 3.16].

Suppose that $f : N \to N/K$ is an epimorphism and $A \overset{cc}{\hookrightarrow} N$. As N is a lifting module, $N = L \oplus L'$, where $L \subseteq K$ and $K = L \oplus (K \cap L')$ with $(K \cap L') \ll L'$. We have an isomorphism $\phi : N/K \to L'/(L' \cap K)$. Also $\phi f = \eta p$, where $p : N \to L'$ is the projection along L, and $\eta : L' \to L'/(L' \cap K)$ the natural map.



Our aim is to prove that if $A \overset{cc}{\hookrightarrow} N$, then $f(A) \overset{cc}{\hookrightarrow} N/K$. It is enough to show that $\phi f(A) = \eta p(A) \overset{cc}{\hookrightarrow} L'/(L' \cap K)$, as ϕ is an isomorphism. Now since $\text{Ker}\eta \ll L'$, $\eta(B) \overset{cc}{\hookrightarrow} L'/(L' \cap K)$, whenever $B \overset{cc}{\hookrightarrow} L'$ [5, 2.6]. By Proposition 3.2, $p(A) \overset{cc}{\hookrightarrow} L'$. Therefore $\eta p(A) \overset{cc}{\hookrightarrow} L'/(L' \cap K)$ and hence $f(A) \overset{cc}{\hookrightarrow} N/K$. Thus N is UCC module. ∎

We recall the definition of M-annihilator submodules. Let M be an R-module. For an R-module N and any subset $X \subseteq \text{Hom}(N, M)$, We put

$$\mathrm{Ker}\,(X) = \bigcap\{\mathrm{Ker}\,g \,|\, g \in X\}.$$

Any submodule of $\mathrm{Ker}(X)$ for some such X is called an *M-annihilator* submodule of N and we denote the set of M-annihilator submodules by $K(N, M)$.

Proposition 3.4. *Let M be a copolyform Σ-lifting module, and N an indecomposable direct summand of M. If $K(N, M) = \{\mathrm{Ker}\,(I) \,|\, I \subseteq \mathrm{Hom}(N, M)\}$ is the set of all M-annihilator submodules of N. Then*
(1) $\mathrm{End}(N)$ *is a division ring and N is local, self-projective;*
(2) $K(N, M)$ *is linearly ordered by inclusion and N has ACC on $K(N, M)$;*
(3) *N has DCC on $K(N, M)$, if M has only finitely many non-isomorphic indecomposable direct summand submodules.*

Proof. (1) Let N be an indecomposable direct summand of M. Then N is lifting and hence a hollow module. Suppose that $f : N \to N$ is a nonzero homomorphism. As N is non-M-cosingular module, f is an epimorphism Lemma 1.1. Let $L = \oplus_{\mathbb{N}} N_i$, for every $i \in \mathbb{N}$, $N_i = N$. Then L is a UCC lifting module, and hence the sum of any family of coclosed submodules of L is coclosed in L [5, 3.16 (3)]. As L is also lifting, any locally direct submmand of L is a direct summand of L Lemma 1.2. Now consider $f_i = f : N_i \to N_{i+1}$ for every $i \in \mathbb{N}$. Then for every family $\{f_i : N_i \to N_{i+1}\}_{\mathbb{N}}$, there exists $r \in \mathbb{N}$ and a nonzero map $h_r : N_{r+1} \to N_r$ such that $f_{r-1} \dots f_1 = h_r f_r \dots f_1$ [13, 43.3]. For any $i \in \mathbb{N}$, f_i is onto, and $h_r f_r$ is the identity map on N_r. Hence f_r is 1-1. Therefore $\mathrm{End}(N)$ is a division ring. Now by Proposition 2.1, N is local and self-projective.
(2) By Proposition 3.3, M is non-M-cosingular. Since every direct summand N of M is non-M-cosingular, so if $f \in \mathrm{Hom}(N, M)$ is a nonzero map then $\mathrm{Im}\, f \not\ll M$. Therefore if $\mathcal{A} = \{\mathrm{Ker} f \,|\, \mathrm{Im}\, f \not\ll M\}$ then $\mathcal{A} \cup N = K(N, M)$ and hence $K(N, M)$ is linearly ordered Proposition 2.3. Suppose that $\mathrm{Ker}\,(I_1) \subsetneqq \mathrm{Ker}\,(I_2)$, where $I_1, I_2 \subsetneqq \mathrm{Hom}(N, M)$. Then there exists $f \in I_1$, such that $\mathrm{Ker} f \subsetneqq \mathrm{Ker} g$, for every $g \in I_2$; for if not, then for every $f_0 \in I_1$, there exists $g_0 \in I_2$, such that $\mathrm{Ker}\, f_0 \supsetneqq \mathrm{Ker} g_0$, and hence $\mathrm{Ker} f_0 \supsetneqq \mathrm{Ker}\,(I_2)$. Therefore $\mathrm{Ker}\,(I_1) \supsetneqq \mathrm{Ker}\,(I_2)$ which is a contradiction.

Consider

$$\mathrm{Ker}\,(I_1) \subsetneqq \mathrm{Ker}\,(I_2) \subsetneqq \cdots \subsetneqq \mathrm{Ker}\,(I_r) \subsetneqq \mathrm{Ker}\,(I_{r+1}) \subsetneqq \cdots$$

a strictly ascending chain of M-annihilator submodules. Fix $r \in \mathbb{N}$. As $\mathrm{Ker}\,(I_r) \subsetneqq \mathrm{Ker}\,(I_{r+1})$, there exists $f_r \in I_r$ such that for every $g \in I_{r+1}$, $\mathrm{Ker} f_r \subsetneqq \mathrm{Ker} g$. Hence we get a strictly increasing chain

$$\mathrm{Ker} f_1 \subsetneqq \mathrm{Ker} f_2 \subsetneqq \cdots \subsetneqq \mathrm{Ker}\, f_r \subsetneqq \mathrm{Ker} f_{r+1} \subsetneqq \cdots,$$

which is a contradiction Proposition 2.3. So N satisfies ACC on $K(N, M)$.
(3) Suppose that

$$\mathrm{Ker}\,(I_1) \supsetneqq \mathrm{Ker}\,(I_2) \supsetneqq \cdots \supsetneqq \mathrm{Ker}\,(I_r) \supsetneqq \mathrm{Ker}\,(I_{r+1}) \supsetneqq \cdots$$

is a strictly descending chain of M-annihilator submodules of N. Then for each $r+1 \in \mathbb{N}$, there exists $f_{r+1} \in I_{r+1}$ such that for every $f_r \in I_r$, $\mathrm{Ker}\, f_{r+1} \subsetneqq \mathrm{Ker}\, f_r$. So we get a strictly descending chain

$$\operatorname{Ker} f_1 \underset{\neq}{\supseteq} \operatorname{Ker} f_2 \underset{\neq}{\supseteq} \cdots \underset{\neq}{\supseteq} \operatorname{Ker} f_r \underset{\neq}{\supseteq} \operatorname{Ker} f_{r+1} \underset{\neq}{\supseteq} \cdots,$$

which is a contradiction Proposition 2.3. ■

The following Lemma regarding copolyform modules has been proved in [11, 2.3].

Lemma 3.5. *Suppose that M is an amply supplemented module. M is a copolyform module if and only if for any nonzero map $f : M \to M/X$, $\operatorname{Im} f \not\ll M/X$.*

Lemma 3.6. *Suppose M is an R-module and $f : P \to M$ a projective cover of M. Then the following are equivalent.*
(1) *P is copolyform;*
(2) *M is copolyform and $0 \overset{cr}{\hookrightarrow} K$ in P, where K is the kernel of f.*

Proof. (1) ⇒ (2). As P is copolyform, M is copolyform [11, 2.2]. Since $K \ll P$, $0 \overset{ce}{\hookrightarrow} K$ in P. Now P is coplyform implies that $0 \overset{cr}{\hookrightarrow} K$ in P.
(2) ⇒ (1). Suppose that $A \overset{ce}{\hookrightarrow} B$ in P. It is easy to see that $(A+K)/K \overset{ce}{\hookrightarrow} (B+K)/K$ in M. As M is copolyform, $(A+K)/K \overset{cr}{\hookrightarrow} (B+K)/K$ in M. Therefore by [11, 1.1 (5)] $(A+K) \overset{cr}{\hookrightarrow} (B+K)$ in P. Since $0 \overset{cr}{\hookrightarrow} K$ in P and $A \overset{cr}{\hookrightarrow} A$ in P, $A \overset{cr}{\hookrightarrow} (A+K)$ in P [11, 1.1 (4)]. Now $(A+K) \overset{cr}{\hookrightarrow} (B+K)$ in P implies $A \overset{cr}{\hookrightarrow} (B+K)$ [11, 1.1 (2)]. Again by [11, 1.1 (2)] $A \overset{cr}{\hookrightarrow} B$ in P. Thus P is a copolyform module. ■

It has been proved in [4] that, M is a polyform module if and only if $\operatorname{End}(\widehat{M})$ is a regular ring. We prove the dual of that result when M is a semiperfect module.

Theorem 3.7. *Let M be a semiperfect module and $f : P \to M$ be the projective cover. Then the following statements are equivalent.*
(1) *M is copolyform and $0 \overset{cr}{\hookrightarrow} \operatorname{Ker} f$ in P;*
(2) *P is copolyform;*
(3) $\operatorname{End}(P)$ *is regular.*

Proof. (1) ⇔ (2) follows from Lemma 3.6.
(2) ⇒ (3). Let P be a copolyform module and $S = \operatorname{End}(P)$. As P is a projective module, $f \in \operatorname{Rad} S$ implies that $\operatorname{Im} f \ll P$. Since P is copolyform $\operatorname{Rad} S = 0$. As M is semiperfect P is semiperfect [8, 5.6]. Then S is f-semiperfect [13, 42.12] and therefore $S/\operatorname{Rad} S$ is a regular ring [13, 42.11]. Now $\operatorname{Rad} S = 0$ implies that S is a regular ring.
(3) ⇒ (2). Let $g : P \to P$ be a homomorphism with $\operatorname{Im} g \ll P$. Since $\operatorname{End}(P)$ is regular, $\operatorname{Im} g$ is a direct summand of P [13, 37.7]. Hence $\operatorname{Im} g = 0$ or $g = 0$. Now by Lemma 3.5 P is copolyform. ■

Recall that a ring R is a left PP-ring (principal projective) if every cyclic left ideal of R is projective. A ring R is a hereditary (semihereditary) ring if every left (finitely generated) ideal is projective.

Harmanci has communicated the following lemma. We give here a proof for the sake of completeness.

Lemma 3.8. *Let M be a copolyform module and $S = \mathrm{End}\,(M)$.*
(1) *If M is lifting, then S is a left and right PP-ring.*
(2) *If M is finitely Σ-lifting, then S is left and right semihereditary.*

Proof. (1) Let $f \in S$. Since M is lifting and copolyform, $\mathrm{Im}\, f \overset{cc}{\hookrightarrow} M$ [11, 2.3]. Therefore $\mathrm{Im}\, f$ is a direct summand of M Lemma 1.2. Hence by [13, 39.11] S is right PP-ring. Now we show that Sf is projective for every $f \in S$. As above $f(M)$ is a direct summand of M and hence $f(M) = e(M)$ for some idempotent $e \in S$. It is enough to prove that the onto map $\phi : S \to Sf$ defined by $\phi(s) = sf$, where $s \in S$ splits. We have $S(1-e) \subseteq \mathrm{Ker}\,\phi$. Let $g \in \mathrm{Ker}\,\phi$. Then $\phi(g) = gf = 0$ and so $gf(M) = ge(M) = 0$. This implies $ge = 0$. Hence $g(1-e) = g \in S(1-e)$. Thus $\mathrm{Ker}\,\phi = S(1-e)$. Therefore S is a left PP-ring.
(2) Since M is copolyform and finitely Σ-lifting, M^n is also copolyform and lifting [11, 2.9]. Therefore by (1), for every $n \in \mathbb{N}$, End $(M^n) \simeq S^{n\times n}$ is a left and right PP-ring. Hence S is left and right semihereditary [13, 39.13]. ∎

Theorem 3.9. *Let M be a copolyform Σ-lifting module and $S = \mathrm{End}(M)$. Suppose that N is any indecomposable direct summand of M. Then we have the following.*
(1) *$M = \oplus_I M_i$ where each M_i is self-projective, local and* End (M_i) *a division ring;*
(2) $\mathrm{Hom}(N, M)$ *is a uniserial, artinian left S-module;*
(3) $\mathrm{Hom}(N, M)$ *is a uniserial left S-module of finite length, if M has only finitely many non-isomorphic indecomposable direct summands;*
(4) $\mathrm{Hom}(M, N)$ *is a uniserial right S-module;*
(5) *if M is finitely generated, then S is left and right serial, right artinian and left and right hereditary ring;*
(6) *if M is finitely generated, then M has a projective cover P in $\sigma[M]$ and* $\mathrm{End}(P)$ *is a semisimple ring.*

Proof. (1). Since in a UCC lifting module M, every local direct summand of M is a direct summand [5], M is a direct sum of indecomposable modules [9, 2.17]. Now (1) follows from Proposion 3.4.
(2) By Proposition 3.3, M is non-M-cosingular. Hence N is non-M-cosingular. Thus for every nonzero $f \in \mathrm{Hom}(N, M)$, $\mathrm{Im}\, f$ is a hollow module which is not small in M. Therefore by Lemma 1.1 $\mathrm{Im}\, f \overset{cc}{\hookrightarrow} M$ and hence a direct summand of M. As $\mathrm{End}(M_i)$ is local, $\mathrm{Im}\, f \simeq M_i$, for some $i \in I$ [1].

Suppose $f, g : N \to M$ with Ker$f \subseteq$ Ker$g$. We claim that $Sg \subseteq Sf$. For there exists an onto map $\psi : \mathrm{Im}\, f \to \mathrm{Im}\, g$ such that $\psi f = g$.

Since $\mathrm{Im}\, f$ is a direct summand of M, we can extend ψ to $\phi : M \to M$ such that $\phi f = g$. Therefore $Sg \subseteq Sf$.

Consider any two nonzero $f, g \in \mathrm{Hom}(N, M)$. Then $\mathrm{Im}\, f \simeq N/\mathrm{Ker} f \simeq M_i$ and $\mathrm{Im}\, g \simeq N/\mathrm{Ker} g \simeq M_j$ for some $i, j \in I$. By Proposition 2.3 (1), either $\mathrm{Ker}\, f \subseteq \mathrm{Ker}\, g$ or $\mathrm{Ker}\, g \subseteq \mathrm{Ker}\, f$. Hence either $Sg \subseteq Sf$ or $Sf \subseteq Sg$. Therefore $\mathrm{Hom}(N, M)$ is a uniserial left S-module.

Let $I_1 \supsetneqq I_2 \supsetneqq \cdots \supsetneqq I_n \supsetneqq \cdots$ be a strictly descending chain of S-submodules of $\mathrm{Hom}(N, M)$. For each $n \in \mathbb{N}$ choose $f_n : N \to M$ such that $f_n \in I_n$ and $f_n \notin I_{n+1}$. We have either $Sf_n \subseteq Sf_{n+1}$ or $Sf_{n+1} \subseteq Sf_n$ and $f_n \notin I_{n+1}$ implies that $Sf_{n+1} \subsetneqq Sf_n$. This along with either $\mathrm{Ker}\, f_{n+1} \subseteq \mathrm{Ker}\, f_n$ or $\mathrm{Ker}\, f_n \subseteq \mathrm{Ker}\, f_{n+1}$ (2.3 (1)) gives us $\mathrm{Ker}\, f_n \subsetneqq \mathrm{Ker}\, f_{n+1}$. Thus we get a strictly ascending chain

$$\mathrm{Ker} f_1 \subsetneqq \mathrm{Ker} f_2 \subsetneqq \cdots \subsetneqq \mathrm{Ker}\, f_n \subsetneqq \mathrm{Ker}\, f_{n+1} \subsetneqq \cdots,$$

which contradicts Proposition 2.3 (2). Therefore $\mathrm{Hom}\,(N, M)$ is an artinian left S-module.

(3) By (2) it is enough to prove that $\mathrm{Hom}(N, M)$ satisfies ACC on S-submodules. Suppose that

$$I_1 \subsetneqq I_2 \subsetneqq \cdots \subsetneqq I_n \subsetneqq \cdots$$

is a strictly ascending chain of S-submodules of $\mathrm{Hom}(N, M)$. For each n there exists $f_{n+1} \in I_{n+1}$ such that $f_{n+1} \notin I_n$. As in the proof of (2) we get a strictly descending chain

$$\mathrm{Ker}\, f_2 \supsetneqq \mathrm{Ker}\, f_3 \supsetneqq \cdots \supsetneqq \mathrm{Ker}\, f_n \supsetneqq \cdots,$$

which contradicts Proposition 2.3 (3). Thus $\mathrm{Hom}(N, M)$ is a uniserial left S-module of finite length.

(4) Since for every $i \in I$, M_i is hollow and N is a local module, any nonzero map $M_i \to N$ is an epimorphism. Hence any nonzero $f \in \mathrm{Hom}(M, N)$ is an epimorphism. Now (4) follows from Theorem 2.4.

(5) Since M is finitely generated $M = \oplus_{i=1}^{k} M_i$, where each M_i is a local module with local endomorphism ring.

Since $S = \oplus_{i=1}^{k} \mathrm{Hom}(M, M_i)$, S is a right serial ring by (4).

Also $S = \oplus_{i=1}^{k} \mathrm{Hom}(M_i, M)$ and for each $i = 1, \cdots, k$, $\mathrm{Hom}(M_i, M)$ is left artinian and left uniserial (by (2) and (3)) imply that S is a left serial and left artinian ring.

We know that every left and right serial, left artinian ring is a right artinian ring [13, 55.16]. Thus S is a left and right artinian serial ring. By Lemma 3.8, S is left and right semihereditary and hence S is a left and right hereditary ring.

(6) As M is finitely generated $M = \oplus_{i=1}^{k} M_i$, where each M_i is a local module with local endomorphism ring. Define $I := \{1, 2, \ldots, k\}$.

For every $i \in I$ put $[i] = \{j \in I | M_i \simeq M_j\}$ and $\mathcal{F} = \{[i] | i \in I\}$. Define an order on $\mathcal{F}$ by $[i] \leq [j]$ if and only if there exists an onto map $M_i \to M_j$. Then $(\mathcal{F}, \leq)$ is a partially ordered set (see the proof of Theorem 2.4). Suppose that

$$J = \{j \in I | \quad [j] \text{ is a minimal elements of } \mathcal{F}\}.$$

Let $N = \oplus_{j \in J} M_j$. For $k, \ell \in J$ any epimorphism from M_k to M_ℓ is an isomorphism and N is lifting, N is self-projective [12, Lemma 2.3].

For any $i \in I$, there exists a $j \in J$ such that $[j] \leq [i]$ and hence there exists an epimorphism from M_j to M_i. Thus $\sigma[M] = \mathbb{N}$. Since N is finitely generated and self-projective, N is projective in $\mathbb{N}$ and hence in $\sigma[M]$.

Given $i \in I$, there exists a $j \in J$ such that there exists an epimorphism from M_j to M_i and hence M_j is a projective cover of M_i. Thus M has a projective cover P which is a direct summand of $N^{(k)}$.

Since P is finitely generated and weakly supplemented, $P/\mathrm{Rad}\,(P)$ is a semisimple module and hence $\mathrm{End}(P/\mathrm{Rad}\,(P))$ is a semisimple ring. By [11, 2.9] $N^{(k)}$ is copolyform and hence P is copolyform. Therefore $\mathrm{Rad}\,(\mathrm{End}(P)) = 0$. By [13, 22.2] $\mathrm{End}(P)/\mathrm{Rad}\,(\mathrm{End}(P)) \simeq \mathrm{End}(P/\mathrm{Rad}\,(P))$ and hence $\mathrm{End}(P)$ is a semisimple ring. ■

4. Endomorphism Rings of $\overline{Z}_M(M)$, When M is Σ-Lifting and Injective

In this section we show that if M is a Σ-lifting injective module and N is an indecomposable direct summand of M, then $\mathrm{Hom}(\overline{Z}_M(N), \overline{Z}_M(M))$ is a uniserial $\overline{S}$-module of finite length, where $\overline{S} = \mathrm{End}(\overline{Z}_M(M))$.

Proposition 4.1. *Let M be a Σ-lifting module which is injective in $\sigma[M]$. Then $M = \oplus_I M_i$, where each M_i is local, self-projective and indecomposable. Also we have the following.*

(1) *For every $k \in I$, $\mathcal{A} = \{Ker(J) \mid J \subseteq \mathrm{Hom}(\overline{Z}_M(M_k), M)\}$ is linearly ordered by set inclusion;*

(2) *the family $\{\overline{Z}_M(M_i)\}_I$ is semi-T-nilpotent.*

Proof. Since M is a lifting and injective module, $M = \oplus_I M_i$ where each M_i is indecomposable [10, 2.4, 2.5]. As M_i is injective and indecomposable, $\mathrm{End}\,(M_i)$ is local. Therefore by Proposition 2.1 each M_i is local and self-projective.

(1) Suppose that $0 \neq I_1$ and $0 \neq I_2 \subseteq \mathrm{Hom}(\overline{Z}_M(M_k), M)$ and $\mathrm{Ker}\,(I_1) \not\subseteq \mathrm{Ker}\,(I_2)$. Then for any nonzero $f \in I_1$, there exists a nonzero $g \in I_2$ such that $\mathrm{Ker}\, f \not\subseteq \mathrm{Ker}\, g$; for if not, then for every $g \in I_2$, $\mathrm{Ker}\, f \subseteq \mathrm{Ker}\, g$ implies $\mathrm{Ker}\, f \subseteq \mathrm{Ker}\,(I_2)$ and hence $\mathrm{Ker}\,(I_1) \subseteq \mathrm{Ker}\,(I_2)$ which is a contradiction.

As M is injective in $\sigma[M]$, homomorphisms f, g can be extended to $\overline{f}, \overline{g} : M_k \to M$, respectively.

Then $\mathrm{Ker}\, f = \overline{Z}_M(M_k) \cap \mathrm{Ker}\, \overline{f}$ and $\mathrm{Ker}\, g = \overline{Z}_M(M_k) \cap \mathrm{Ker}\, \overline{g}$. If $\mathrm{Im}\, \overline{f}$ is small in M, then $\mathrm{Im}\, \overline{f}$ is an M-small module and hence is M-cosingular. Thus $M_k/\mathrm{Ker}\, \overline{f} \simeq \mathrm{Im}\, \overline{f}$ is M-cosingular and hence $\overline{Z}_M(M_k) \subseteq \mathrm{Ker}\, \overline{f}$. Therefore $\mathrm{Ker}\, f = \overline{Z}_M(M_k)$ which is contradiction. Hence $\mathrm{Im}\, \overline{f} \not\ll M$. Similarly $\mathrm{Im}\, \overline{g} \not\ll M$.

Since M_k is an indecomposable direct summand of M with local endomorphism ring, by Proposition 2.3 (1) either $\mathrm{Ker}\, \overline{f} \subseteq \mathrm{Ker}\, \overline{g}$ or $\mathrm{Ker}\, \overline{g} \subseteq \mathrm{Ker}\, \overline{f}$. As $\mathrm{Ker}\, \overline{f} \not\subseteq \mathrm{Ker}\, \overline{g}$, so $\mathrm{Ker}\, \overline{g} \subseteq \mathrm{Ker}\, \overline{f}$ and hence $\mathrm{Ker}\, g \subseteq \mathrm{Ker}\, f$. Thus $\mathrm{Ker}\,(I_2) \subseteq \mathrm{Ker}\, f$. Since f is any nonzero element of I_1, $\mathrm{Ker}\,(I_2) \subseteq \mathrm{Ker}\,(I_1)$.

(2) Suppose that $f : \overline{Z}_M(M_i) \to \overline{Z}_M(M_j)$ is a nonzero non-isomorphism. As M is injective in $\sigma[M]$, M_j is M_i-injective and hence f can be extended to an homomorphism $\overline{f} : M_i \to M_j$. We claim that $\overline{f}$ is also a non-isomorphism.

If $\overline{f}$ is not onto, then $\operatorname{Im}\overline{f}$ is an M-small module and hence is M-cosingular. Thus $M_i/\operatorname{Ker}\overline{f} \simeq \operatorname{Im}\overline{f}$ is M-cosingular. Therefore $\operatorname{Ker}\overline{f} \supseteq \overline{Z}_M(M_i)$ and hence $f = 0$, a contradiction. Thus $\overline{f}$ is an surjective map. Suppose $\operatorname{Ker}\overline{f} = 0$. Then $\overline{f}$ is an isomorphism. Hence $M_i \simeq M_j$ and so $\overline{Z}_M(M_i) \simeq \overline{Z}_M(M_j)$ and $\operatorname{Ker} f = 0$. $\overline{Z}_M(M_i)$ is fully invariant in M_i and hence is self-injective. Thus $\operatorname{Im} f \simeq \overline{Z}_M(M_i)$ is $\overline{Z}_M(M_j)$-injective implies that $\operatorname{Im} f$ is a direct summand of $\overline{Z}_M(M_j)$. As $\overline{Z}_M(M_j)$ is uniform $\operatorname{Im} f = \overline{Z}_M(M_j)$. Therefore f is an isomorphism, a contradiction. Thus $\operatorname{Ker}\overline{f} \neq 0$.

By Theorem 1.5 the family $\{M_i\}_I$ is semi-T-nilpotent. Since any nonzero non-isomorphism from $\overline{Z}_M(M_i) \to \overline{Z}_M(M_j)$ can be extended to a non-isomorphism from $M_i \to M_j$, the family $\{\overline{Z}_M(M_i)\}_I$ is also semi-T-nilpotent. ∎

Proposition 4.2. *Let M be a Σ-lifting module which is injective in $\sigma[M]$. Then $M = \oplus_I M_i$ where each M_i is a local and self-projective module. If $\overline{S} = \operatorname{End}(\overline{Z}_M(M))$, then we have*

(1) *for any $k \in I$, $\operatorname{Hom}(\overline{Z}_M(M_k), \overline{Z}_M(M))$ is a uniserial, artinian left $\overline{S}$-module;*

(2) *if $\{\overline{Z}_M(M_i)\}_I$ contains only a finite number of non-isomorphic modules, then each $\operatorname{Hom}(\overline{Z}_M(M_k), \overline{Z}_M(M))$ is a uniserial $\overline{S}$-module of finite length;*

(3) *$\operatorname{Hom}(\overline{Z}_M(M), \overline{Z}_M(M_k))$ is a uniserial right $\overline{S}$-module.*

Proof. The first assertion follows from Proposition 4.1.

(1) Let f, g be two nonzero homomorphisms in $\operatorname{Hom}(\overline{Z}_M(M_k), \overline{Z}_M(M))$. By the injectivity of M they can be extended to $\overline{f}, \overline{g} : M_k \to M$. As in the proof of Proposition 4.1 (1), $\operatorname{Im}\overline{f}$ and $\operatorname{Im}\overline{g}$ are not small in M. By Proposition 2.3 (1) either $\operatorname{Ker}\overline{f} \subseteq \operatorname{Ker}\overline{g}$ or $\operatorname{Ker}\overline{g} \subseteq \operatorname{Ker}\overline{f}$.

Suppose $\operatorname{Ker}\overline{f} \subseteq \operatorname{Ker}\overline{g}$. We claim that $S\overline{g} \subseteq S\overline{f}$, where $S = \operatorname{End}(M)$. For, there exists a surjective map $\psi : \operatorname{Im}\overline{f} \to \operatorname{Im}\overline{g}$ such that $\psi\overline{f} = \overline{g}$. Since $\operatorname{Im}\overline{f}$ is a direct summand of M, we can extend ψ to $\phi : M \to M$ such that $\phi\overline{f} = \overline{g}$. Therefore $S\overline{g} \subseteq S\overline{f}$. Hence either $S\overline{f} \subseteq S\overline{g}$ or $S\overline{g} \subseteq S\overline{f}$.

Suppose that $S\overline{f} \subseteq S\overline{g}$. As any homomorphism $\overline{Z}_M(M) \to \overline{Z}_M(M)$ can be extended to a homomorphism from $M \to M$, and the restriction of any map from $M \to M$ to $\overline{Z}_M(M)$ can be considered as a map from $\overline{Z}_M(M) \to \overline{Z}_M(M)$, we get so $\overline{S}f \subseteq \overline{S}g$.

Similarly if $S\overline{g} \subseteq S\overline{f}$, then $\overline{S}g \subseteq \overline{S}f$. Hence $\operatorname{Hom}(\overline{Z}_M(M_k), \overline{Z}_M(M))$ is a uniserial left $\overline{S}$-module.

We have to prove that $\operatorname{Hom}(\overline{Z}_M(M_k), \overline{Z}_M(M))$ is an artinian left $\overline{S}$-module. Let

$$I_1 \supsetneqq I_2 \supsetneqq \cdots \supsetneqq I_r \supsetneqq \cdots$$

be a strictly descending chain of $\overline{S}$-submodules of $\operatorname{Hom}(\overline{Z}_M(M_k), \overline{Z}_M(M))$.

For any $r \in \mathbb{N}$, there exists $f_r \in I_r$ such that $f_r \notin I_{r+1}$. We have either $\overline{S}f_r \subseteq \overline{S}f_{r+1}$ or $\overline{S}f_{r+1} \subseteq \overline{S}f_r$. Since $f_r \notin I_{r+1}$ we get $\overline{S}f_r \not\subseteq \overline{S}f_{r+1}$. Hence $\overline{S}f_{r+1} \subsetneqq \overline{S}f_r$.

As $f_{r+1} \in \overline{S}f_r$, there exists $\overline{\alpha} \in \overline{S}$ such that $f_{r+1} = \overline{\alpha}f_r$. Therefore $\operatorname{Ker} f_r \subsetneqq \operatorname{Ker} f_{r+1}$ which implies that $\operatorname{Ker}\overline{f}_r \cap \overline{Z}_M(M_k) \subsetneqq \operatorname{Ker}\overline{f}_r \cap \overline{Z}_M(M_k)$, where $\overline{f}_r$ and $\overline{f}_{r+1}$ are extensions of f_r and f_{r+1} respectively from M_k to M.

Suppose $\operatorname{Ker}\overline{f}_r = \operatorname{Ker}\overline{f}_{r+1}$. As f_r and f_{r+1} are nonzero maps, the images of the maps $\overline{f}_r$ and $\overline{f}_{r+1}$ are not small in M and hence are isomorphic to direct summands of M. Hence we can define a map $\phi : M \to M$ such that $\overline{f}_r = \phi\overline{f}_{r+1}$. The restriction to $\overline{Z}_M(M_k)$ gives us $f_r = \phi|_{\overline{Z}_M(M_k)} f_{r+1}$. Since $\phi|_{\overline{Z}_M(M_k)}$ can be considered as an element of $\overline{S}$, we get $\overline{S}f_r \subseteq \overline{S}f_{r+1}$, a contradiction. Hence $\operatorname{Ker}\overline{f}_r \subsetneqq \operatorname{Ker}\overline{f}_{r+1}$.

Hence we get a strictly ascending chain

$$\operatorname{Ker}\overline{f}_1 \subsetneqq \operatorname{Ker}\overline{f}_2 \subsetneqq \cdots \subsetneqq \operatorname{Ker}\overline{f}_r \subsetneqq \cdots .$$

For all $r \in \mathbb{N}$, $\operatorname{Im}\overline{f}_r$ is not small in M. Hence by Proposition 2.3 (2) the above chain becomes stationary after finitely many steps and hence this is also true for the chain

$$I_1 \supsetneqq I_2 \supsetneqq \cdots \supsetneqq I_r \supsetneqq \cdots .$$

Therefore $\operatorname{Hom}(\overline{Z}_M(M_k), \overline{Z}_M(M))$ is an artinian left $\overline{S}$-module.

(2) By (1) it is enough to prove that $\operatorname{Hom}(\overline{Z}_M(M_k), \overline{Z}_M(M))$ satisfies ACC on $\overline{S}$-submodules. Suppose that

$$I_1 \subsetneqq I_2 \subsetneqq \cdots \subsetneqq I_r \subsetneqq \cdots$$

is a strictly ascending chain of $\overline{S}$-submodules of $\operatorname{Hom}(\overline{Z}_M(M_k), \overline{Z}_M(M))$.

As in the proof of (1) we get homomorphisms $\overline{f}_r : M_k \to M$ such that $\operatorname{Im}\overline{f}_r \not\ll M$ and

$$\operatorname{Ker}\overline{f}_1 \supsetneqq \operatorname{Ker}\overline{f}_2 \supsetneqq \cdots \supsetneqq \operatorname{Ker}\overline{f}_r \supsetneqq \cdots .$$

By Proposition 2.3 (3) this chain stops. Therefore $\operatorname{Hom}(\overline{Z}_M(M_k), \overline{Z}_M(M))$ is a uniserial $\overline{S}$-module of finite length.

(3) Any two nonzero maps $\phi, \psi \in \operatorname{Hom}(\overline{Z}_M(M), \overline{Z}_M(M_k))$ can be extended to nonzero maps $\overline{\phi}, \overline{\psi} \in \operatorname{Hom}(M, M_k)$. Since $\operatorname{Im}\overline{\phi}$ and $\operatorname{Im}\overline{\psi}$ are not small in M_k (as in the proof of Proposition 4.1 (1)), $\overline{\phi}, \overline{\psi}$ are surjective maps. By Theorem 2.4, either $\overline{\phi}S \subseteq \overline{\psi}S$ or $\overline{\psi}S \subseteq \overline{\phi}S$ where $S = \operatorname{End}(M)$. Suppose that $\overline{\phi}S \subseteq \overline{\psi}S$.

As any homomorphism $\overline{Z}_M(M) \to \overline{Z}_M(M)$ can be extended to a homomorphism from $M \to M$, and the restriction of any map from $M \to M$ to $\overline{Z}_M(M)$ can be considered as a map from $\overline{Z}_M(M) \to \overline{Z}_M(M)$, $\phi\overline{S} \subseteq \psi\overline{S}$. Hence $\operatorname{Hom}(\overline{Z}_M(M), \overline{Z}_M(M_k))$ is a uniserial right S-module. ■

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