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Prime Kernel Functors of Group Graded Rings and Their Identity Components

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Abstract. The basic Krull relations of Going Up and Going Down theorems for prime kernel functors of a group graded ring R and its identity component R_1 are proved herein.

1. Introduction

Let R be a k-algebra with 1, over a commutative ring k with 1 and G be a finite group, whose identity is also denoted by 1 such that the order of G is a unit in R.

Throughout this paper R is assumed to be a right seminoetherian ring (c.f. Definition 2.11) graded by G, that is, $R = \sum_{g \in G} \oplus R_g$ where R_g's are k-subspaces of R and $R_g R_h \subseteq R_{gh}$ for all $g, h \in G$. For any k-algebra R graded by a finite group G, we can construct the smash product $R\#k[G]^*$ [1]. The smash product $R\#k[G]^*$ and the identity component R_1 play the same role for graded rings that the skew group ring $R * G$ and the G-fixed subring R^G play for group actions. The most interesting application of skew ring methods to Galois theory is the correspondence obtained between the prime ideals of R and of R^G. A similar correspondence was derived by Cohen and Montgomery [1] for the prime ideals of a group graded ring R and of R_1. The aim of the present paper is to obtain a similar correspondence in the context of the prime kernel functors of R and R_1.

In [2], we proved that if R is right seminoetherian, then so is $R\#k[G]^*$. Further, we studied the prime kernel functors of R and $R\#k[G]^*$ under the assumption that R is right seminoetherian. In view of the right seminoetherian restriction on R, the primeness of a kernel functor (torsion theory) of R is the same as lattice theoretic primeness (c.f. Lemma 2.12). This enabled us to establish certain relationships between prime kernel functors of R and prime kernel functors of $R\#k[G]^*$.

Keeping in view the foregoing facts, first, we prove that if R is right seminoetherian, then R_1 is right seminoetherian and therefore, using the results of [2], derive a correspondence between the prime kernel functors of R and of R_1. Finally, we prove Going Up and Going Down theorems for prime kernel functors of R and R_1.

2. Preliminaries

Let R be graded by G and $R\#k[G]^*$ be the smash product defined in [1]. For $a, b \in R$ and p_{g_1}, p_{g_2} basis elements of $k[G]^*$, the product is given by

$$(ap_{g_1})(bp_{g_2}) = ab_{g_1g_2^{-1}}p_{g_2}.$$

We shall use the following formulae given in [1, Proposition 1.4].

(i) For $a \in R, p_{g_1}a = \sum_{g_2\in G} a_{g_1g_2^{-1}}p_{g_2}$.

(ii) For $a_{g_1} \in R_{g_1}, p_{g_2}a_{g_1} = a_{g_1}p_{g_1g_2^{-1}}$.

(iii) Each p_g centralizes R_1.

An action of the group G on $R\#k[G]^*$ is given by $(ap_{g_1})^g = ap_{g_1g}, ap_{g_1} \in R\#k[G]^*$, $g \in G$ [1, 3.3]. For convenience, throughout this paper we write $S = R\#k[G]^*$. Let J be any right ideal of S. Then $J^g = \{x^g | x \in J\}$ and $J^G = \bigcap_{g\in G} J^g$ is the largest G-invariant right ideal of S contained in J. More precisely J is G-invariant if and only if $J = J^G$.

For the familiar notations, definitions and results we mainly follow [1-4]. However, before proceeding further, we recall some definitions and record some simple facts for convenience of future reference.

Definition 2.1. [4] *Let $\tau \in$ tors-S. Then τ^g is a torsion theory given by*

$$\tau^g(S) = \{J^g \mid J \in \tau(S)\},$$

$\tau(S)$ denotes the Gabriel filter (Gabriel topology) of the torsion theory τ.

Definition 2.2. [4] *For $\tau \in$ tors-S, $\tau^G = \bigcap_{g\in G} \tau^g$ is a G-invariant torsion theory and $\tau^G \leq \tau$. The equality holds if and only if τ is G-invariant.*

Definition 2.3. [4] *$\Gamma \in$ tors-S is said to be G-prime if Γ is G-invariant and for any G-invariant $\Gamma_1, \Gamma_2 \in$ tors-S, $\Gamma \geq \Gamma_1 \wedge \Gamma_2$ implies that either $\Gamma \geq \Gamma_1$ or $\Gamma \geq \Gamma_2$.*

Note. If $\Gamma \in$ tors-S is prime, then Γ^G is G-prime.

Definition 2.4. *Let $\sigma \in$ tors-R. We recall from* [5, II. 11.4] *that σ_G is the graded torsion theory given by*

$$\sigma_G(R) = \{I \in \sigma(R) | I_G \in \sigma(R)\},$$

for a right ideal I of R. Here I_G denotes the largest graded right ideal of R contained in I. $\sigma_G \leq \sigma$ and the equality holds if and only if σ is graded.

Definition 2.5. [2] *$\sigma \in$ tors-S is said to be graded-prime if σ is a graded torsion theory and for any two graded torsion theories $\sigma_1, \sigma_2 \in$ tors-R, $\sigma \geq \sigma_1 \wedge \sigma_2$ implies that either $\sigma \geq \sigma_1$ or $\sigma \geq \sigma_2$.*

We observed the following in [2].

Let $\gamma : R \to S$ be the inclusion map $(a :\to a \sum_{g\in G} p_g)$, then we get the induced functors

$$\gamma_* : \text{mod-}S \to \text{ Mod-}R \text{ (restriction of scalars)}$$

and

$$\gamma^* : \text{mod-}R \to \text{ Mod-}S \text{ (extension of scalars)}$$

respectively defined on the objects by $\gamma_*(N_S) = N_R$ and $\gamma^*(M_R) = M \oplus_R S$. We get a function $\gamma_{\#}$: tors-$R \to$ tors-S which assigns to each $\sigma \in$ tors-R, the torsion theory $\gamma_{\#}(\sigma) \in$ tors-S given by

$$\Im_{\gamma\#}(\sigma) = \{N \in \text{ mod-}S \,|\, N_R \in \Im_\sigma\}.$$

Since S is flat as a right R-module (follows from [1, Proposition 1.4]), we also get $\gamma^{\#}$: tors-$S \to$ tors-R which assigns to each $\tau \in$ tors-S, the torsion theory $\gamma^{\#}(\tau) \in$ tors-R given by

$$\Im_\gamma^{\#}(\tau) = \{M \in \text{ mod-}R \,|\, M \oplus_R S \in \Im_\tau\}.$$

We have the following lemma from [2].

Lemma 2.6. *Let $\sigma \in$ sp-R, where sp-R is the set of all prime members of tors-R,* [3, Sec. 19]. *Then*
(i) *σ_G is graded prime.*
(ii) *$\gamma_{\#}(\sigma_G)$ is G-prime.*

Definition 2.7. [6] *Let R and R' be rings with unities. A Morita context between the rings R and R' is (R, R', M, N) where $M = {}_{R'}M_R$ and $N = {}_RN_{R'}$ are bimodules together with two bimodule homomorphisms*

$(\, , \,) : N \otimes_{R'} M \to R$, $[\, , \,] : M \otimes_R N \to R'$, satisfying the associativity conditions $n[m, n'] = (n, m)n'$ and $m(n, m') = [m, n]m'$ for all $m, m' \in M$ and $n, n' \in N$.

The images of (,) and [,] which are denoted by T_R and $T_{R'}$ are called the trace ideals of the Morita context.

Proposition 2.8. [7] *Let (R, R', M, N) be a Morita context with the trace ideals T_R and $T_{R'}$. Then there exists a lattice structure preserving bijection between the Gabriel topologies on R containing T_R and Gabriel topologies on R' containing $T_{R'}$.*

Remark 1. [6, 7] If T_R and $T_{R'}$ are idempotent, then the correspondence in the above proposition are as follows:

If $F(R)$ and $F(R')$ are the corresponding Gabriel topologies on R and R' containing T_R and $T_{R'}$ respectively, then

$$F(R') = \{I \subseteq R' \mid [I^+ : m]_R \in F(R) \text{ for all } m \in M\},$$

and

$$F(R) = \{J \subseteq R \mid ((J^{(n)})^+ : n)_{R'} \in F(R') \text{ for all } n \in N\};$$

where

$$[I^+ : m]_R = \{r \in R \mid mr \in I^+\} \quad \text{and} \quad I^+ = \{m \in M \mid [m, n] \in I \text{ for all } n \in N\};$$

and

$$\left(J^{(n)}\right)^+ = \{r' \in R' \mid r'M \subseteq J^{(n)}\} \quad \text{and} \quad J^{(n)} = \{m \in M \mid (n, m) \in J\}.$$

Definition 2.9. [3] *Let $\tau \in$ tors-R. A nonzero right R-module M is said to be τ-cocritical if and only if M is τ-torsion free and every nonzero submodule of M is τ-dense in M. For example, a simple right R-module M is $\chi(M)$-cocritical, where $\chi(M)$ is the torsion theory cogenerated by M.*

Definition 2.10. [3] *A torsion theory $\tau \in$ tors-R is said to be proper if and only if $\tau \neq \chi$, where χ is the member of tors-R defined by $\Im_\chi =$ mod-R. The set of all proper torsion theories on mod-R is denoted by prop-R.*

Definition 2.11. [3] *If every $\tau \in$ Prop-R has a τ-cocritical right R-module, then R is known as a right seminoetherian. This condition is equivalent to the condition that R has right Gabriel dimension k for some ordinal k.*

Lemma 2.12. [3] *Let R be a right seminoetherian ring. Then the following conditions are equivalent.*
(1) *$\tau \in$ sp-R.*
(2) *For $\tau_1, \tau_2 \in$ tors-R, $\tau_1 \wedge \tau_2 \leq \tau$ implies that either $\tau_1 \leq \tau$ or $\tau_2 \leq \tau$.*

Proof. It is clear from [3, 20.11, 20.12].

3. Prime Kernel Functors of R and R_1

Let R be graded by a finite group G and let R_1 be its identity component and $S = R\#k[G]^*$. We use the following notations:

$$\text{tors}_{p_1}\text{-}R = \{\sigma \in \text{ tors-}R \mid (Sp_1S) \cap R \in \sigma(R)\},$$
$$\text{tors}_{p_1}\text{-}S = \{\Gamma \in \text{ tors-}S \mid Sp_1S \in \Gamma(S)\}.$$

We also use the notations sp_{p_1}-R and sp_{p_1}-S replacing tors by *sp*.

We study the relationships between kernel functors of the rings S and R_1 with the help of Morita context between them.

Lemma 3.1. *There exists a lattice structure preserving bijection between tors$_{p_1}$-S and tors-R_1.*

Proof. (S, p_1Sp_1, Sp_1, Sp_1) is a Morita context and the trace ideals of S and p_1Sp_1 are Sp_1S and p_1Sp_1 respectively. Therefore, it follows from Proposition 2.8 that there exists a lattice structure preserving bijection between tors$_{p_1}$-S and tors-p_1Sp_1. Now the required result follows from the fact that $p_1Sp_1 = R_1p_1 \cong R_1$ [1].

Notations. The lattice structure preserving isomorphisms between tors$_{p_1}$-S and tors-p_1Sp_1 and tors-p_1Sp_1 and tors$_{p_1}$-S, given by the above lemma which are inverse of each other, will be denoted by

$$\eta^{\#} : \text{ tors}_{p_1}\text{-}S \to \text{ tors-}p_1Sp_1$$

and

$$\eta_{\#} : \text{ tors-}p_1Sp_1 \to \text{ tors}_{p_1}\text{-}S.$$

Lemma 3.2. *Let P be a right ideal of p_1Sp_1. Then $P = P_1p_1$ for some right ideal P_1 of R_1. Conversely, if P_1 is a right ideal of R_1, then P_1p_1 is a right ideal of p_1Sp_1.*

Proof. Define $P_1 = \{r_1 \in R_1 \mid r_1p_1 \in P\}$, then P_1 is a right ideal of R_1 and $P = P_1p_1$ since $p_1Sp_1 = R_1p_1$. The converse follows from the fact $p_1Sp_1 = R_1p_1 \cong R_1$.

Proposition 3.3. *Let $\tau \in$ tors$_{p_1}$-S and $\mu \in$ tors-p_1Sp_1. Then we have the following characterizations:*

(i) $\eta^{\#}(\tau)(p_1Sp_1) = \{I_1p_1 \subseteq p_1Sp_1 \mid (I_1p_1^+ : p_1) \in \tau(S)$, *where* $(I_1p_1^+ : p_1)$ *is as in Remark 1.*

(ii) $\eta_{\#}(\mu)(S) = \{J \subseteq S) \mid (J : n) \cap p_1Sp_1 \in \mu(p_1Sp_1)$ *for all* $n \in Sp_1\}$.

Proof.
(i) Since $(I_1p_1^+ : p_1s)_S = ((I_1p_1^+ : p_1) : s)_S$ for all $s \in S$, the required result follows from Remark 1.
(ii) By Remark 1, we have

$$n_{\#}(\mu)(S) = \{J \subseteq S | (J^{(n)})^+ \in \mu(p_1Sp_1) \text{ for all } n \in Sp_1\},$$

where

$$(J^{(n)})^+ = \{r_1p_1 \in p_1Sp_1 \mid r_1p_1(p_1S) \subseteq J^n\},$$

and

$$J^n = \{m \in p_1S \,|\, (n,m) \in J\}.$$

That is,

$$(J^{(n)})^+ = \big\{r_1p_1 \in p_1Sp_1 \,|\, (n, r_1p_1s) \in J \text{ for all } p_1s \in p_1S\big\},$$
$$= \big\{r_1p_1 \in p_1Sp_1 \,|\, nr_1p_1s \in J \text{ for all } p_1s \in p_1S\big\}.$$

But $nr_1p_1s \in J$ for all $s \in S$ if and only if $nr_1p_1 \in J$. Thus

$$\begin{aligned}(J^{(n)})^+ &= \{r_1p_1 \in p_1Sp_1 \,|\, nr_1p_1 \in J\}\\ &= \{r_1p_1 \in p_1Sp_1 \,|\, r_1p_1 \in (J:n)\}\\ &= (J:n) \cap p_1Sp_1.\end{aligned}$$

Hence

$$\eta_{\#}(\mu)(S) = \big\{J \subseteq S \,|\, (J:n) \cap p_1Sp_1 \in \mu(p_1Sp_1) \text{ for all } n \in Sp_1\big\}.$$

Remark 2. Since $p_1Sp_1 = R_1p_1 \cong R_1$ [1], therefore in the light of Lemma 3.2 a Gabriel topology $\mu(p_1Sp_1)$ of p_1Sp_1 is identified with a Gabriel topology $\mu(R_1)$ of R_1 as follows: Let $I_1 \subseteq R_1.I_1 \in \mu(R_1)$ if and only if $I_1p_1 \in \mu(p_1Sp_1)$.

To have a one-one correspondence between the torsion theories of sp_{p_1}-S and sp-R_1, we need the following.

Theorem 3.4. *If R is right seminoetherian, then so is R_1.*

Proof. Since $p_1Sp_1 = R_1p_1 \cong R_1$, it suffices to show that p_1Sp_1 is right seminoetlherian. Let $\mu \in prop\text{-}p_1Sp_1$. Then by Lemma 3.1, there exists $\tau \in \text{tors}_{p_1}\text{-}S$ such that $\eta^{\#}(\tau) = \mu$. R is right seminoetherian, thus so is S (c.f. [2]) and therefore there exists an S-module M which is τ-cocritical. We shall show that p_1Sp_1-module Mp_1 is μ-cocritical.

Let $mp_1 \in Mp_1$ and $\text{ann}_{p_1Sp_1}(mp_1) \in \mu(p_1Sp_1)$. Then $\text{ann}_{p_1Sp_1}(mp_1)^+ : p_1)_s \in \tau(S)$, by Proposition 3.3. We claim that $(\text{ann}_{p_1Sp_1}(mp_1)^+ : p_1)_s = \text{ann}_s(mp_1)$. For, let $\lambda \in \text{ann}_{p_1Sp_1}(mp_1)^+ : p_1)_s$. Then $mp_1\lambda sp_1 = 0$ for all $sp_1 \in Sp_1$. Thus $mp_1\lambda Sp_1S = 0$. Since $Sp_1S \in \tau(S)$ and M is τ-torsionfree, $mp_1\lambda = 0$, which implies that $\lambda \in \text{ann}_s(mp_1)$. Moreover, it is obvious that $\text{ann}_s(mp_1) \subseteq (\text{ann}_{p_1Sp_1}(mp_1)^+ : P_1)_s$, proving the claim. Thus $\text{ann}_s(mp_1) \in \tau(S)$. Therefore $mp_1 = 0$ because M is τ-torsionfree. This proves that Mp_1 is μ-torsionfree.

Let Np_1 be a p_1Sp_1-submodule of Mp_1, where N is an S-submodule of M. Using Proposition 3.3 and the fact that M/N is τ-torsion, it is easy to see that $(Np_1 : mp_1) \in \mu(p_1Sp_1)$ for all $mp_1 \in Mp_1$, which proves that Mp_1/Np_1 is μ-torsion and Mp_1 is μ-cocritical.

Using Theorem 3.4 and Remark 2, now, we are able to prove

Theorem 3.5. *Let R be a right seminoetherian. Then there exists a lattice structure preserving bijection between sp_{p_1}-S and sp-R_1.*

Proof. Since R is right seminoetherian, by [2, Theorem 3.1] and Theorem 3.4, S and R_1 are right seminoetherian. Let $\Gamma \in sp_{p_1}$-S and $\mu_1, \mu_2 \in$ *tors*-R_1 such that $\eta^{\#}(\Gamma) \geq \mu_1 \wedge \mu_2$. Then $\Gamma = \eta_{\#}((\eta^{\#}(\Gamma))) \geq \eta_{\#}(\mu_1) \wedge \eta_{\#}(\mu_2)$. Now the fact that Γ is prime gives that either $\Gamma \geq \eta_{\#}(\mu_1)$ or $\Gamma \geq \eta_{\#}(\mu_2)$ implying that either $\eta^{\#}(\Gamma) \geq \mu_1$ or $\eta^{\#}(\Gamma) \geq \mu_2$. Therefore $\eta^{\#}(\Gamma) \in$ sp-R_1 (c.f. [3], 20.11 and 20.12). Similarly, we can prove that if $\mu \in$ *sp*-R_1 then $\eta_{\#}(\mu) \in sp_{p_1}$-$S$. Hence the theorem follows because $\eta_{\#}$ and $\eta^{\#}$ are inclusion preserving maps which are inverse of each other.

Now, we are able to prove a result analogous to ([1, 7.3]) proved by Cohen and Montgomery.

Theorem 3.6. *Let R be a right seminoetherian ring.*

(i) *Let $\sigma \in sp_{p_1}$-R. Then there are r primes $r \leq |G|)\mu_1, \mu_2, \ldots, \mu_r$ belonging to sp-R_1, which are minimal over $\eta^{\#}(\gamma_{\#}(\sigma_G))$ and $\eta^{\#}(\gamma_{\#}(\sigma_G)) = \mu_1 \wedge \mu_2 \wedge \cdots \wedge \mu_r$. The set $\{\mu_1, \mu_2, \ldots, \mu_r\}$ is uniquely determined by σ.*

(ii) *Let $\mu \in$ sp-R_1. Then there are k primes $(k \leq |G|)$ $\sigma_1, \sigma_2, \ldots, \sigma_k$ belonging to sp-R, minimal over $(\gamma^{\#}(\eta_{\#}(\mu)))_G$ and $(\gamma^{\#}(\eta_{\#}(\mu)))_G = \sigma_1 \wedge \sigma_2 \wedge \ldots \wedge \sigma_k$. They are precisely the primes satisfying $(\sigma_i)_G = (\gamma^{\#}(\eta_{\#}(\mu)))_G$.*

Proof. Since Sp_1S is a two sided ideal of S, it follows from [1, Lemma 6.1] that $(Sp_1S) \cap R$ is a graded ideal of R. Thus $(Sp_1S) \cap R \in \sigma(R)$ implies that $(Sp_1S) \cap R \in \sigma_G(R)$ (c.f. Definition 2.4). Therefore by [2, Theorem 2.5], $((Sp_1S) \cap R)\#K[G]^* \subseteq Sp_1S \in \gamma_{\#}(\sigma_G)(S)$. Moreover, by [2, Theorem 3.3 (i)], there exists $\tau \in$ sp-S such that

$$\gamma_{\#}(\sigma_G) = \bigwedge_{g \in G} \tau^g.$$

Since $Sp_1S \in \gamma_{\#}(\sigma_G)(S)$, $Sp_1S \in \tau^g(S)$ for all $g \in G$. Hence $\eta^{\#}(\gamma_{\#}(\sigma_G)) = \bigwedge_{g \in G} \eta^{\#}(\tau^g)$, by definition of $\eta^{\#}$. Write $\mu_i = \eta^{\#}(\tau^g)$ and throw away which are redundant. We have the desired set of r minimal primes over $\eta^{\#}(\gamma_{\#}(\sigma_G))$. The uniqueness of the G-orbit $\{\tau^g\}$ of τ determines the uniqueness of the set $\{\mu_i\}$.

(ii) Let $\mu \in$ sp-R_1. Then $\eta_{\#}(\mu)$ is prime so that $(\eta_{\#}(\mu))^G$ is G-prime in tors-S (c.f. Note after Definition 2.3). By [2, Lemma 3.4], $\gamma^{\#}((\eta_{\#}(\mu))^G)$ is a graded-prime torsion theory in tors-R. Now the required result follows from [2, Theorem 3.4].

4. Going Up and Going Down Theorems

Let $\sigma \in$ sp-R. Then by [2, Theorem 3.3], there exists $\tau \in$ sp-S such that $\gamma_{\#}(\sigma_G) = \bigwedge_{g \in G} \tau^g$. To pass on to sp-$R_1$ we need the following

Lemma 4.1. *Let $\tau \in$ tors-S. Then $Sp_1S \in \tau^g(S)$ for some $g \in G$.*

Proof. We have

$$\xi = \xi(0) = \xi\Big(S/\sum_{g\in G} Sp_g S\Big) \le \tau.$$

Since Sp_gS are two sided ideals of S and G is a finite group, we have

$$\bigwedge_{g\in G} \xi(S/Sp_gS) = \xi(\Big(S/\sum_{g\in G} Sp_g S\Big) \le \tau.$$

Now, the primeness of τ implies that $\xi(S/Sp_gS) \le \tau$ for some $g \in G$ that is, $Sp_gS \in \tau(S)$ for some $g \in G$ and hence $Sp_1S \in \tau^g(S)$, proving the lemma.

Note. It follows from Lemma 4.1 that for $\sigma \in$ sp-R, there exists $\mu \in$ sp-R_1($\mu = \eta^{\#}(\tau^{g-1})$), where $\gamma_{\#}(\sigma_G) = \bigwedge_{g\in G} \tau^g$. Hence we give the following

Definition 4.2. *Let $s \in$ sp-R and $\mu \in$ sp-R_1. We say that σ lies over μ if $\eta_{\#}(\mu)$ is a minimal element of pgen($\gamma_{\#}(\sigma_G)$).*

Proposition 4.3.

(i) *Let $\sigma \in$ sp-R. Then there exists a prime $\mu \in$ sp-R_1 such that σ lies over μ. In particular, if $\sigma \in sp_{p_1}$-R, then σ lies over k-primes $\mu_1, \mu_2, ..., \mu_k (k \le |G|)$ of tors-R_1.*

(ii) *Let $\mu \in$ sp-R_1. Then there exists a prime $\sigma \in$ sp-R such that σ lies over μ, more precisely, there exist k such primes $(k \le |G|)\sigma_1, \sigma_2, \ldots, \sigma_k$ lying over μ and $\gamma^{\#}((\eta_{\#}(\mu))^G = \bigwedge_{i=1}^{k} \sigma_i$.*

Proof.

(i) The first part follows from Lemma 4.1 and the second from Theorem 3.6.

(ii) Let $\mu \in$ sp-R_1. Then $\Gamma = \eta_{\#}(\mu) \in$ sp-S. By [2, Theorem 3.3], there exists $\sigma \in$ sp-R such that $\gamma^{\#}(\Gamma^G) = \sigma_G$ and $\gamma_{\#}(\sigma_G) = \bigwedge_{g\in G} \Gamma^g$. Thus $\Gamma = \eta_{\#}(\mu)$ is a minimal element of pgen($\gamma_{\#}(\sigma_G)$), proving that σ lies over μ. Since σ_G is graded prime (c.f. [2, Lemma 3.1, Theorem 3.4]) there exist k-primes $\sigma_1, \sigma_2, \ldots, \sigma_k$ ($k \le |G|$) such that $\sigma_G = \bigwedge_{i=1}^{k} \sigma_i = (\sigma_i)_G$ and hence $\gamma_{\#}(\sigma_G) = \gamma_{\#}((\sigma_i)_G)$. Therefore, all these primes lie over μ, clearly, $\sigma_G = \bigwedge_{i=1}^{k} \sigma_i$ implies that

$$\gamma^{\#}((\eta_{\#}(\mu))^G) = \bigwedge_{i=1}^{k} \sigma_i.$$

We define an equivalence relation on sp-R_1 as follows:

Definition 4.4. *$\mu, \mu' \in$ sp-R_1 are said to be equivalent if and only if there exists a prime $\sigma \in$ sp-R such that σ lies over both μ and μ'.*

With the foregoing machinery at our disposal, we, now, prove Going Up and Going Down theorems.

Theorem 4.5.

(i) *Let R be graded by a finite group G.*
Let $\sigma_1 \underset{+}{<} \sigma_2$ be prime kernel functors of tors-R *and $\mu_2 \in sp\text{-}R_1$ such that σ_2 lies over μ_2. If $\sigma_1 \in sp_{p_1}\text{-}R$, then there exists a prime μ_1 of tors-R_1 such that σ_1 lies over μ_1 and $\mu_1 \underset{+}{<} \mu_2$.*

(ii) *Let $\sigma_1 \underset{+}{<} \sigma_2$ be prime kernel functors of tors-R and $\mu_1 \in sp\text{-}R_1$ such that σ_1 lies over μ_1. If $\sigma_2 \in sp_{p_1}\text{-}R$, then there exists a prime μ_2 of tors-R_1 such that σ_2 lies over μ_2 and $\mu_1 \underset{+}{<} \mu_2$.*

Proof. Since σ_2 lies over μ_2, we have $\gamma_{\#}((\sigma_2)G) = \bigwedge_{g \in G} \Gamma_2^g$, for some $\Gamma_2 \in$ sp-S and $\eta_{\#}(\mu_2) = \Gamma_2^g$ for some $g \in G$. For $\sigma_1 \in$ sp-R, there exists $\mu \in$ sp-R_1 such that σ_1 lies over μ that is, $\gamma_{\#}((\sigma_1)_G) = \bigwedge_{g \in G} \Gamma_1^g$, for some $\Gamma_1 \in$ sp-S and $\Gamma_1^h = \eta_{\#}(\mu)$ for some $h \in G$. Now, $\sigma_1 \underset{+}{<} \sigma_2$ implies that

$$\bigwedge_{g \in G} \Gamma_1^g \leq \bigwedge_{g \in G} \Gamma_2^g \leq \Gamma_2^g.$$

The prime character of Γ_2^g implies that $\Gamma_1^{g'} \leq \Gamma_2^g$ for some $g' \in G$. Since $(Sp_1S) \cap R \in \sigma_1(R)$ implies that, the ideal Sp_1S is in $\gamma_{\#}((\sigma_1)_G)(S) = \big(\bigwedge_{g \in G} \Gamma^g\big)(S)$ (as observed in Theorem 3.6 (i)). This proves that $Sp_1S \in \Gamma_1^g(S)$ for all $g \in G$ and hence $Sp_1S \in \Gamma_1^{g'}(S)$.

Taking $\mu_1 = \eta^{\#}(\Gamma_1^{g'})$, we have σ_1 lies over μ_1 and $\mu_1 \leq \mu_2$.

Finally, we show that $\mu_1 = \mu_2$ does not enable. Suppose that $\mu_1 = \mu_2$ then reversing the steps we get $(\sigma_1)^G = (\sigma_2)^G$. Using [2, Theorem 3.4 (ii)] we get that σ_2 is a minimal prime over $(\sigma_1)_G$. But this is a contradiction to $(\sigma_1)_G \leq \sigma_1 \underset{+}{<} \sigma_2$ and $\sigma_1 \in sp\text{-}R$. Thus, $\mu_1 \underset{+}{<} \mu_2$.

(ii) This also follows in a manner similar to (i). ■

Finally, we prove a Going Down Theorem.

Theorem 4.6. *Let $\mu_1 \underset{+}{<} \mu_2$ (μ_1 not equivalent to μ_2) be primes of tors-R_1 and $\sigma_2 \in sp\text{-}R$ such that σ_2 lies over μ_2. Then there exists a prime σ_1 of tors-R such that σ_1 lies over μ_1 and $\sigma_1 \underset{+}{<} \sigma_2$.*

Proof. Clearly, $\mu_1 \underset{+}{<} \mu_2$ and σ_2 lying over μ_2 give that

$$\gamma^{\#}((\eta_{\#}(\mu_1))^G \leq \gamma^{\#}((\eta_{\#}(\mu_2))^G) = (\sigma_2)_G \leq \sigma_2. \tag{1}$$

By Proposition 4.3 (ii), there exist k primes $\Gamma_1, \Gamma_2, \ldots, \Gamma_k$ of tors-R lying over μ_1 such that

$$\gamma^{\#}((\eta_{\#}(\mu_1))^G) = \bigwedge_{i=1}^{k} \Gamma_i. \tag{2}$$

Now, the prime character of σ_2 together with (1) and (2) implies that $\Gamma_i \leq \sigma_2$ for some i, $1 \leq i \leq k$. Taking $\sigma_1 = \Gamma_i$, we get the required result because $\sigma_1 = \sigma_2$ does not enable as μ_1 is not equivalent to μ_2.

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