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On Quasi-Linear Implicit Difference Equations

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Abstract. This paper is concerned with the solvability and approximate solutions of a class of quasi-linear implicit difference equations. Thanks to the index-1 (quasi-index-1) property of linear parts, an initial infinite system can be decoupled. Then the Banach's and Brouwer's fixed point theorems are applied to ensure the unique solvability (solvability) of the IVP for quasi linear implicit difference equations.

1. Introduction

In this paper we consider the following system of quasi-linear implicit difference equation:

$$A_n x_{n+1} = B_n x_n + f_n(x_{n+1}, x_n), \quad (n = 0, 1, 2, ...), \tag{1}$$

where A_n and B_n are given matrices of order $m \times m$. In what follows we assume that the matrices A_n are all singular, such that $1 \leq r_n \leq m - 1$ where $r_n = \operatorname{rank} A_n$. Further, suppose that $f_n \in C^1(\mathbb{R}^m \times \mathbb{R}^m, \mathbb{R}^m)$ and moreover

$$\operatorname{Ker} A_n \subset \operatorname{Ker} \frac{\partial f_n}{\partial y}(y, x), \tag{2}$$

for all $n \geq 0$ and $x, y \in \mathbb{R}^m$. Obviuosly, under condition (2), each equation (1) is implicit with respect to x_{n+1}.

Note that equation (1) may be regarded as a discrete analogue of quasi-linear differetial-algebraic equations (DAEs) which have already attracted much attention of researchers [2, 4].

Together with equation (1) we consider the corresponding system of linear implicit difference equation (LIDE)

$$A_n x_{n+1} = B_n x_n + q_n, \quad (n \geq 0) \tag{3}$$

which has been investigated in our recent works [1, 3].

Let $A_n = U_n \Sigma_n V_{n+1}^T$ be an SVD of A_n, where U_n (V_{n+1}) are orthogonal matrices, whose columns are left (right) singular vectors of A_n respectively, $\Sigma_n = \text{diag}\,(\sigma_1^{(n)}, \sigma_2^{(n)}, ..., \sigma_{r_n}^{(n)}, 0, ..., 0)$ are diagonal matrices with singular values $\sigma_1^{(n)} \geq \sigma_2^{(n)} \geq ... \geq \sigma_{r_n}^{(n)} > 0$ on their diagonals. We say that the LIDE (3) is of *quasi-index-1* if the matrices $G_n = A_n + B_n V_n Q^{(n)} V_{n+1}^T$, where $Q^{(n)} = \text{diag}\,(O_{r_n}, I_{m-r_n})$, are nonsingular for all $n \geq 0$. Here O_k, I_k stand for $k \times k$ zero matrix and $k \times k$ identity matrix, respectively. For $k = m$ we put simply $O := O_m; I := I_m$. If equation (3) is of quasi-index 1 and the rank A_n are the same for all n, i.e. $r_n \equiv r (n \geq 0)$ then the LIDE (3) is said to be of *index-1*. In this case, $Q := Q^{(n)} = \text{diag}\,(O_r, I_{m-r})$. It has been shown in [3] that the index-1 property of equation (3) does not depend on the choice of SVDs of A_n. Furthermore, the solvability and the unique solvability of some IVPs, as well as multipoint boundary value problems (MPBVPs) for index 1 LIDEs have been established in [1, 3].

Let $P^{(n)} := I - Q^{(n)}; Q_n := V_{n+1} Q^{(n)} V_{n+1}^T; P_n := I - Q_n (n \geq 0)$. For definiteness, we can set $V_0 := V_1; Q_{-1} := Q_0; P_{-1} := P_0$. Clearly, Q_n are projections onto $\text{Ker}\, A_n (n \geq 0)$, i.e. $Q_n^2 = Q_n; A_n Q_n = 0$, and besides $P_n Q_n = Q_n P_n = 0$.

The aim of this paper is to establish some existence theorems for systems of nonlinear difference equations involving linear index-1 or quasi-index-1 parts. The paper is organized as follows. In Sec. 2 we study the unique solvability and approximate solutions of IVPs for IDEs with linear index-1 parts. Section 3 deals with the solvability of implicit difference equations, whose linear parts are of quasi-index-1. Finally, in the last section, some illustrative examples are considered.

2. Unique Solvability of IVPs for IDEs Involving Linear Index-1 Parts

Suppose we want to find a solution of system (1), satisfying the initial condition

$$P_0 x_0 = \gamma \tag{4}$$

with a given vector $\gamma \in \mathbb{R}^m$. In what follows $\|.\|$ denotes any norm in $\mathbb{R}^m$.

Theorem 1. *Let the equation* (3) *be of index* 1. *Further assume that the nonlinear functions* $f_n(y, x)$ *satisfy condition* (2) *and the inequality*

$$\|f_n(y, x) - f_n(\xi, \zeta)\| \leq \alpha_n \|y - \xi\| + \beta_n \|x - \zeta\|, \tag{5}$$

with nonnegative constants α_n *and* β_n *for all* $y, x, \xi, \zeta \in \mathbb{R}^m$. *If*

$$\omega_n := \alpha_n \|P_n G_n^{-1}\| + \beta_n \|V_n V_{n+1}^T Q_n G_n^{-1}\| < 1, \tag{6}$$

then the IVP (1), (4) *has a unique solution.*

Proof. Taking into acount relation (2) and observing that

$$f_n(y,x) - f_n(P_n y, x) = \int_0^1 \frac{\partial f_n}{\partial y}(P_n y + tQ_n y, x) Q_n y dt = 0$$

for all $n \geq 0, x, y \in \mathbb{R}^m$ we come to the equation

$$f_n(y,x) = f_n(P_n y, x), \qquad \forall y, x \in \mathbb{R}^m. \tag{7}$$

First, observing that $A_n P_n = A_n$, we find that $G_n P_n = A_n P_n + B_n V_n Q V_{n+1}^T P_n = A_n + B_n V_n V_{n+1}^T Q_n P_n = A_n$, hence

$$G_n^{-1} A_n = P_n. \tag{8}$$

Similarly, $G_n Q_n = B_n V_n Q V_{n+1}^T Q_n = B_n V_n Q V_{n+1}^T V_{n+1} Q V_{n+1}^T = B_n V_n Q V_{n+1}^T$, hence $Q_n = G_n^{-1} B_n V_n Q V_{n+1}^T = G_n^{-1} B_n Q_{n-1} V_n V_{n+1}^T$, therefore $G_n^{-1} B_n Q_{n-1} = Q_n V_{n+1} V_n^T$. Thus

$$Q_n G_n^{-1} B_n Q_{n-1} = Q_n V_{n+1} V_n^T; P_n G_n^{-1} B_n Q_{n-1} = 0. \tag{9}$$

Now applying $P_n G_n^{-1}$ and $Q_n G_n^{-1}$ to both sides of equation (1) respectively and using relations (7), we find

$$P_n x_{n+1} = P_n G_n^{-1} B_n (P_{n-1} x_n + Q_{n-1} x_n) + P_n G_n^{-1} f_n(P_n x_{n+1}, x_n), \tag{10}$$

$$0 = Q_n G_n^{-1} B_n (P_{n-1} x_n + Q_{n-1} x_n) + Q_n G_n^{-1} f_n(P_n x_{n+1}, x_n). \tag{11}$$

Denoting $u_n := P_{n-1} x_n; v_n := Q_{n-1} x_n (n \geq 0)$ and using (9), from (10) we find

$$u_{n+1} = P_n G_n^{-1} B_n u_n + P_n G_n^{-1} f_n(u_{n+1}, x_n). \tag{12}$$

Replacing $Q_n G_n^{-1} B_n Q_{n-1} x_n$ with $Q_n V_{n+1} V_n^T x_n$ and taking into account the fact that $Q_n V_{n+1} V_n^T x_n = V_{n+1} Q V_{n+1}^T V_{n+1} V_n^T x_n = V_{n+1} Q V_n^T x_n = V_{n+1} V_n^T Q_{n-1} x_n = V_{n+1} V_n^T v_n$, from (11) we get

$$v_n = -V_n V_{n+1}^T Q_n G_n^{-1} \{B_n u_n + f_n(u_{n+1}, x_n)\}.$$

Since $x_n = u_n + v_n$, from the last relation we have

$$x_n = (I - V_n V_{n+1}^T Q_n G_n^{-1} B_n) u_n - V_n V_{n+1}^T Q_n G_n^{-1} f_n(u_{n+1}, x_n). \tag{13}$$

Now suppose that $u_n (n \geq 0)$ is found (u_0 is given, since $u_0 = P_{-1} x_0 = P_0 x_0 = \gamma$), we have to find $x_n = u_n + v_n$ and u_{n+1} from equations (12), (13). For the sake of simplicity, we denote $\xi := u_{n+1}, \zeta := x_n, r_n := P_n G_n^{-1} B_n u_n$ and $s_n := (I - V_n V_{n+1}^T Q_n G_n^{-1} B_n) u_n$. Thus we come to the system

$$\xi = P_n G_n^{-1} f_n(\xi, \zeta) + r_n, \tag{14}$$

$$\zeta = -V_n V_{n+1}^T Q_n G_n^{-1} f_n(\xi, \zeta) + s_n. \tag{15}$$

For establishing the unique solvability of system (14), (15) we can use an iteration method, namely

$$\xi^{k+1} = P_nG_n^{-1}f_n(\xi^k, \zeta^k) + r_n, \tag{16}$$
$$\zeta^{k+1} = -V_nV_{n+1}^TQ_nG_n^{-1}f_n(\xi^k, \zeta^k) + s_n, \tag{17}$$

where $k \geq 0$ and $\xi^0, \zeta^0 \in \mathbb{R}^m$ are arbitrarily chosen.
Clearly, for $k \geq 1$ we have

$$\|\xi^{k+1} - \xi^k\| \leq \|P_nG_n^{-1}\|(\alpha_n\|\xi^k - \xi^{k-1}\| + \beta_n\|\zeta^k - \zeta^{k-1}\|) \tag{18}$$

and

$$\|\zeta^{k+1} - \zeta^k\| \leq \|V_nV_{n+1}^TQ_nG_n^{-1}\|(\alpha_n\|\xi^k - \xi^{k-1}\| + \beta_n\|\zeta^k - \zeta^{k-1}\|). \tag{19}$$

Denoting $\mu_k := \alpha_n\|\xi^k - \xi^{k-1}\| + \beta_n\|\zeta^k - \zeta^{k-1}\|$ and multiplying both sides of inequalities (18) and (19) by α_n and β_n, respectively, then taking the sum of both sides of the obtained inequalities, we find

$$\mu_{k+1} \leq (\alpha_n\|P_nG_n^{-1}\| + \beta_n\|V_nV_{n+1}^TQ_nG_n^{-1}\|)\mu_k = \omega_n\mu_k \leq ... \leq \omega_n^k\mu_1.$$

Since $\|\xi^{k+1}-\xi^k\| \leq \|P_nG_n^{-1}\|\mu_k \leq \mu_1\|P_nG_n^{-1}\|\omega_n^{k-1}$, $\|\zeta^{k+1}-\zeta^k\| \leq \mu_1\|V_nV_{n+1}^TQ_n G_n^{-1}\|\omega_n^{k-1}$, it follows that $\xi^k \longrightarrow \xi^*$ and $\zeta^k \longrightarrow \zeta^*$ when $k \longrightarrow \infty$. Evidently, (ξ^*, ζ^*) is a solution of system (14), (15).

Now assume that (ξ, ζ) and (ξ', ζ') are two solutions of system (14), (15). Then $\|\xi - \xi'\| = \|P_nG_n^{-1}\{f_n(\xi, \zeta) - f_n(\xi', \zeta')\}\| \leq \|P_nG_n^{-1}\|\{\alpha_n\|\xi - \xi'\| + \beta_n\|\zeta - \zeta'\|\}$. Similarly, $\|\zeta - \zeta'\| = \|V_nV_{n+1}^TQ_nG_n^{-1}\{f_n(\xi, \zeta) - f_n(\xi', \zeta')\}\| \leq \|V_nV_{n+1}^TQ_nG_n^{-1}\|\{\alpha_n\|\xi - \xi'\| + \beta_n\|\zeta - \zeta'\|\}$. Acting as before, we get $\alpha_n\|\xi - \xi'\| + \beta_n\|\zeta - \zeta'\| \leq \omega_n(\alpha_n\|\xi - \xi'\| + \beta_n\|\zeta - \zeta'\|)$. From the last inequality, it follows $\alpha_n\|\xi - \xi'\| + \beta_n\|\zeta - \zeta'\| = 0$ hence $\xi = \xi'$ and $\zeta = \zeta'$. Thus the unique solvability of the IVP (1), (4) is proved. ■

Remark 1.
* Condition (5) holds if for all $x, y \in \mathbb{R}^m$, $\|\frac{\partial f_n}{\partial y}(y, x)\| \leq \alpha_n$ and $\|\frac{\partial f_n}{\partial x}(y, x)\| \leq \beta_n$
* Condition (2) is satisfied if $f_n(y, x) = g_n(A_ny, x)$ where $g_n \in C^1(\mathbb{R}^m \times \mathbb{R}^m, \mathbb{R}^m)$.

Theorem 2. *The conclusion of Theorem* 1 *remains valid if instead of condition* (5) *we require*

$$\operatorname{Im}\frac{\partial f_n}{\partial y}(y, x) \subset \operatorname{Im} A_n, \forall x, y \in \mathbb{R}^m, \tag{20}$$
$$\max\{\alpha_n\|P_nG_n^{-1}\|, \beta_n\|V_nV_{n+1}^TQ_nG_n^{-1}\|\} < 1, (n \geq 0). \tag{21}$$

Proof. First, observe that

$$\begin{aligned} Q_nG_n^{-1}[f_n(y, x) - f_n(0, x)] &= Q_nG_n^{-1}\int_0^1 \frac{\partial f_n}{\partial y}(ty, x)ydt \\ &= \int_0^1 Q_nG_n^{-1}\frac{\partial f_n}{\partial y}(ty, x)ydt. \end{aligned}$$

Taking into account relation (20) we can rewrite $\frac{\partial f}{\partial y}(ty,x)y$ as $A_n z_n(y,x,t)$ for some $z_n(y,x,t) \in \mathbb{R}^m$. From (8) it follows $Q_n G_n^{-1} A_n = 0$, hence

$$\int_0^1 Q_n G_n^{-1} \frac{\partial f_n}{\partial y}(ty,x) y dt = 0.$$

Therefore

$$Q_n G_n^{-1} f_n(y,x) = Q_n G_n^{-1} f_n(0,x), \forall x, y \in \mathbb{R}^m. \tag{22}$$

Thus instead of system (14), (15) we have

$$\xi = P_n G_n^{-1} f_n(\xi,\zeta) + r_n, \tag{23}$$

$$\zeta = \; - V_n V_{n+1}^T Q_n G_n^{-1} f_n(0,\zeta) + s_n. \tag{24}$$

Obviously, the contractivity condition $\beta_n \|V_n V_{n+1}^T Q_n G_n^{-1}\| < 1$ ensures the unique solvability of equation (24). Substituting the unique solution ζ of equation (24) into equation (23) and using the assumption $\alpha_n \|P_n G_n^{-1}\| < 1$ we can easily deduce the unique solvability of equation (23). ■

Remark 2. Both conditions (2), (20) hold, for example, if

$$f_n(y,x) = \Sigma_{i=1}^{k_n} a_{in} A_n^i y + g_n(x),$$

where $g_n \in C^1(\mathbb{R}^m, \mathbb{R}^m)$.

Next, we present an algorithm for finding an approximate solution of IVP (1), (4).

Theorem 3. *Let the LIDE* (3) *be of index*-1 *and the nonlinear functions* f_n *satisfy condition* (2), (5). *Moreover, suppose the following conditions are satisfied:*

(i) *the matricies* $I - V_n V_{n+1}^T Q_n G_n^{-1} B_n$ *are uniformly bounded, i.e.*

$$\|I - V_n V_{n+1}^T Q_n G_n^{-1} B_n\| \le C_1; \tag{25}$$

(ii) *the norm of the matrices* $P_n G_n^{-1} B_n$ *are uniformly bounded by a constant less than 1, i.e.*

$$\|P_n G_n^{-1} B_n\| \le \delta_0 < 1; \tag{26}$$

(iii) *the coefficients* α_n, β_n *are small enough, such that*

$$\alpha_n \|P_n G_n^{-1}\| + \beta_n \|V_n V_{n+1}^T Q_n G_n^{-1}\| \le \omega < 1, \tag{27}$$

$$(1-\omega)^{-1} \|P_n G_n^{-1}\| (\delta_0 \alpha_n + C_1 \beta_n) \le \delta_1 < 1 - \delta_0, \tag{28}$$

and

$$(1-\omega)^{-1} \|V_n V_{n+1}^T Q_n G_n^{-1}\| (\delta_0 \alpha_n + C_1 \beta_n) \le C_2. \tag{29}$$

Then within a given tolerance $\epsilon > 0$, *we can always find an approximate solution* $\{\overline{x}_n\}(n \ge 0)$ *via iterations, such that* $\|\overline{x}_n - x_n\| \le \epsilon (n \ge 0)$, *where* $\{x_n\}(n \ge 0)$ *is a unique solution of IVP* (1), (4).

Proof. Theorem 1 ensures the unique solvability of IVP (1), (4), therefore system (14), (15) has the unique solution $\{u_{n+1}, x_n\}(n \ge 0)$. Starting with $\overline{u}_0 = \gamma$ and performing some iterations by formulae (16), (17) for $n = 0$, we can find $\overline{x}_0$ and $\overline{u}_1$ such that $\|\overline{u}_1 - u_1\| \le \epsilon/C$ and $\|\overline{x}_0 - x_0\| \le \epsilon$ where $C = (C_1 + C_2)(1-\delta)^{-1}$ and $\delta = 1 - \delta_0 - \delta_1 > 0$. Now suppose by induction that for $k = 0, 1, ..., n-1$ we have found approximate solution $\overline{u}_{k+1}, \overline{x}_k$ of system (14), (15), such that

$$\|\overline{u}_{k+1} - u_{k+1}\| \le \epsilon/C; \|\overline{x}_k - x_k\| \le \epsilon; (k = 0, 1, ..., n-1). \tag{30}$$

Let $\tilde{x}_n, \tilde{u}_{n+1}$ be the unique solution of the system

$$\tilde{u}_{n+1} = P_n G_n^{-1} f_n(\tilde{u}_{n+1}, \tilde{x}_n) + \overline{r}_n, \tag{31}$$

$$\tilde{x}_n = -V_n V_{n+1}^T Q_n G_n^{-1} f_n(\tilde{u}_{n+1}, \tilde{x}_n) + \overline{s}_n, \tag{32}$$

where $\overline{r}_n = P_n G_n^{-1} B_n \overline{u}_n$ and $\overline{s}_n = (I - V_n V_{n+1}^T Q_n G_n^{-1} B_n)\overline{u}_n$. The unique solvability of system (31), (32) is established by the same manner as in Theorem 1. Using relations (14), (15), (31), (32) and taking into account assumptions (25), (26), (30), we get $\|\tilde{u}_{n+1} - u_{n+1}\| \le \|P_n G_n^{-1}\| \{\alpha_n \|\tilde{u}_{n+1} - u_{n+1}\| + \beta_n \|\tilde{x}_n - x_n\|\} + \|P_n G_n^{-1} B_n\| \|\overline{u}_n - u_n\|$ or

$$\|\tilde{u}_{n+1} - u_{n+1}\| \le \mu_n \|P_n G_n^{-1}\| + \delta_0 \epsilon/C, \tag{33}$$

where μ_n denotes the quantity $\alpha_n \|\tilde{u}_{n+1} - u_{n+1}\| + \beta_n \|\tilde{x}_n - x_n\|$. Analogously, $\|\tilde{x}_n - x_n\| \le \|V_n V_{n+1}^T Q_n G_n^{-1}\| \{\alpha_n \|\tilde{u}_{n+1} - u_{n+1}\| + \beta_n \|\tilde{x}_n - x_n\|\} +$
$+ \|I - V_n V_{n+1}^T Q_n G_n^{-1} B_n\| \|\overline{u}_n - u_n\|$, i.e.

$$\|\tilde{x}_n - x_n\| \le \mu_n \|V_n V_{n+1}^T Q_n G_n^{-1}\| + C_1 \epsilon/C. \tag{34}$$

Multiplying both sides of inequalities (33) and (34) by α_n and β_n respectively and then taking the sum of both sides of the obtained inequalities we have $\mu_n \le (\alpha_n \|P_n G_n^{-1}\| + \beta_n \|V_n V_{n+1}^T Q_n G_n^{-1}\|)\mu_n + (\delta_0 \alpha_n + C_1 \beta_n)\epsilon/C \le \omega \mu_n + (\delta_0 \alpha_n + C_1 \beta_n)\epsilon/C$. Thus,

$$\mu_n \le \frac{(\delta_0 \alpha_n + C_1 \beta_n)\epsilon}{C(1-\omega)}. \tag{35}$$

From relations (33) and (35) it follows

$$\|\tilde{u}_{n+1} - u_{n+1}\| \le \frac{(\delta_0 \alpha_n + C_1 \beta_n)\|P_n G_n^{-1}\|\epsilon}{C(1-\omega)} + \frac{\delta_0 \epsilon}{C}.$$

Using estimate (28), from the last relation we find

$$\|\tilde{u}_{n+1} - u_{n+1}\| \le \frac{\delta_1 + \delta_0}{C}\epsilon = \frac{(1-\delta)\epsilon}{C}. \tag{36}$$

Similarly, from inequalities (34), (35), using assumption (29), we get

$$\|\tilde{x}_n - x_n\| \le \{(1-\omega)^{-1} \|V_n V_{n+1}^T Q_n G_n^{-1}\| (\delta_0 \alpha_n + C_1 \beta_n) + C_1\}\epsilon/C \le (C_1 + C_2)\epsilon/C,$$

hence

$$\|\tilde{x}_n - x_n\| \le (1-\delta)\epsilon. \tag{37}$$

Now performing some iterations by the formulae

$$\xi^{k+1} = P_n G_n^{-1} f_n(\xi^k, \zeta^k) + \overline{r}_n, \tag{38}$$

$$\zeta^{k+1} = -V_n V_{n+1}^T Q_n G_n^{-1} f_n(\xi^k, \zeta^k) + \overline{s}_n, \tag{39}$$

we can find $\overline{u}_{n+1} = \xi^{k+1}, \overline{x}_n = \zeta^{k+1}$ such that $\|\overline{u}_{n+1} - \tilde{u}_{n+1}\| \le \delta\epsilon/C$ and $\|\overline{x}_n - \tilde{x}_n\| \le \delta\epsilon$. From relations (36), (37) and the last estimates, we come to the disired inequality $\|\overline{u}_{n+1} - u_{n+1}\| \le \epsilon/C$ and $\|\overline{x}_n - x_n\| \le \epsilon$. The proof of Theorem 3 is complete. ∎

3. Solvability of IVPs in Quasi-Index-1 Cases

Suppose the LIDE (3) is of quasi-index-1. Moreover, assume that

$$r_{n+1} \ge r_n (n \ge 0). \tag{40}$$

Clearly, equation (1) can be reduced to the form

$$\Sigma_n y_{n+1} = \tilde{B}_n y_n + g_n(y_{n+1}, y_n), \tag{41}$$

where $y_n = V_n^T x_n, \tilde{B}_n = U_n^T B_n V_n$ and $g_n(y, x) = U_n^T f_n(V_{n+1} y, V_n x)$. The following lemma collects some obvious facts needed for our further consideration.

Lemma. *The following 3 assertions hold:*

1. *The matrices* $G_n = A_n + B_n V_n Q^{(n)} V_{n+1}^T$ *and* $\tilde{G}_n := \Sigma_n + \tilde{B}_n Q^{(n)}$ *are singular or nonsingular simultaneously.*
2. *Condition (2) is equivalent to the inclusion*

$$Ker \Sigma_n \subset Ker \frac{\partial g_n}{\partial y}(y, x). \tag{42}$$

3.

$$g_n(y, x) = g_n(P^{(n)} y, x) \tag{43}$$

for all $n \ge 0$ *and* $y, x \in \mathbb{R}^m$.

In the following theorem, for simplicity, we use the Euclid norm.

Theorem 4. *Suppose the LIDE* (3) *is of quasi-index* 1 *and condition* (40) *holds. Further, assume that the nonlinear functions* $f_n(y, x)$ *satisfy condition* (2) *and the following growth condition:*

$$\|f_n(\xi, \zeta)\| \le a_n \|\xi\|^{\nu_n} + b_n \|\zeta\|^{\mu_n} + c_n \tag{44}$$

with nonegative constants $a_n, b_n, c_n, \nu_n, \mu_n$, *where* $\theta_n := max\{\nu_n, \mu_n\} \le 1$. *Additionally, we suppose that if* $\theta_n = 1$ *then* $(a_n + b_n)\|G_n^{-1}\| < 1$. *Then the IVP* (1), (4) *possesses a solution.*

Proof. Applying $P^{(n)} \tilde{G}_n^{-1}$ and $Q^{(n)} \tilde{G}_n^{-1}$ to both sides of equation (41) respectively and using the relations $P^{(n)} + Q^{(n)} = I, \tilde{G}_n^{-1} \tilde{B}_n Q^{(n)} = Q^{(n)}, \tilde{G}_n^{-1} \Sigma_n = P^{(n)}$ as well as taking into account the previous lemma, we get

$$P^{(n)} y_{n+1} = P^{(n)} \tilde{G}_n^{-1} \tilde{B}_n P^{(n)} y_n + P^{(n)} \tilde{G}_n^{-1} g_n(P^{(n)} y_{n+1}, y_n),$$
$$0 = Q^{(n)} \tilde{G}_n^{-1} \tilde{B}_n P^{(n)} y_n + Q^{(n)} y_n + Q^{(n)} \tilde{G}_n^{-1} g_n(P^{(n)} y_{n+1}, y_n).$$

Putting $\xi_n = P^{(n)}y_n$ and observing that $P^{(n)}y_{n+1} = P^{(n)}P^{(n+1)}y_{n+1} = P^{(n)}\xi_{n+1}$ we come to the following system:

$$P^{(n)}\xi_{n+1} = P^{(n)}\tilde{G}_n^{-1}\tilde{B}_n\xi_n + P^{(n)}\tilde{G}_n^{-1}g_n(P^{(n)}\xi_{n+1}, y_n), \tag{45}$$

$$0 = Q^{(n)}\tilde{G}_n^{-1}\tilde{B}_n\xi_n + Q^{(n)}y_n + Q^{(n)}\tilde{G}_n^{-1}g_n(P^{(n)}\xi_{n+1}, y_n). \tag{46}$$

From the last equation, it follows

$$y_n = (P^{(n)} + Q^{(n)})y_n = (I - Q^{(n)}\tilde{G}_n^{-1}\tilde{B}_n)\xi_n - Q^{(n)}\tilde{G}_n^{-1}g_n(P^{(n)}\xi_{n+1}, y_n). \tag{47}$$

On the other hand, equation (45) gives

$$\xi_{n+1} = \tilde{G}_n^{-1}\tilde{B}_n\xi_n + \tilde{G}_n^{-1}g_n(P^{(n)}\xi_{n+1}, y_n) + Q^{(n)}\eta, \tag{48}$$

where $\eta \in \mathbb{R}^m$ is an arbitrary vector. Denoting $p_n(\eta) := \tilde{G}_n^{-1}\tilde{B}_n\xi_n + Q^{(n)}\eta$ and $q_n{:=}(I - Q^{(n)}\tilde{G}_n^{-1}\tilde{B}_n)\xi_n$ and simplifying our notations by setting $\xi{=}\xi_{n+1}, \zeta{=}y_n$ we come to the final system that should be investigated.

$$\xi = \tilde{G}_n^{-1}g_n(P^{(n)}\xi, \zeta) + p_n(\eta), \tag{49}$$

$$\zeta = \, - Q^{(n)}\tilde{G}_n^{-1}g_n(P^{(n)}\xi, \zeta) + q_n. \tag{50}$$

In $\mathbb{R}^m \times \mathbb{R}^m$ endowed with the norm $\|(\xi, \eta)\| = \max\{\|\xi\|, \|\eta\|\}$ we consider the map

$$T_n(\xi, \zeta) := (\tilde{G}_n^{-1}g_n(P^{(n)}\xi, \zeta) + p_n(\eta), -Q^{(n)}\tilde{G}_n^{-1}g_n(P^{(n)}\xi, \zeta) + q_n).$$

Since $\|U_n\| = \|V_{n+1}\| = \|P^{(n)}\| = \|Q^{(n)}\| = 1$, from the growth condition (44) it follows for $\|(\xi, \zeta)\| \geq 1$ that

$$\|g_n(P^{(n)}\xi, \zeta)\| \leq a_n\|\xi\|^{\nu_n} + b_n\|\zeta\|^{\mu_n} + c_n \leq (a_n + b_n)\|(\xi, \zeta)\|^{\theta_n} + c_n. \tag{51}$$

Clearly, relation (51) implies that

$$\|T_n(\xi, \zeta)\| = \max\{\|\tilde{G}_n^{-1}g_n(P^{(n)}\xi, \zeta) + p_n(\eta)\|, \|Q^{(n)}\tilde{G}_n^{-1}g_n(P^{(n)}\xi, \zeta) + q_n\|\}$$
$$\leq \|\tilde{G}_n^{-1}\|(a_n + b_n)\|(\xi, \zeta)\|^{\theta_n} + \tilde{c}_n,$$

where $\tilde{c}_n = \max\{\|p_n(\eta)\|, \|q_n\|\} + c_n\|\tilde{G}_n^{-1}\|$. Observing that

$$\|\tilde{G}_n^{-1}\| = \|V_{n+1}^T G_n^{-1} U_n\| \leq \|G_n^{-1}\| = \|V_{n+1}\tilde{G}_n^{-1}U_n^T\| \leq \|\tilde{G}_n^{-1}\|$$

we get $\|G_n^{-1}\| = \|\tilde{G}_n^{-1}\|$. When $\|(\xi, \zeta)\|$ is sufficiently large, there holds the inequality

$$\frac{\|T_n(\xi, \zeta)\|}{\|(\xi, \zeta)\|} \leq (a_n + b_n)\|G_n^{-1}\|\|(\xi, \zeta)\|^{\theta_n - 1} + \tilde{c}_n\|(\xi, \zeta)\|^{-1}. \tag{52}$$

The estimate (52) implies that, in both cases, when $\theta_n < 1$ or $\theta_n = 1$ and $(a_n + b_n)\|G_n^{-1}\| < 1, \overline{\lim}_{\|(\xi,\zeta)\|\to\infty} \frac{\|T_n(\xi,\zeta)\|}{\|(\xi,\zeta)\|} < 1$. Thus $T_n(\xi, \zeta)$ maps a closed ball in $\mathbb{R}^m \times \mathbb{R}^m$ centered at $(0, 0)$ with a sufficiently large radius R in to itself. The Brouwer fixed point theorem ensures the existence of solutions of the system (49), (50). Starting with the given initial value $u_0 = P_0x_0 = \gamma$ or $u_0 = V_1P^{(0)}V_1^Tx_0 = \gamma$ we get $P^{(0)}V_1^Tx_0 = V_1^T\gamma$, hence, $\xi_0 = P^{(0)}y_0 = P^{(0)}V_0^Tx_0 = P^{(0)}V_1^Tx_0 = V_1^T\gamma$ is known. From system (49), (50) we can find ξ_1, y_0 and then ξ_2, y_1, etc... Thus $x_n = V_ny_n$ can be found, which was to be proved. ■

4. Examples

4.1. Consider system (1) with the following data

$$A_n = \begin{pmatrix} \frac{1}{\sin(n+1)} & -\frac{1}{4\sin(n+1)} \\ \frac{1}{\cos(n+1)} & -\frac{1}{4\cos(n+1)} \end{pmatrix}, \qquad B_n = \begin{pmatrix} \frac{1}{17\sin(n+1)} & 0 \\ \frac{-1}{34\cos(n+1)} & \frac{1}{17\cos(n+1)} \end{pmatrix},$$

$$f_n(y, x) = g_n(A_ny, A_nx) = g_n(\xi, \zeta) = \begin{pmatrix} a_n \cos\xi^{(1)} + b_n \sin\zeta^{(2)} \\ c_n \cos\zeta^{(1)} + d_n \sin\xi^{(2)} \end{pmatrix},$$

where

$$a_n = c_n = \frac{1}{2^{10}}\|A_n\|^{-1}; b_n = \frac{1}{2^{10+n}}\|A_n\|^{-1}; d_n = \frac{1}{2^{10+n+1}}\|A_n\|^{-1} \quad \text{and}$$

$\|.\|$ denotes the max-norm in $\mathbb{R}^m$ or $R^{m\times m}$. Obviously,

$$\text{Ker } A_n \subset \text{Ker } \frac{\partial f_n}{\partial y}(y, x),$$

and

$$V_n^T = \begin{pmatrix} \frac{-4}{\sqrt{17}} & \frac{1}{\sqrt{17}} \\ \frac{1}{\sqrt{17}} & \frac{4}{\sqrt{17}} \end{pmatrix}, \qquad G_n^{-1} = \begin{pmatrix} \frac{-466}{85\sin(n+1)} & \frac{546}{85\cos(n+1)} \\ \frac{-468}{17\sin(n+1)} & \frac{-464}{17\cos(n+1)} \end{pmatrix},$$

$$\|f_n(y, x) - f_n(z, t)\| \leq \alpha_n\|y - z\| + \beta_n\|x - t\|,$$

where

$$\alpha_n = \beta_n = \frac{1}{2^{10}}.$$

Further,

$$\|P_nG_n^{-1}\| = \frac{112}{85}|\sin(n + 1)| + \frac{32}{85}|\cos(n + 1)| \leq \frac{144}{85},$$

$$\|V_nV_{n+1}^TQ_nG_n^{-1}B_n\| = \frac{136}{5}(|\sin(n + 1)| + |\cos(n + 1)|) \leq \frac{272}{5},$$

$$\omega_n = \alpha_n\|P_nG_n^{-1}\| + \beta_n\|V_nV_{n+1}^TQ_nG_n^{-1}B_n\| \leq \frac{144}{85}\alpha_n + \frac{272}{5}\beta_n < 1.$$

Thus, all conditions of theorem 1 are satisfied, hence the IVP (1), (4) has a unique solution.

Since

$$\|P_n G_n^{-1} B_n\| = \frac{32}{289} = \delta_0 < 1, \ \ \|I - V_n V_{n+1}^T Q_n G_n^{-1} B_n\| = 3 = C_1, \omega_n \leq \frac{57}{1024} = \omega,$$

the inequalities (28),(29) hold with the left sides $\frac{129456}{23754355}$ and $\frac{14384}{82195}$ respectively.

Let $\gamma = (1,1)^T$.

According to Theorem 3 we can find approximate solutions $\{\bar{x}_n\}, n \geq 0$ via interations such that $\|\bar{x}_n - x_n\| \leq 10^{-6}, n \geq 0$. (see table).

n	$x_n^{(1)}$	$x_n^{(2)}$	n	$x_n^{(1)}$	$x_n^{(2)}$
0	1.2040725	1.8162902	5	−0.0016114	−0.0070262
1	0.1175060	0.1792858	6	0.0001727	−0.0005747
2	0.0115993	0.0196851	7	0.0010052	0.0027487
3	0.0005366	0.0002837	8	0.0036480	0.0146555
4	−0.0020460	−0.0075569	...	...	...

4.2. Consider system (1) with the data

$$A_n x_{n+1} = B_n x_n + f_n(x_{n+1}, x_n) \quad (n = 0, 1, 2, ...) \tag{53}$$

with

$$A_n = \begin{pmatrix} 0 & \frac{\sqrt{2}n(n+1)}{2\sqrt{n^2+1}} & \frac{-\sqrt{2}n(n+1)}{2\sqrt{n^2+1}} \\ 0 & \frac{\sqrt{2}(n+1)}{2\sqrt{n^2+1}} & \frac{-\sqrt{2}(n+1)}{2\sqrt{n^2+1}} \\ n & 0 & 0 \end{pmatrix}$$

$$B_n = \begin{pmatrix} \frac{3n^2}{\sqrt{n^2+1}} & \frac{-\sqrt{2}}{2\sqrt{n^2+1}} & \frac{\sqrt{2}(6n-1)}{2\sqrt{n^2+1}} \\ \frac{3n}{\sqrt{n^2+1}} & \frac{3\sqrt{2}}{2\sqrt{n^2+1}} & \frac{5\sqrt{2}}{2\sqrt{n^2+1}} \\ n+1 & \frac{\sqrt{2}n(-2\sqrt{n^2+1}+3n+1)}{2\sqrt{n^2+1}} & \frac{-\sqrt{2}n(4\sqrt{n^2+1}+n-1)}{2\sqrt{n^2+1}} \end{pmatrix}$$

and $f_n(y,x) = U_n g_n(V_{n+1}^T y, V_n^T x)$, where

$$U_n = \begin{pmatrix} \frac{n}{\sqrt{n^2+1}} & 0 & \frac{-1}{\sqrt{n^2+1}} \\ \frac{1}{\sqrt{n^2+1}} & 0 & 0 \\ 0 & 1 & \frac{n}{\sqrt{n^2+1}} \end{pmatrix} \quad \text{and} \quad V_{n+1}^T = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

are orthogonal matrices, such as $A_n = U_n \Sigma_n V_{n+1}^T$ with $\Sigma_n = \text{diag}\,(n+1, n, 0)$ and

$$g_n(\xi,\zeta) = \begin{pmatrix} g_n^{(1)}(\xi,\zeta) \\ g_n^{(2)}(\xi,\zeta) \\ g_n^{(3)}(\xi,\zeta) \end{pmatrix}$$

$$= \begin{pmatrix} a_n \cos(\xi^{(1)} + \xi^{(2)})(\zeta^{(3)})^{1/3} \\ b_n \sin(\zeta^{(1)} + \zeta^{(2)})(\xi^{(1)})^{1/5} \\ b_n \cos(\zeta^{(1)} + \zeta^{(2)})(\xi^{(1)})^{1/5} + a_n \sin(\xi^{(1)} + \xi^{(2)})(\zeta^{(3)})^{1/3} + c_n \end{pmatrix}.$$

Here we put

$$a_n = \frac{\sqrt{2}\big[-4n^2 + \big(\pi/4 - 7\big)n + \pi/2 - 5\big]}{(n+1)^{1/3}}, \quad b_n = \frac{\sqrt{2}[4n^2 + (3-\pi/4)n]}{(\pi/4 - n - 1)^{1/5}},$$

$$c_n = n^2 + \Big(2 - \frac{\pi}{2}\Big)n + 4 - \frac{\pi}{2}.$$

Then system (1) is reduced to the system (41) with

$$\tilde{B}_n = U_n^T B_n V_n = \begin{pmatrix} -1 & 3n & 4 \\ n & n+1 & -3n \\ 2n & 0 & n+1 \end{pmatrix}.$$

Since $\frac{\partial g_n}{\partial \xi} Q^{(n)} = (0,0,0)^T$, condition (2) is satisfied. Moreover, we have

$$\|f_n(y,x)\| = \|U_n g_n(V_{n+1}^T y, V_n^T x)\| \le \|g_n(\xi,\zeta)\|$$
$$\le \sqrt{3}\{|a_n|\|\xi\|^{1/5} + |b_n|\|\zeta\|^{1/3} + |c_n|\} \le \sqrt{3}|a_n|\|x\|^{1/5} + \sqrt{3}|b_n|\|y\|^{1/3} + \sqrt{3}|c_n|.$$

Further, the matrices $\tilde{G}_n = \Sigma_n + \tilde{B}_n Q^{(n)}$ are nonsingular,

$$\tilde{G}_0^{-1} = \begin{pmatrix} 1 & 0 & -4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \tilde{G}_n^{-1} = \begin{pmatrix} \frac{1}{n+1} & 0 & \frac{-4}{(n+1)^2} \\ 0 & \frac{1}{n} & \frac{3}{n+1} \\ 0 & 0 & \frac{1}{n+1} \end{pmatrix}.$$

Thus all conditions of Theorem 4 are fulfilled. A short computation shows that

$$x_n = V_n y_n = \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \pi/4 - n \\ n \\ n+1 \end{pmatrix} = \begin{pmatrix} n \\ \frac{1}{\sqrt{2}}(\pi/4+1) \\ \frac{1}{\sqrt{2}}(\pi/4+1) \end{pmatrix}.$$

References

1. P. K. Anh and L. C. Loi, On multipoint boundary-value problems for linear implicit non-autonomous systems of difference equations, *Vietnam J. Math.* **29** (2001) 281–286.
2. V. F. Čistjakov, *Differential-Algebraic Operators with Finite Dimensional Kernels*, Nauka, Moscow, 1996 (Russian).
3. L. C. Loi, N. H. Du, and P. K. Anh, On linear implicit non-autonomous system of difference equations, *J. Differen. Eq. Appl.* **8** (2002) 1085–1105.
4. R. März, On linear differential-algebraic equations and linearizations, *Appl. Num. Math.* **18** (1995) 267–292.