Vietnam Journal of Mathematics **32:** *SI* (2004) 19–22

Vietnam Journal
of
MATHEMATICS

A Cauchy Like Problem in Plane Elasticity: A Moment Theoretic Approach*

Dang Dinh Ang[1], Chu Duc Khanh[2], and Masahiro Yamamoto[3]

[1] *Dept. of Math. and Inform., Ho Chi Minh City National University, 227 Nguyen Van Cu, Ho Chi Minh City, Vietnam*
[2] *Graduate School of Mathematical Sciences, The University of Tokyo, Japan*

Abstract. Let Ω be an elastic body represented by a bounded domain in the (x, y)-plane, the outer boundary of which contains an open straight line segment Γ_0 parallel to the x-axis. Assuming plane stress with no body forces involved, and given the normal stress, shear, and displacements on Γ_0, and with the additional hypothesis that the domain of solutions can be extended so as to contain Γ_0 in its interior, the authors develop a method for constructing the solution in the whole of Ω.

Let Ω be an elastic body represented by a bounded plane domain. We propose to determine the stress field in Ω from displacements and surface stresses given on an open portion Γ_0 of the outer boundary of Ω. It is assumed that Γ_0 is a straight line segment parallel to the x-axis (see Fig. 1). Cauchy like problems in plane elasticity are treated by the method of quasireversibility in [2] where the geometries and the boundary conditions are more general than those considered below but the results obtained are less specific.

Assume plane stress. We shall follow the notations in Timoshenko and Goodier [5]. Denote the displacements in the x - and y -directions respectively by u and v and the stress components by σ_x, σ_y and τ_{xy} (cf [5] loc.cit.).

Assume that no body forces are involved. Then we have the equations of equilibrium

*The work of the first two named authors is supported by the Council for National Sciences of Vietnam.

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0, \tag{1}$$

$$\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} = 0. \tag{2}$$

Eqs. (1)–(2) are subject to the boundary conditions on Γ_0

$$\ell \sigma_x + m\tau_{xy} = p(x), \tag{3}$$

$$m\sigma_y + \ell \tau_{xy} = q(x), \tag{4}$$

$$u = \overline{u}(x), \tag{5}$$

$$v = \overline{v}(x), \tag{6}$$

where (ℓ, m) is the outer unit normal $\vec{n}$ to the boundary portion Γ_0 with $\ell = x$ - component, $m = y$ - component of $\vec{n}$. Note that the given data (3)-(6) are sufficient to ensure uniqueness of the stress field in Ω (cf. [1]). We finally assume that the domain of the solutions can be extended so as to contain Γ_0 in its interior.

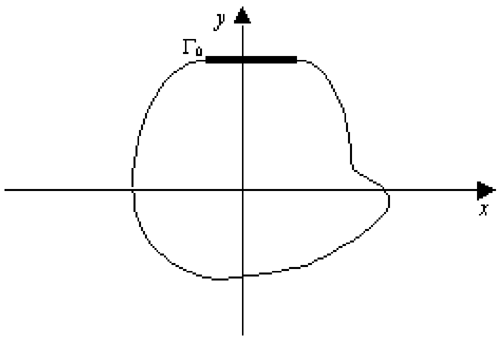

Fig. 1

Define

$$\varepsilon_x = \frac{\partial u}{\partial x}, \quad \varepsilon_y = \frac{\partial v}{\partial y}, \quad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}. \tag{7}$$

Recalling that plane stress is assumed, we have

$$\varepsilon_x = \frac{1}{E}(\sigma_x - v\sigma_y), \tag{8}$$

$$\varepsilon_y = \frac{1}{E}(\sigma_y - v\sigma_x), \tag{9}$$

$$\gamma_{xy} = \frac{1}{G}\tau_{xy} = \frac{2(1+v)}{E}, \tag{10}$$

and

$$\Delta(\sigma_x + \sigma_y) = 0. \tag{11}$$

The next steps will be

Step 1. Determination of $\sigma_x + \sigma_y$.

Step 2. Calculation of the Airy stress function ϕ to be defined later.

Step 1. Note first that $\sigma_x + \sigma_y$ is a harmonic function on Ω. We shall show that $\sigma_x + \sigma_y$ is solution of a Cauchy problem with Cauchy data prescribed on Γ_0. From (3) we have

$$\tau_{xy} = p(x) \quad \text{on} \quad \Gamma_0 \tag{12}$$

or equivalently

$$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{2(1+v)}{E} p(x). \tag{13}$$

Since $\frac{\partial v}{\partial x}$ is known on Γ_0, it follows that on Γ_0

$$\frac{\partial u}{\partial y} = \frac{2(1+v)}{E} p(x) - \frac{\partial v}{\partial x} \equiv \alpha(x) \quad (\text{known on } \Gamma_0). \tag{14}$$

By (5),

$$\sigma_y = q(x)(\text{known on } \Gamma_0). \tag{15}$$

Since τ_{xy} is known on Γ_0, it follows from (2) that

$$\frac{\partial \sigma_y}{\partial y} = -\frac{\partial \tau_{xy}}{\partial x} = -p\ (x). \tag{16}$$

Thus

$$\sigma_y \ \text{ and } \ \frac{\partial \sigma_y}{\partial y} \ \text{ are known on } \Gamma_0. \tag{17}$$

Similarly, it can be shown that

$$\sigma_x \ \text{ and } \ \frac{\partial \sigma_x}{\partial y} \ \text{ are known on } \Gamma_0. \tag{18}$$

Thus $\sigma_x + \sigma_y$ is seen as solution of a Cauchy problem on Ω. As is well-known, the problem admits at most one solution. It is also known that the problem is ill-posed. Since by (11), $\sigma_x + \sigma_y$ is harmonic on Ω, it is analytic on Γ_0. Then if $z_n = (x_n, k), n = 1, 2, ...$ is a sequence of points of Γ_0 with $x_i \neq x_j$ for $i \neq j$ and accumulating at a point interior to Γ_0, then $\sigma_x + \sigma_y$ is uniquely determined by its values on (z_n). Hence the Cauchy problem for the Laplace equation can be formulated as a moment problem, it has been regularized by various methods (cf, e.g., [3, Chapt. 6], and [4]).

Step 2. We introduce the Airy stress function satisfying

$$\Delta\phi = \sigma_x + \sigma_y = f \tag{19}$$

where $\sigma_x + \sigma_y$ was constructed as the solution of a Cauchy problem for the Laplace equation as in step 1.

We define

$$\overline{\sigma}_x = \frac{\partial^2\phi}{\partial y^2}, \quad \overline{\sigma}_y = \frac{\partial^2\phi}{\partial x^2}, \quad \overline{\tau}_{xy} = -\frac{\partial^2\phi}{\partial x\partial y}. \tag{20}$$

Then $\overline{\sigma}_x, \overline{\sigma}_y$ and $\overline{\tau}_{xy}$ satisfy the equations of equilibrium (1)–(2) and the compatibility equation (11). Hence $\overline{\sigma}_x = \sigma_x, \overline{\sigma}_y = \sigma_y$, and $\overline{\tau}_{xy} = \tau_{xy}$. Note that from (19)

$$\phi = \frac{1}{4\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(\xi,\eta)\ell n\big[(x-\xi)^2 + (y-\eta)^2\big]d\xi\eta \tag{21}$$

where $f(\xi,\eta)$ is taken to be 0 in the complement of Ω.

To conclude, it should be noted that the approach we followed in the "contruction" of the stress field satisfying (1) - (6), involves a series of ill-posed problems, namely, differentiation and solution of a Cauchy problem for the Laplace equation, and hence requires regularization at various steps. The actual calculations which are rather involved will be published elsewhere.

References

1. D. D. Ang, M. Ikehata, D. D. Trong, and M. Yamamoto, Unique continuation for a stationary isotropic lame system with variable coefficients, *Communications in Partial Diff. Eqs.* **23** (1998) 371–385.
2. D. D. Ang, D. D. Trong, and M. Yamamoto, A Cauchy like problem in plane elasticity: Regularization by quasireversibility with error estimates, *Vietnam J. Math.* **23:SI** (2004) 9–17.
3. D. D. Ang, R. Gorenflo, V. K. Le, and D. D. Trong, *Moment Theory and some Inverse Problems in Potential Theory and Heat Conduction,* Lecture Notes in Mathematics, Vol. 1792, Springer-Verlag, Berlin - Heidelberg - Berlin, 2002.
4. J. Cheng, Y. C. Hon, J. Wei, and M. Yamamoto, Numerical computation of a Cauchy problem for Laplace's equation, *Angew. Math. Mech.* **81** (2001) 665–674.
5. S. Timoshenko and J. N. Goodier, *Theory of Elasticity*, Mc Graw Hill, 1951.