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A Cauchy Problem for Elliptic Equations: Quasi-Reversibility and Error Estimates*

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Abstract. In this paper, we consider a Cauchy problem for an elliptic equation in a plane domain. The problem is ill-posed. Using the method of quasi-reversibility, an approximation to the exact solution is given. Using Carleman's inequatily, we derive a sharp error estimate.

1. Introduction

Let Ω be a bounded domain in R^2 and let Γ be an open subset of $\partial\Omega$. We consider the problem of finding a function $u = u(x,y)$ satifying the following equation:

$$Au = \sum_{i,j=1}^{2} D_i(a_{ij}D_j u) + b_1 D_1 u + b_2 D_2 u + a_0 u = f,\ \forall (x,y) \in \Omega, \tag{1}$$

subject to the conditions

$$u|_\Gamma = g,\ \frac{\partial u}{\partial \nu_A}\Big|_\Gamma \equiv \sum_{i,j=1}^{2} a_{ij} D_j u \nu_i |_\Gamma = h, \tag{2}$$

where $\nu = (\nu_1, \nu_2)$: unit outer normal vector to $\partial\Omega$ and $D_1 = \partial/\partial x,\ D_2 = \partial/\partial y$.

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As is well-known, this is an ill-posed problem. In Lattès–Lions' book [2, Chap. 4], a special form of (1) - (2) is regularized by the method of quasi-reversibility (QR method for short). Solutions of (1) - (2) are approximated by a family (u_ϵ), $\epsilon \downarrow 0$, of solutions of well-posed problems. Henceforth, we call u_ϵ the quasi-reversibility solution (QR for short). In [2] it is proved that under the assumption of existence of a solution u, the family (u_ϵ) converges to u as $\epsilon \downarrow 0$ but the case of non existence of a solution is not discussed. It should be noted however that exact solutions usually do not exist. In fact, the set of boundary data (g, h) for which the system (1) - (2) has no solution is dense in $L^2(\Gamma) \times L^2(\Gamma)$. Indeed, if (1) - (2) has a solution u in $H^2(\Omega)$, say, then g is in $H^{3/2}(\Gamma)$. Thus if g, h are step functions, then (1) - (2) has no solution in $H^2(\Omega)$ (which is a natural solution space). Now, in practice g, h are results of experimental measurements and thus are given as finite sets of points. It is realistic therefore not to assume existence of solutions, and this is the approach of Klibanov and Santosa [1]. However, in the latter work, the given Cauchy data are assumed to be highly smooth (as traces of Sobolev functions of higher order), whereas, as pointed out above, in practice, available data for the exact Cauchy data are just finite sets of points that are patched up into L^2-functions. In the present paper, we take the available data as L^2-functions. For our construction of QR solutions, we regularize the given data into smooth functions. Thus our QR approach departs from the usual ones in that not only the equation is perturbed but also the given data are regularized. We shall regularize the problem as it is given, without any existence assumption, and shall derive estimates of the error between the regularized solution and an exact solution corresponding to an exact right hand side.

The remainder of the paper consists of two sections. Sec. 2 gives some preliminary results of the Carleman-type estimate. The results are prepared to the proof of the main theorem (Sec. 3). However, they are also of independent interest. The final Sec. 3 is devoted to a construction by quasi-reversibility of regularized solutions and a derivation of error estimates. For the shortness of the paper, all of proofs of the results in the paper will be omitted and in a forthcoming paper, we will give the proofs.

2. An Inequality of the Carleman Type and Construction of Boundary Functions

This section is devoted to derivation of an inequality of the Carleman type and construction of the boundary functions. Both results are fundamental for the development of Sec. 3 which deals with regularization by quasi-reversibility (QR regularization for short).

We first construct the boundary functions. These functions will be used to prove the existence of QR solutions (i.e. solutions regularized by quasi-reversibility), and, in particular, to establish error estimates between the exact solution and a QR solution.

For simplicity, we assume that Ω is a subset of the rectangle

$$Q = \{(x, y) : 0 < x < \pi, 0 < y < T\}.$$

We denoted by Γ the following subset of $\partial\Omega$:

$$\Gamma = \{(x, 0) : \alpha < x < \pi,\ \ 0 < y < T\}.$$

Obviously, a general domain Ω (with Γ smooth enough) can be transformed into the above form by an appropriate map.

For $(\phi, \psi) \in (L^2(\Gamma))^2$, we shall construct a function $\Phi(\phi, \psi)$ such that $\Phi(\phi, \psi) \in C^2(Q)$ and that under an appropriate condition,

$$\Phi(\phi, \psi)|_{\Gamma_0} = \phi, \quad D_2\Phi(\phi, \psi)|_{\Gamma_0} = \psi,$$

where

$$\Gamma_0 = \{(x, 0) : \alpha' < x < \beta'\}$$

with $0 < \alpha < \alpha' < \beta' < \beta < \pi$.

Since $\overline{\Gamma}_0 \subset \Gamma \subset (0, \pi) \times \{0\}$, we can find a function $\chi \in C^2(R)$ satisfying

$$\begin{aligned} \chi(x) &= 1, \quad x \in [\alpha', \beta'], \\ &= 0, \quad x \notin (\alpha, \beta). \end{aligned}$$

For each $(g, h) \in L^2(\Gamma) \times L^2(\Gamma)$, one has the following Fourier expansion:

$$\chi g(x) = \sum_{n=0}^{\infty} a_n \sin nx, \quad a_n = \frac{1}{\pi} \int_0^{\pi} \chi g(x) \sin nx dx,$$

$$\chi h(x) = \sum_{n=0}^{\infty} b_n \sin nx, \quad b_n = \frac{1}{\pi} \int_0^{\pi} \chi h(x) \sin nx dx.$$

For $\delta > 0$, we put

$$g_\delta(x) = \sum_{n=0}^{\infty} \frac{a_n}{1 + \delta n^{3/2}} \sin nx,\ \ h_\delta(x) = \sum_{n=0}^{\infty} \frac{b_n}{1 + \delta n^{1/2}} \sin nx.$$

Now, for each $(\phi_0, \psi_0) \in L^2(0, \pi) \times L^2(0, \pi)$ we put

$$\Phi(\phi_0, \psi_0)(x, y) = \sum_{n=0}^{\infty} \big\{\alpha_n(1 + \sin ny) + \beta_n \sin ny/n\big\} e^{-ny} \sin nx,$$

where

$$\alpha_n = \frac{1}{\pi} \int_0^{\pi} \phi_0(x) \sin nx dx, \quad \beta_n = \frac{1}{\pi} \int_0^{\pi} \psi_0(x) \sin nx dx.$$

Then we have the following proposition:

Proposition 2.1. *The operator Φ has the following properties:*

(i) $\Phi(\phi_0, \psi_0) \in C^2(Q)$ *for every* $(\phi_0, \psi_0) \in (L^2(0, \pi))^2$.

Moreover, if

$$\sum_{n=0}^{\infty}(n^3|\alpha_n|^2+n|\beta_n|^2)<\infty,$$

then $\Phi(\phi_0,\psi_0)\in H^2(Q)$ *and*

$$\Phi(\phi_0,\psi_0)|_{(0,\pi)\times\{0\}}=\phi_0,\quad D_2\Phi(\phi_0,\psi_0)\big|_{(0,\pi)\times\{0\}}=\psi_0,$$

and there is a constant $C>0$ *independent of* ϕ_0,ψ_0 *such that*

$$\left\|\Phi(\phi_0,\psi_0)\right\|^2_{H^2(Q)}\le C\sum_{n=0}^{\infty}(n^3|\alpha_n|^2+n|\beta_n|^2).$$

(ii) *If*

$$(g_0,h_0)\in H^{3/2+s}(\Gamma)\times H^{1/2+s}(\Gamma)\quad for\ some\quad 0\le s<1/2,$$

then there is a constant C *independent of* g_0,h_0 *such that*

$$\sum_{n=0}^{\infty}\left(n^{3+2s}|a_{0n}|^2+n^{1+2s}|b_{0n}|^2\right)\le C\left(\|g_0\|^2_{H^{3/2+s}(\Gamma)}+\|h_0\|^2_{H^{1/2+s}(\Gamma)}\right)\tag{3}$$

where

$$a_{0n}=\frac{1}{\pi}\int_0^{\pi}\chi g_0(x)\sin nxdx,\ \ b_{0n}=\frac{1}{\pi}\int_0^{\pi}\chi h_0(x)\sin nxdx.$$

Moreover $\Phi(\chi g_0,\chi h_0)\in H^2(Q)$ *and*

$$\Phi(\chi g_0,\chi h_0)|_{\Gamma_0}=g_0,\ \ D_2\Phi(\chi g_0,\chi h_0)|_{\Gamma_0}=h_0.$$

(iii) *If* $(g,h)\in L^2(\Gamma)\times L^2(\Gamma)$, *then* $\Phi(g_\delta,h_\delta)$ *is in* $H^2(Q)$ *for every* $\delta>0$. *For* $\epsilon>0$ *and* $(g_0,h_0)\in H^{3/2}(\Gamma)\times H^{1/2}(\Gamma)$, *we put*

$$\eta(\epsilon)=$$
$$\left\{\epsilon+\sqrt{\epsilon}\left(\|g_0\|^2_{H^{3/2}(\Gamma)}+\|h_0\|^2_{H^{1/2}(\Gamma)}\right)+\sum_{n=[\epsilon^{-1/6}]+1}^{\infty}\left(n^3|a_{0n}|^2+n|b_{0n}|^2\right)\right\}^{1/2}.$$

Then

$$\lim_{\epsilon\downarrow 0}\eta(\epsilon)=0,$$

and if

$$\|g_0-g\|_{L^2(\Gamma)}+\|h_0-h\|_{L^2(\Gamma)}<\epsilon,\tag{4}$$

then there is a constant C *independent of* g_0,h_0,g,h,ϵ *such that*

$$\left\|\Phi(g_{\sqrt{\epsilon}},h_{\sqrt{\epsilon}})-\Phi(\chi g_0,\chi h_0)\right\|_{H^2(Q)}<C\eta(\epsilon).$$

(iv) *Let* $(g_0, h_0) \in H^{3/2+s}(\Gamma) \times H^{1/2+s}(\Gamma)$, $0 < s < 1/2$. *If* (4) *holds then there is a constant* C *independent of* g_0, h_0, g, h, ϵ *such that*

$$\|\Phi(g_{\sqrt{\varepsilon}}, h_{\sqrt{\varepsilon}}) - \Phi(\chi g_0, \chi h_0)\|_{H^2(Q)} \le \epsilon^{\frac{s}{6}}\big(\|g_0\|_{H^{3/2+s}(\Gamma)} + \|h_0\|_{H^{1/2+s}(\Gamma)} + 1\big).$$

Now, we turn to the derivation of an inequality of the Carleman type (cf. [3]). Consider an elliptic operator

$$Lv = \gamma_2 v_{tt} + \gamma_1 v_{zz} + \beta_1 v_z + \beta_2 v_t + \gamma v, \ \ \forall (z,t) \in D,$$

where D is a domain such that ∂D is smooth and $D \subset (a,b) \times (0,1)$, $a < b$, and $\gamma_1, \gamma_2 \in C^1(\bar{D}), \beta_1, \beta_2, \gamma \in C(\bar{D})$ and

$$\gamma_i(z,t) \ge \alpha_0 > 0 \ \ \forall (z,t) \in D, \ \ i = 1,2.$$

Then we have the following proposition:

Proposition 2.2. *Let* $v \in H^2(D)$,

$$v|_{\partial D} = v_z|_{\partial D} = v_t|_{\partial D} = 0.$$

Then there exist $C, \lambda_0 > 0$ *depending on* $\|\gamma_i\|_{C^1(\bar{D})}$, $\|\beta_i\|_{C(\bar{D})}$ $(i = 1,2)$ *and* $\|\gamma\|_{C(\bar{D})}$ *such that*

$$\lambda^3 \int\limits_D v^2 e^{2\lambda(t+1/2)^{-m}} dzdt + \lambda \int\limits_D (v_z^2 + v_t^2) e^{2\lambda(t+1/2)^{-m}} dzdt$$
$$\le C \int\limits_D |Lv|^2 e^{2\lambda(t+1/2)^{-m}} dzdt, \quad \forall \lambda \ge \lambda_0,$$

where

$$a_1 = \gamma_1\gamma_2^{-1}, \quad \tilde{\alpha}_0 = \inf_{(z,t)\in\bar{D}} a_1(z,t),$$
$$m = \max\big\{1, 3 \sup_{(z,t)\in D} |a_{1t}(z,t)| \tilde{\alpha_0}^{-1} - 2\big\}.$$

3. Regularization by Quasi-Reversibility

Consider the following Cauchy problem for the elliptic equation $Au = f$, where

$$Au = \sum_{i,j=1}^{2} D_i(a_{ij} D_j u) + b_1 D_1 u + b_2 D_2 u + cu \quad \text{on} \quad \Omega \subset R^2 \tag{5}$$

with the Cauchy data

$$u|_\Gamma = g, \quad \frac{\partial u}{\partial \nu_A}\Big|_\Gamma = h.$$

As pointed out in Introduction, this is an ill-posed problem, that is, solutions do not always exist and even in the case of existence, there is no continuous dependence on given data. The purpose of this section is to regularize the problem, that is, to construct a family of approximate solutions depending continuously on given data. The strategy is as follows. We let u_0 be the solution of the equation with the Cauchy data (g_0, h_0) such that (g, h) is an appoximation to (g_0, h_0) with a known degree of accuracy (the data (g_0, h_0) may not be known). We regularize the problem by a variant of the method of quasi- reversibility, where both the equation and the "measured" data (g, h) are perturbed. The regularized solution will be denoted by $(u_\epsilon)_{\epsilon>0}$ and an estimate of the error $\|u_0 - u_\epsilon\|$ will be derived.

Before proceeding to "construct" the regularized solution we list below our standing assumptions:
(i) $a_{ij} \in C_1(\overline{\Omega}), b_i, c \in C(\overline{\Omega}), \quad i, j = 1, 2,$
(ii) there is a $C_0 > 0$ such that

$$a_{11}(x,y)\xi_1^2 + (a_{12}(x,y) + a_{21}(x,y))\xi_1\xi_2 + a_{22}(x,y)\xi_2^2 \geq C_0(\xi_1^2 + \xi_2^2)$$

for every $(\xi_1, \xi_2) \in R^2$ and $(x, y \in \overline{\Omega})$.

The main results of this section are Theorems 3.3 and 3.4. To prove these theorems, we need Lemmas 3.1 and 3.2 below.

Lemma 3.1. *Assume that Ω is simply connected, bounded by a Jordan curve. Let*

$$\Gamma_0 = \{(x, 0) : \alpha' < x < \beta'\},$$
$$\Gamma_1 = \partial\Omega \setminus \Gamma_0.$$

Then there is a homeomorphism $\Phi_1 : \bar{\Omega} \to \Phi_1(\bar{\Omega}) \subset R^2$ such that

$$\Phi_1(\Gamma_1) \subset \{(z, 1) : z \in R\}, \tag{6}$$
$$\Phi_1(\bar{\Omega}) \subset \{(z, t) : 0 \leq z \leq 1, \ t \leq 1\}, \tag{7}$$

and that there exist functions

$$\gamma_1, \gamma_2 \in C^1(\Phi_1(\overline{\Omega})), \quad \beta_1, \beta_2, \gamma \in C(\Phi_1(\overline{\Omega})),$$

satisfying

$$Au = \gamma_2 v_{tt} + \gamma_1 v_{zz} + \beta_1 v_z + \beta_2 v_t + \gamma v, \quad (z, t) \in \Phi_1(\Omega) \tag{8}$$

for all $u \in H^2(\Omega)$, where

$$v = u \circ \Phi_1^{-1}.$$

We now propose to give an approximation of $u_y|_{\Gamma_0}$. From the second equality in (2), we get

$$a_{21}D_1 u + a_{22}D_2 u = -h \ \text{ on } \Gamma,$$

which gives in view of (2):

$$u_y(x, 0) = -a_{22}^{-1}(x, 0)\{h(x) + g'(x)a_{21}(x, 0)\}. \tag{9}$$

If g is not in $H^1(\alpha, \beta)$, Equality (9) does not hold. Hence, we shall use an approximation of the exact data (g_0, h_0)

$$u_y(x, 0) = -a_{22}^{-1}(x, 0)\{h_0(x) + g_0'(x)a_{21}(x, 0)\}. \tag{10}$$

Let $\varphi \in C_c^\infty(R)$ satisfy

$$\varphi \geq 0, \quad \int_R \varphi(x)dx = 1, \quad \operatorname{supp}\varphi \subset (-1, 1).$$

Put

$$\begin{aligned} \varphi_\delta &= \delta^{-1}\varphi(x\delta^{-1}) \ ; \ \ \forall \delta > 0, \\ \tilde{h_0}(x) &= \ - a_{22}^{-1}(x, 0)\{h_0(x) + g_0'(x)a_{21}(x, 0)\}. \end{aligned} \tag{11}$$

Choose α", β" such that

$$0 < \alpha < \alpha" < \alpha' < \beta' < \beta" < \beta < \pi.$$

Let $\chi_1 \in C_c^\infty(R)$ be a function satisfying

$$\begin{aligned} \chi_1(x) &= 1, \quad x \in [\alpha", \beta"], \\ &= 0, \quad x \notin (\alpha, \beta). \end{aligned}$$

Put

$$\tilde{h}(x) = -a_{22}^{-1}(x, 0)\{h(x) + (\varphi_{\sqrt{\epsilon}} * (\chi_1 g))'(x)a_{21}(x, 0). \tag{12}$$

Then we have

Lemma 3.2. *Let $\epsilon > 0$, and*

$$\begin{aligned} (g_0, h_0) &\in H^{3/2}(\Gamma) \times H^{1/2}(\Gamma), \\ (g, h) &\in L^2(\Gamma) \times L^2(\Gamma), \end{aligned}$$

satisfy

$$\|g - g_0\|_{L^2(\Gamma)} + \|h - h_0\|_{L^2(\Gamma)} < \epsilon,$$

then there is a positive constant C independent from ϵ such that

$$\|\tilde{h_0} - \tilde{h}\|_{L^2(\Gamma_2)} < C\sqrt{\epsilon}(1 + \|g_0\|_{H^{3/2}(\Gamma)}) \tag{13}$$

where

$$\Gamma_2 = \{(x, 0) : \alpha" < x < \beta"\}$$

and $\tilde{h_0}, \tilde{h}$ are as in (11) *and* (12).

We now construct the regularized equation. Put:

$$\Omega_\delta = \Phi_1^{-1}\{(x, y) \in \Phi_1(\Omega) : y < 1 - \delta\}.$$

Let ρ_δ be a nonnegative function such that:

$$\begin{aligned} \rho_\delta(x) &= 1, \quad (x, y) \in \Omega_\delta, \\ &= 0, \quad (x, y) \notin \Omega_{\delta/2}. \end{aligned}$$

By the definition of Ω_δ

$$\rho_\delta(x, y) = 0, \ \ \forall (x, y) \in \Gamma_1 = \partial\Omega \setminus \Gamma_0.$$

Put:

$$V_\delta = \{v : v \in L^2(\Omega),\ \rho_\delta D_j v \in L^2(\Omega),\ j = 1, 2,$$
$$\rho_\delta A v \in L^2(\Omega) \quad \text{and} \quad v|_{\Gamma_{0\delta}} = D_2 v|_{\Gamma_{0\delta}} = 0\},$$

where

$$\Gamma_{0\delta} = \bar{\Omega_\delta} \cap \Gamma_0.$$

In V_δ, we consider the norm:

$$\|v\|^2_{V_\delta} = \|v\|^2_{L^2(\Omega)} + \sum_{i=1}^{2} \|\rho_\delta D_j v\|^2_{L^2(\Omega)} + \|\rho_\delta A v\|^2_{L^2(\Omega)}.$$

It can be shown (cf. [2]) that V_δ is a Hilbert space. Then we have

Theorem 3.3. *Let $\delta > 0, \epsilon_1 > 0$, let $f_0 \in L^2(\Omega)$ and let $(g_0, h_0) \in H^{3/2}(\Gamma) \times H^{1/2}(\Gamma)$. Then the system*

$$\frac{1}{\epsilon_1^2} A^*(\rho_\delta^2 A u_{\epsilon_1}) - \sum_{i,j=1}^{2} D_i(\rho_\delta^2 a_{ij} D_j u_{\epsilon_1}) + \delta u_{\epsilon_1} = \frac{1}{\epsilon_1^2} A^*(\rho_\delta^2 f_0), \tag{14}$$

$$u_{\epsilon_1}|_{\Gamma_{0\delta}} = g_0, \quad D_2 u_{\epsilon_1}|_{\Gamma_{0\delta}} = \tilde{h_0}, \tag{15}$$

has a unique solution u_{ϵ_1} satifying $u_{\epsilon_1} - \Phi(g_0, h_0) \in V_\delta$, where Φ is the operator defined in Proposition 2.1, *$\tilde{h_0}$ is in* (11).

Now, we consider the following system:

$$\frac{1}{\epsilon_1^2} A^*(\rho_\delta^2 A u_\epsilon) - \sum_{i,j=1}^{2} D_i(\rho_\delta^2 a_{ij} D_j u_\epsilon) + \delta u_\epsilon = \frac{1}{\epsilon_1^2} A^*(\rho_\delta f), \tag{16}$$

$$u_\epsilon|_{\Gamma_{0\delta}} = g_{\sqrt{\epsilon}}, \quad D_2 u_\epsilon|_{\Gamma_{0\delta}} = \tilde{h}_{\sqrt{\epsilon}}, \tag{17}$$

where $\epsilon_1 > 0$ and the function $g_{\sqrt{\epsilon}}, \tilde{h}_{\sqrt{\epsilon}}$ are defined from $g, \tilde{h}$ as in Proposition 2.1.

We are ready to state the principal result of our paper:

Theorem 3.4. *Let $K > 0,\ 0 < s < 1/2,\ 0 < \epsilon < 1,\ \delta > 0,\ \epsilon_1 = \epsilon^{s/12}$. Suppose*

$$f, f_0 \in L^2(\Omega), \quad g, h \in L^2(\Gamma),$$
$$(g_0, h_0) \in H^{3/2+s}(\Gamma) \times H^{1/2+s}(\Gamma).$$

Then the system (16)-(17) *has a unique solution u_ϵ such that*

$$u_\epsilon - \Phi(g_{\sqrt{\epsilon}}, \tilde{h}_{\sqrt{\epsilon}}) \in V_\delta,$$

and that if $u_0 \in H^{2+s}(\Omega)$ with

$$\|u_0\|_{H^2(\Omega)} \le K$$

is the solution of (1) *subject to the Cauchy conditions*

$$u|_{\Gamma_{0\delta}} = g_0, \qquad \frac{\partial u}{\partial \nu_A}\bigg|_{\Gamma_{0\delta}} = h_0,$$

then there exists a $C > 0,\ m > 0$ *independent from* ϵ *and depending only on* K *such that:*

$$\|u_\epsilon - u_0\|_{L^2(\Omega_{3\delta})} \leq C\Big(\ln\frac{1}{\epsilon}\Big)^{-3}\epsilon^{\sigma(\delta)},$$

where

$$\sigma(\delta) = \frac{s}{3}\frac{(3/2-3\delta)^{-m} - (3/2-2\delta)^{-m}}{2^m - (3/2-2\delta)^{-m}}.$$

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