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Collisions and Fracture

Elena Bonetti[1] **and Michel Fremond**[2]

[1] *Dipartimento di Matematica "F. Casorati", Università di Pavia,*
Via Ferrata, 1, 27100, Pavia, Italy
[2] *Laboratoire Lagrange, LCPC, 58 boulevard Lefebvre,*
75732 Paris cedex 15, France

Abstract. When a plate falls on the ground, it breaks. We study this phenomenon at the macroscopic level. We restrict ourselves to 1-D problems and illustrate the theory with a chandelier to which a falling stone is tied. The collisions are assumed instantaneous. The equations of motion and constitutive laws give a set of differential equations which may be solved in SBV spaces made of smooth or special velocities with bounded variation. These velocities exhibit a countable number of fractures (a fracture is characterized by a spatial discontinuity of the velocity). The example shows how the theory applies and gives realistic results.

1. Introduction

Consider a plate falling on the floor: it breaks! Consider a rock avalanching from a mountain on a concrete protecting wall: depending on the circumstances either both the rock and the concrete wall break, or only one breaks, or none of them breaks. We address this problem at the engineering macroscopic level. We restrict ourselves to one dimensional problems (similar problems may be studied in 2-D or 3-D [4]). To present and illustrate the theory, we consider a chandelier fixed to the ceiling and have a stone fixed to a string tied to the lower part of the chandelier. We let the stone fall and study what occurs...

Collision phenomenons we consider are very short compare to the duration of the other phenomenons: thus we assume the collisions are instantaneous. A collision is characterized by a time discontinuity of the velocity field $U(x,t)$. At collision time t, the chandelier is schematised by a bar which occupies the segment $[0,l]$, the velocity field is discontinuous with respect to time: $U^-(x,t)$

is the smooth velocity field before the collision and $U^+(x,t)$ the velocity field after the collision. A fracture resulting from the collision is characterized by a spatial discontinuity of the velocity field $U^+(x,t)$. The mechanical predictive theory is based on continuum mechanics collision theory [6, 7]. The equations of motion (paragraph 2) are derived from the principle of virtual work at time t. It introduces interior percussions which account for the very large stresses and forces which result from the cinematic incompatibilities. They are interior volume percussion stresses and interior surface percussions. The latter appear at the ends of the bar together at the fracture points. The constitutive laws (paragraph 3) are derived in order to fulfill the second law of thermodynamics, namely the Clausius-Duhem inequality. For this purpose, we use the elegant and productive technique of dissipative potentials.

The system of differential equations resulting from the equations of motion and from the constitutive laws is investigated in a variational framework (paragraph 4 and 5). The variational problem has solutions (paragraph 5) which are smooth (i.e., special bounded variation functions in terms of mathematics). A special velocity with bounded variation has a countable number of fractures (in terms of mathematics a fracture is a point of discontinuity) and is smooth between these fractures. As an example of the theory, we describe what occurs either when a percussion is applied to the stone at rest (paragraph 6) or when the stone tied to the chandelier falls (paragraph 7). The results are the expected ones.

2. The Equations of Motion

The system we consider is made of the ceiling which does not move, of the bar which occupies segment $(0,l)$ and of the stone. At the time we consider this system, the string under the chandelier is tightened, i.e., the distance of the stone to the bar is equal to the length of the string, and either the stone is falling or an exterior percussion is applied to the stone.

There are three velocities describing the velocity of the whole system: $U_{ext}(0)$, the velocity of the ceiling which is in the present situation equal to 0; the velocity of the bar $U_{int}(x)$ and the velocity of the stone $U_{ext}(l)$. The velocities of the two ends of the bar are $U_{int}(0)$ and $U_{int}(l)$; in terms of mathematics they are the traces of $U_{int}(x)$ as x goes to 0 and l, respectively. The fixation device to the ceiling can break: thus the velocities $U^+_{ext}(0)$ and $U^t_{int}(0)$ may be different. It is the same for the velocities $U^+_{ext}(l)$ and $U^+_{int}(l)$ when the string breaks. We denote $V = \{V_{int}, V_{ext}(0), V_{ext}(l)\}$ a triplet of virtual velocities, U is the actual velocity triplet. The equations of motion are derived from the principle of virtual work which introduces different works (duality pairings in mathematical terms). For the sake of simplicity, we assume that there is only one fracture at point $s \in]0,l[$, even though, of course, there may be more. The virtual work of the interior percussions is defined by

$$\mathcal{T}_{int}(V^+,V^-) = -\int_{]0,s[\cup]s,l[} \Sigma \frac{d}{dx}\Big(\frac{V^+_{int}+V^-_{int}}{2}\Big)dx + R(0)\Big[\frac{V^+(0)+V^-(0)}{2}\Big] + R(s)\Big[\frac{V^+_{int}(s)+V^-_{int}(s)}{2}\Big]_s + R(l)\Big[\frac{V^+(0)+V^-(0)}{2}\Big]_s,$$

where dV_{int}/dx is the classical strain rate, V^+, V^- are virtual velocity triplets after and before the collision. The spatial velocity discontinuity, $[W]_s = W_r - W_l$, is given by the difference between the right (or down) velocity W_r and the left (or up) velocity W_l. At point 0, the left velocity is the velocity of the ceiling, $V_{ext}(0)$. At point l, the right velocity is the velocity of the stone $V_{ext}(l)$. This work defines the lineic percussion stress Σ in the bar and the point percussions R on the fracture and the two ends of the bar. The virtual work of the acceleration forces is

$$\mathcal{T}_{acc}(V^+,V^-) = \int_{]0,l[} \rho[U_{int}]\frac{V^+_{int}+V^-_{int}}{2}dx + m[U_{ext}(l)]\frac{V^+_{ext}(l)+V^-_{ext}(l)}{2},$$

where ρ is the density of the solid, m the mass of the stone with velocity U_{ext} and $[U] = (U^+ - U^-)$ is the time discontinuity. An exterior percussion P may be applied to the stone, its virtual work is

$$\mathcal{T}_{ext}(V^+,V^-) = P\frac{V^+_{ext}(l)+V^-_{ext}(l)}{2}.$$

The equations of motion result from the principle of virtual work

$$\forall V^+, V^-, \mathcal{T}_{acc}(V^+,V^-) = \mathcal{T}_{int}(V^+,V^-) + \mathcal{T}_{ext}(V^+,V^-).$$

They are

$$\begin{aligned} \rho[U] &= \frac{d\Sigma}{dx}, \quad \text{in }]0,s[\cup\, s,l[, \\ \Sigma(0) &= -R(0), \\ \big[\Sigma(s)\big]_s &= 0, \;\; \Sigma(s) = -R(s), \\ m[U_{ext}(l)] + \Sigma(l) &= P, \;\; \Sigma(l) = -R(l). \end{aligned} \tag{1}$$

3. The Constitutive Laws

The first and second laws of thermodynamics give, by a classical computation taking advantage that the state quantities are constant, the Clausius–Duhem inequalities

$$\begin{aligned} 0 &\le \Sigma\frac{d}{dx}\Big(\frac{U^+_{int}+U^-_{int}}{2}\Big), \quad \text{in }]0,s[\cup]s,l[, \\ 0 &\le -R(x)\Big[\frac{U^+(x)+U^-(x)}{2}\Big]_s, \quad \text{at } x = 0,s,l. \end{aligned} \tag{2}$$

The constitutive laws which have to satisfy the Clausius-Duhem inequalities, are defined by the lineic dissipative function Φ, and fracture dissipative functions Φ_s.

3.1. The Bar and the Ceiling Dissipative Functions

We choose the lineic dissipative function as

$$\Phi\Big(\frac{d}{dx}(U^+_{int}+U^-_{int})\Big)=k_0\Big|\frac{d}{dx}(U^+_{int}+U^-_{int}\Big|+\frac{k_1}{2}\Big|\frac{d}{dx}(U^+_{int}+U^-_{int}\Big|^2,$$

where k_0 and k_1 are positive constants. The fracture dissipative function at point $x\in[0,l)$ is chosen as

$$\Phi_x\big([U^+(x)+U^-(x)]_s\big)=2k_2(x)\sqrt{\big|\big[U^+(x)+U^-(x)\big]_s\big|}$$
$$+\,k_3(x)\big|\big[U^+(x)+U^-(x)\big]_s\big|+I_+\big(\big[U^+(x)+U^-(x)\big]_s\big),\quad \text{at } x=0,\ s,$$

where $[U(0)]_s=U_{int}(0)-U_{ext}(0)$ and $k_2(x),k_3(x),x\in[0,l)$ are positive functions. The volume dissipative function ensures a classical behaviour away from the fractures. The effect of the function $2k_2\sqrt{\big|\big[U^++U^-\big]_s\big|}$ is to avoid to have many fractures with small discontinuities. The indicator function I_+ of R^+ takes into account the impenetrability condition on the fractures at points $x=0$ and s, i.e.,

$$[U^+(x)]_s\geq 0. \tag{3}$$

3.2. The String and Stone Dissipative Function

The dissipative function Φ_l at point $x=l$ describes the behaviour of the stone and string tied to the chandelier. It is different from the dissipative functions at other points x because there is no interpenetration condition: the stone can go either upward or downward. The behaviour is completely different depending on the direction of the motion, upward or downward, after the collision due either to the fall of the stone or to an external percussion.

Let recall that at the time we consider the system, the string is tightened. Thus, the distance of the stone to the bar being equal to the length of the string, the relative velocity of the stone and the chandelier before the collision

$$\chi=\big(U^-_{ext}(l)-U^-_{int}(l)\big)$$

is non negative

$$\chi\geq 0, \tag{4}$$

and cannot be negative because the distance of the stone to the chandelier is at its maximum (the left time derivative of this distance cannot be negative! $\chi>0$ if the stone is falling, $\chi=0$ if it is not).

Let us consider the future relative velocity

$$W^+=U^+_{ext}(l)-U^+_{int}(l)=[U^+(l)+U^-(l)]_s-\chi.$$

If $W^+<0$, the string slackens, if $W^+\geq 0$, the string remains tightened or breaks. The potential Φ_l has to take into account this unilateral property. Thus the dissipative function Φ_l, which is already a function of $[U^+(l)+U^-(l)]_s$, depends also on quantity $\chi = U^-_{ext}(l) - U^-_{int}(l)$ which is obviously depending on the history of the system [6]. Thus we may choose $\Phi_l = 0$ when $W^+ < 0$, assuming no interaction between the stone and chandelier when the string slackens, and P_l similar to a fracture dissipative function when $W^+ \geq 0$, i.e., when the string remains tightened or breaks

$$\begin{gathered}\Phi_l([U^+(l) + U^-(l)]_s,\ \chi) = 2k_2(l)\sqrt{pp\{W^+\}} + k_3(l)pp\{W^+\},\\ W^+ = [U^+(l) + U^-(l)]_s - \chi,\end{gathered} \tag{5}$$

where the positive part function is defined by

$$pp\{x\} = \begin{cases} x & \text{if } x \geq 0 \\ 0 & \text{if } x \leq 0. \end{cases}$$

Let us note that $\Phi_l(0, \chi) = 0$, which is important to satisfy the Clausius–Duhem inequality.

The chandelier is homogeneous, thus

$$\forall x \in]0, l[, \quad k_2(x) = k_2^i, \quad k_3(x) = k_3^i.$$

We let

$$\begin{aligned} k_2(0) = k_2^0, \quad k_3(0) = k_3^0,\\ k_2(l) = k_2^l, \quad k_3(l) = k_3^l,\end{aligned}$$

and assume for the sake of simplicity but without loss of generality

$$k_2^0 = k_2^i = k_2^l = k_2,$$

(cf. Remark (3)).

3.3. The Constitutive Laws

Dissipative function Φ is a pseudo-potential of dissipation, i.e., a convex, positive function with value 0 at the origin [8], but Φ_x is not because it is not a convex function of $[U^+ + U^-]_s$. Nevertheless Φ_x may be split into a convex part, Φ_x^c, and a differentiable part, Φ_x^{nc}. Its generalized subdifferential set is the sum of the subdifferential set of the convex part and of the extended derivative of the non convex part. It is for $x = 0, s$

$$\begin{aligned}\overline{\partial}\Phi_x(X) &= \frac{\partial \Phi_x^{nc}}{\partial X}(X) + \partial\Phi_x^c(X)\\ &= \begin{cases} \dfrac{k_2(x)}{\sqrt{X}} + k_3(x), & \text{if } X > 0,\\ R, & \text{if } X = 0,\\ \Phi, & \text{if } X < 0,\end{cases}\end{aligned}$$

and for $x = l$

$$\begin{aligned}\overline{\partial}\Phi_l(X) &= \frac{\partial \Phi_x^{nc}}{\partial X}(X, \chi) + \partial\Phi_x^c(X, \chi)\\ &= \begin{cases} \dfrac{k_2}{\sqrt{Y}} + k_3(l), & \text{if } Y = X - \chi > 0,\\ R^+, & \text{if } Y = X - \chi = 0,\\ \Phi, & \text{if } Y = X - \chi < 0.\end{cases}\end{aligned}$$

Remark 1. The generalized subdiffertial set of $\Phi_l(X,\chi)$ with respect to X depends on χ. For the sake of simplicity, we do not mention χ and denote $\overline{\partial}\Phi_l(X)$ the generalized subdifferential set.

The constitutive laws are

$$\Sigma \in \partial\Phi\Big(\frac{d}{dx}(U_{int}^+ + U_{int}^-)\Big) = k_0 sg\Big(\frac{d}{dx}(U_{int}^+ + U_{int}^-)\Big) + k_1\frac{d}{dx}(U_{int}^+ + U_{int}^-), \quad (6)$$

$$-R(x) \in \overline{\partial}\Phi_x\big([U^+(x)+U^-(x)]_s\big), \text{ at } x=0,\ s,\ l, \quad (7)$$

where the sign graph sg is defined by

$$sg(x) = \begin{cases} 1, & \text{if } x>0,\\ [-1,1], & \text{if } x=0,\\ -1, & \text{if } x<0. \end{cases}$$

It is easy to see that they satisfy Clausius–Duhem inequalities (2) by applying the classical properties of pseudo-potentials and to prove following theorem.

Theorem 1. *We have*

$$\forall R^d \in \overline{\partial}\Phi_x(X),\ \ -R^dX \geq 0, \quad x=0,\ s,\ l.$$

4. The Equations for the Chandelier and Stone

The chandelier, schematised by the bar with length l, is at rest at collision time. Its velocity is $U^-(x)=0$, $x\in[0,l]$. The velocity of the ceiling remains $0 : U_{exl}^-(0) = U_{exl}^+(0) = 0$. In this paragraph and following, the number of fractures is no more specified, there may be many or none. We define $S(U_{int})$, the jump set, the set of the abscissas of the fractures (or the set of the discontinuity points of the velocity)

$$S(U_{int}) = \big\{s \in]0,l[| x \to U_{int}(x) \text{ is dis continuous at point } s\big\}.$$

The equations follow from equations of motion (1), constitutive laws (6), (7) and initial condition

$$U^- = \big\{U_{int}^-,\ U_{ext}^-(0),\ U_{ext}^-(l)\big\} = \{0,0,U_{ext}^-(l)\}.$$

with $U_{ext}^-(l) \geq 2$ satisfying (4), (i.e., the stone is falling or at rest). They are

$$\begin{gathered}
\rho U_{int}^+ - \frac{d\Sigma}{dx} = 0, \quad \text{a.e. in }]0,l[\backslash S(U_{int}^+),\\
[\Sigma]_s(s) = 0, \quad \text{in } S(U_{int}^+),\\
\Sigma \in \partial\Phi\Big(\frac{d}{dx}(U_{int}^+)\Big), \quad \text{a.e. in }]0,l[\backslash S(U_{int}^+),\\
\Sigma(x) \in \overline{\partial}\Phi_x\big([U^+(x)+U^-(x)]_s\big), \quad \text{in } S(U_{int}^+)\cup\{0,l\},\\
U_{ext}^+(0) = 0,\\
mU_{ext}^+(l) + \Sigma(l) = mU_{ext}^-(l) + P. \qquad (8)
\end{gathered}$$

The unknown is the triplet $U^+ = \{U^+_{int},\ U^+_{ext}(0),\ U^+_{ext}\}$ depending on the known triplet U^-.

4.1. A Variational Formulation

We look for triplet $U^+ = \{U^+_{int}, 0, U^+_{ext}(l)\}$ which minimizes a functional whose Euler equations are the previous equations. We may choose

$$\begin{aligned}\mathcal{F}(U^+_{int}, U^+_{ext}(l)) = &\int_{]0,l[\setminus(U^+_{int})} \Phi\Big(\frac{d}{dx}(U^+_{int})\Big)dx \\ + &\sum_{x\in S(U^+_{int})} \big\{\Phi_x([U^+(c)]_s)\big\} + \Phi_0\big([U^+(0)]_s\big) + \Phi_l\big([U^+(l) + U^-(l)]_s, \chi\big) \\ + &\int_0^l \frac{\rho}{2}(U^+_{int})^2 dx + \frac{m}{2}(U^+_{ext}(l))^2 - mU^+_{ext}(l)U^-_{ext}(l) - PU^+_{ext}(l).\end{aligned} \tag{9}$$

The problem we want to solve is

$$\inf\big\{\mathcal{F}(V_{int}, V_{ext}) | V_{int} \in SBV(0,l),\ V_{ext} \in R\big\}, \tag{10}$$

where $SBV(0,l)$ is the space of the velocities which may be discontinuous [2, 5, 9]. This problem is given a precise mathematical treatment below.

5. Mathematics

5.1. The Cinematically Admissible Velocities

Let the triplet $V = \{V_{int}, V_{ext}(0), V_{ext}(l)\}$ with $V_{int} \in SBV(0,l)$. We denote $\{dV_{int}/dx\}$ the Lebesgue part of the differential dV_{int} and $S(V_{int})$ the jump set, so that dV_{int} may be split as follows

$$dV_{int} = \Big\{\frac{dV_{int}}{dx}\Big\}dx + \sum_{s\in S(V_{int})} [V_{int}]_s \delta(x-s),$$

where δ is the Dirac measure. We define the space of the smooth cinematically admissible velocities

$$\begin{aligned}&S_{CV}(0,l) = \\ &\big\{V = \{V_{int}, V_{ext}(0), V_{ext}(l)\} | V_{int} \in SBV(0,l) \cap L^2(0,l),\ V_{ext}(l) \in R,\ V_{ext}(0) = 0\big\},\end{aligned}$$

equipped with the norm

$$\|V\| = \|V_{int}\|_{BV} + \|V_{int}\|_{L^2(0,l)} + |V_{ext}(l)|,$$

$$\|V_{int}\|_{BV} = \Big\|\Big\{\frac{dV_{int}}{dx}\Big\}\Big\|_{L^1(0,l)} + \sum_{s\in S(V_{int})} |[V_{int}]_s| + \|V_{int}\|_{L^1(0,l)}.$$

5.2. The Functionals

From now on, we assume that

$$k_3^i \geq \max\{k_3^0, k_3^l\}. \tag{11}$$

This assumption means that a fracture close to an end of the bar is advantageously replaced by a fracture at this end of the bar. This is a kind of continuity (or lower semi-continuity) which is not satisfied with the opposite assumption.

Remark 2. When assumption (11) is not satisfied, the strength of the string or the strength of the ceiling chandelier connecting device is larger than the strength of the chandelier material. In practice, a fracture has a tendency to appear close to the connections: think of two pieces of concrete strongly glued on one another (when breaking the fracture is close to the glued part, but it is distinct). A way to overcome the difficulty is to take into account the lengths of the string and device (for the concrete the connecting part has some thickness due to glue diffusion).

Let the triplet $U^+ = \{U_{int}, U_{ext}(0) = 0, U^+_{ext}(l)\}$. The functional (9) is

$$\begin{aligned}
&\mathcal{F}(U^+_{int}, U^+_{ext}(0) = 0, U^+_{ext}(l)) = \\
&\int_0^1 \Big\{k_0\Big|\Big\{\frac{dU^+_{int}}{dx}\Big\}\Big| + \frac{k_1}{2}\Big(\Big\{\frac{dU^+_{int}}{dx}\Big\}\Big)^2\Big\}dx \\
&\qquad + 2k_2\sqrt{|U^+_{int}(0)|} + k_3^0|U^+_{int}(0)| + I_+(U^+_{int}(0)) \\
&\qquad + \sum_{s\in S(U^+_{int})} \Big\{2k_2\sqrt{\big|[U^+_{int}(s)]_s\big|} + k_3^i\big|[U^+_{int}(s)]_s\big| + I_+\big(\big[U^+_{int}(s)\big]_s\big)\Big\} \\
&\quad + 2k_2\sqrt{pp\big\{U^+_{ext}(l) + U^-_{ext}(l) - U^+_{int}(l)\big\}} + k_3^l pp\big\{U^+_{ext}(l) + U^-_{ext}(l) - U^+_{int}(l)\big\} \\
&\quad + \int_0^l \frac{\rho}{2}(U^+_{int})^2 dx + \frac{m}{2}(U^+_{ext}(l))^2 - mU^-_{ext}(l)U^+_{ext}(l) - PU^+_{ext}(l),
\end{aligned} \tag{12}$$

where $S(U^+_{int})$ is the jump set of U^+_{int} which, of course, may contain many unknowns jump points (in fact a countable quantity).

Let $V = (V_{int}, V_{ext}(0) = 0, V_{ext}(l))$ and define

$$\mathcal{F}_1(V) = \int_0^1 \left\{ k_0 \left| \left\{ \frac{dV_{int}}{dx} \right\} \right| + \frac{k_1}{2} \left(\left\{ \frac{dV_{int}}{dx} \right\} \right)^2 \right\} dx$$

$$+2k_2 \left\{ \sqrt{pp\{V_{int}(0)\}} + \sum_{s \in S(V_{int}^+)} \sqrt{pp\{[V_{int}(s)]_s\}} + \sqrt{pp\{V_{ext}(l) + U_{ext}^-(l) - V_{int}(l)\}} \right\}$$

$$+ k_3^i \left\{ pp\{V_{int}(0)\} + \sum_{s \in S(V_{int}^+)} pp\{ \left[V_{int}(s) \right]_s \} + pp\{ V_{ext}(l) + U_{ext}^-(l) - V_{int}(l) \} \right\},$$

$$\mathcal{F}_2(V) = \int_0^2 \frac{\rho}{2} (V_{int})^2 dx,$$

$$\mathcal{F}_3(V) = \frac{m}{2} (V_{ext}(l))^2 - mU_{ext}^-(l) V_{ext}(l) - PV_{ext}(l).$$

$$\mathcal{F}_3(V) = (k_3^0 - k_3^i) pp\{V_{int}(0)\} + (k_3^l - k_3^i) pp\{V_{ext}(l) + U_{ext}^-(l) - V_{int}(l)\}.$$

5.3. The Impenetrability Condition

In $SBV(0,l)$ a trace operator may be defined at points 0 and l but it is not weakly continuous if SBV is endowed with the weak topology [1,2]. To overcome this difficulty, we treat the traces as jump points in agrement with mechanics which consider the fixation device and the stone as parts of the system. We let $a > 0$ and define the measure $\mathcal{A}(V)$ on $(-a,l)$ for $V \in S_{CV}(0,l)$ by

$$\forall \varphi \in C_c^0(-a,l),$$
$$\langle \mathcal{A}(V), \varphi \rangle = \sum_{s \in S(V_{int}^+)} \varphi(s)[V_{int}(s)]_s + \varphi(0)(V_{int}(0) - V_{ext}(0)).$$

The impenetrability conditions are formally equivalent to

$$\forall \varphi \in C_c^0(-a,l), \;\; \varphi \geq 0,$$
$$\langle \mathcal{A}(U^+), \varphi \rangle \geq 0,$$

or

$$\mathcal{A}(U^+) \geq 0, \;\text{ in }\; \mathcal{M}(-a,l),$$

where $\mathcal{M}(-a,l)$ denotes the set of Radon measures on $(-a,l)$. Let us denote

$$K = \{V \in S_{CV}(0,l) | \mathcal{A}(V) \geq 0 \;\;\text{ in }\;\; \mathcal{M}(-a,l)\}.$$

It turns out that this weak formulation of the impenetrability condition has good properties in $SEV(-a,l)$.

5.4. Existence of Solutions

Let us note that

$$\sqrt{|X|} + I_+(X) = \sqrt{pp\{X\}} + I_+(X).$$

It results that we can provide a weak formulation of (10) letting (12) rewritten as follows

$$\mathcal{F}(U) = \sum_{i=1}^{4} \mathcal{F}_i(U), \quad \text{if } \mathcal{A}(U^+) \geq 0,$$

$$\mathcal{F}(U) = +\infty, \quad \text{if } \mathcal{A}(U^+) \ngeq 0.$$

Thus the problem we want to solve is

$$\inf \big\{ \mathcal{F}(V) = \sum_{i=1}^{4} \mathcal{F}_i(V) | V \in S_{CV}(0,l), V \in K, \; V_{ext}(0) = 0 \big\}. \tag{13}$$

We prove that (13) has solutions:

Theorem 2. *If assumption* (11) *is satisfied, then problem* (13) *has solutions.*

The complete proofs are given in [3]. The direct method of calculus of variations are applied. The main points are to prove the weak lower semicontinuity of functionals $\mathcal{F}_i(V)$ and to prove that K is weakly closed in S_{CV}.

Remark 3. We may also assume that the coefficient k_2^0 and k_2^l are different from k_2^i but in order to keep the existence theorem, they have to satisfy

$$k_2^i \geq \max\{k_2^0, k_2^l\},$$

which has the same mechanical meaning than (11).

6. A Numerical Example: a Percussion is Applied to the Stone

We investigated the case where $U_{ext}^-(0) = 0$ and $P \neq 0$. We assume there is no fracture inside the bar (it is proved below that if P is not too large, it is possible to have solutions without fractures inside). We look for a solution such that $dU_{int}^+/dx = 0$. Inside the chandelier $]0, l[$ the velocity is constant and positive to satisfy the non-interpenetration condition at point $x = 0 : U_{int}^+(0) = U_r^+(0) = U \geq 0$. At that point, the left velocity, the ceiling velocity or exterior velocity is zero, $U_l^+(0) = U_l^+(0) = U_{ext}^+(0) = 0$. At point $x = l$, the right velocity which is the velocity of the stone, is $U_r^+(l) = U_{ext}^+(l) = V$. The equations are

$$\rho U^+ = \frac{d\Sigma}{dx}, \;\; |\Sigma| \leq k_0, \;\; U^+(x) = U \geq 0, \;\; \text{on }]0, l[,$$

$$\Sigma(0) = \frac{k_2}{\sqrt{U}} + k_3^0 + \partial I_+(U),$$

$$mV + \Sigma(l) = p,$$

$$\Sigma(l) = \frac{k_2}{\sqrt{V-U}} + k_3^l, \;\; \text{if } (V-U) > 0,$$

$$\Sigma(l) \geq 0, \text{ if } (V-U)=0,$$

$$\Sigma(l) = 0, \text{ if } (V-U)<0.$$

To have $|\Sigma| \leq k_0$, we need that

$$\Sigma(l) \leq k_0,$$

because $0 \leq \Sigma(0) \leq \Sigma(l)$. When $V-U>0$ and $U>0$, we have

$$0 \leq \Sigma(0) \leq \Sigma(l) = \frac{k_2}{\sqrt{V-U}} + k_3',$$

and

$$\frac{\partial \mathcal{F}}{\partial V} = \frac{k_2}{\sqrt{V-U}} + k_3^l + mV - P = 0.$$

Thus the condition is

$$P - mV \leq k_0. \tag{14}$$

When $V-U=0$ and $U>0$, we need

$$\Sigma(l) = \frac{k_2}{\sqrt{U}} + k_3^l + \rho l U \leq k_0,$$

but

$$\frac{\partial \mathcal{F}}{\partial U} = \frac{k_2}{\sqrt{U}} + k_3^0 + \rho l U + mU - P = 0,$$

thus the condition is

$$P - mU \leq k_0. \tag{15}$$

When $V-U=0$ and $U=0$, we need

$$\Sigma(x) = P \leq k_0. \tag{16}$$

When $V-U<0$ and $U=0$, we have $\Sigma(x)=0$ and there is no condition. In the sequel, we assume that k_0 is large enough for $P \leq k_0$, thus conditions (14), (15) and (16) are satisfied. When one of those conditions is not satisfied, the assumption that $dU_{int}^+/dx = 0$ or that there is only one fracture is no more satisfied.

The functional to minimize becomes

$$\mathcal{F}(U,V) = 2k_2\sqrt{U} + k_3^0 U + 2k_3\sqrt{pp\{V-U\}} + k_3^l pp\{V-U\} + \frac{\rho l}{2}U^2 + \frac{m}{2}V^2 - PV + I_+(U).$$

The behavior depends on the relative strength of the string and fixation to the ceiling device. We investigate the two situations: first the string is the solidest, then the fixation to the ceiling is the solidest.

6.1. The Chandelier is not Well Fixed to the Ceiling and the String is Solid

Let us assume that the strength of the string is larger than the strength of the fixation device of the chandelier to the ceiling. We quantify this difference by having $k_3^l \gg k_3^0$. We choose

$$k_2 = 1,\ \ m = 1,\ \ \rho l = 1,\ \ k_3^0 = 1,\ \ k_3^l = 10^6.$$

Remark 4. The assumption (11) is satisfied by choosing k_3^i large enough.

Let define

$$Y = V - U.$$

Function $\mathcal{F}(U,V)$ becomes

$$\mathcal{F}(U, Y+U) = \mathcal{G}(U,Y)$$
$$= 2\big(\sqrt{U} + \sqrt{pp\{Y\}}\big) + U + 10^6 pp\{Y\} + \frac{U^2}{2} + \frac{(U+Y)^2}{2} - P(U+Y) + I_+(U).$$

We solve

$$\inf\big\{\mathcal{G}(U,Y)\,|\,U \in R,\ Y \in R\big\},$$

and discuss the results depending on the values of percussion P. There are four cases:

6.1.1. Percussion P is Large and Positive

In this case the absolute minimum occurs for U and Y large. For $P = 3\times 10^6$, $\mathcal{G}$ is minimum for $U \simeq Y = 10^6$. The Figs. 1, 2 show the values of 9 in the neighborhood of this point.

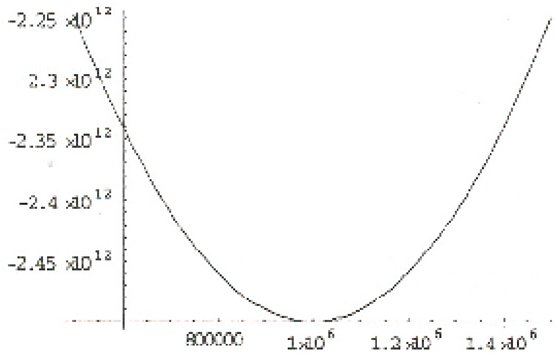

Fig. 1. $P = 3\times 10^6$ variation of $\mathcal{G}$ with respect to U for $Y = 10^6$.

This absolute minimum at $U > Y > 0$ is to be compared with the relative minimum which exists at $Y = 0$ and $U > 0$. The value of $\mathcal{G}$ at this relative minimum increases when P decreases.

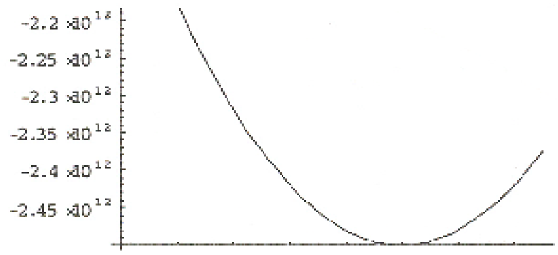

Fig. 2. $P = 3 \times 10^6$ variation of $\mathcal{G}$ with respect to Y for $U = 10^6$.

6.1.2. Percussion P is Medium and Positive

The absolute minimum occurs at $Y = 0$ when P has sufficiently decreased. For instance, for $P = 100$, the absolute minimum is at point $U = 49.4$, $Y = 0$. Figs. 3, 4 show the values of g in the neighborhood of this point.

This absolute minimum is to be compared with the relative minimum at the origin, $U = Y = 0$.

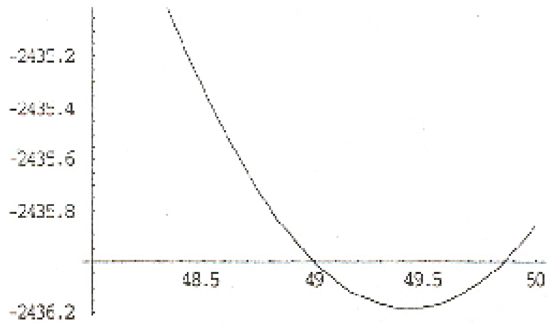

Fig. 3. $P = 3 \times 100$ variation of $\mathcal{G}$ with respect to U for $Y = 0$.

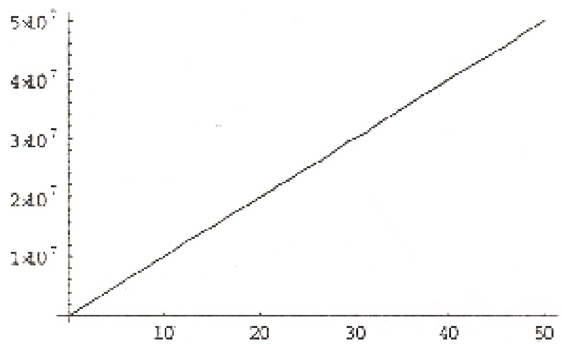

Fig. 4. $P = 3 \times 100$ variation of $\mathcal{G}$ with respect to Y for $U = 49.4$.

6.1.3. Percussion P is Small and Positive

When P decreases, the absolute minimum occurs at $U = Y = 0$. This switch occurs for $P = 4$. For this value there are two absolute minima at $U = Y = 0$ and $U = 1, Y = 0$. For $0 < P < 4$, the absolute minimum occurs at $U = Y = 0$. Figs. 5 to 9 illustrate this property.

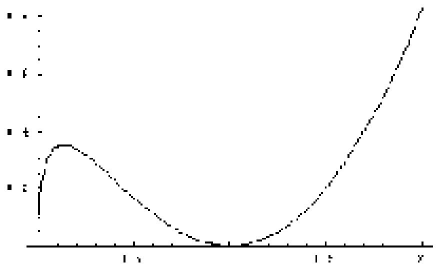

Fig. 5. $P = 4$ variation with respect to U for $Y = 0$.
There are two absolute minimum at $U = 1$ and $U = 0$.

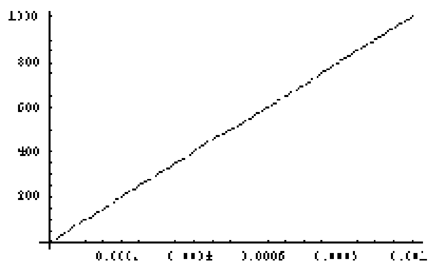

Fig. 6. $P = 4$ variation with respect to $U = 0$

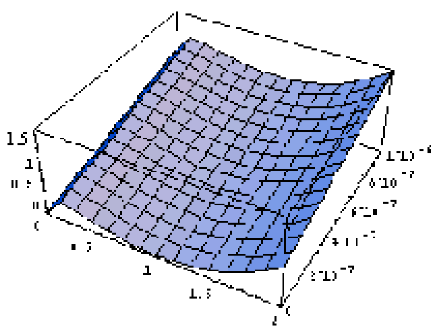

Fig. 7. $P = 4$ variation with respect to U and Y

6.1.4. Percussion P is Negative

The minimum of $\mathcal{G}(U, Y)$ is obtained for $U = 0$ and $Y < 0$. The stress $\Sigma(x) = 0$

and nothing occurs to the chandelier.

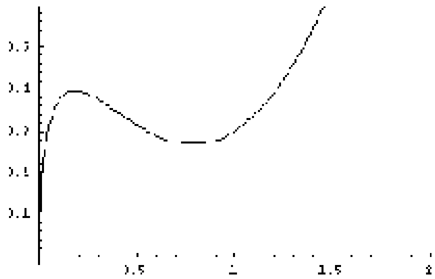

Fig. 8. $P = 3.7$ variation of $\mathcal{G}$ with respect U for $Y = 0$

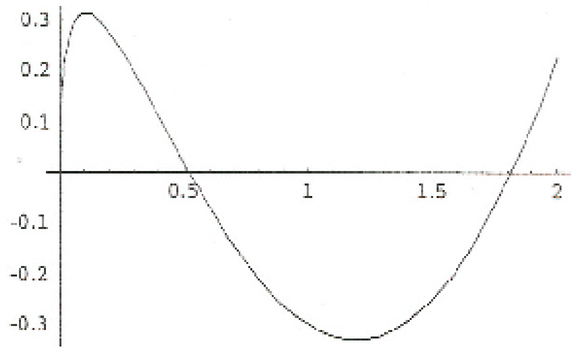

Fig. 9. $P = 4.3$ variation of $\mathcal{G}$ with respect U for $Y = 0$.
The absolute minimum is at $U = 1.2$.

The motion of the stone is given by minimizing

$$\mathcal{G}(0, Y) = \frac{Y^2}{2} - PY,$$

or by the last equation (8)

$$mU^+_{ext}(l) + \Sigma(l) = mU^-_{ext}(l) + P,$$

which is

$$mU^+_{ext}(l) = P.$$

The stone blown by percussion P goes upward without interfering with the chandelier in agreement with experiments.

Remark 5. It is possible to have a more sophisticated constitutive law for the evolution of the stone when $W^+ < 0$. Our choice is the simplest: $\Phi_l = 0$! For instance, we may choose

$$\Phi_l\big([U^+(l)+U^-(l)]_s,\chi\big)$$
$$=\begin{cases} 0, & \text{if } [U^+(l)+U^-(l)]_s \le 0,\\ \dfrac{k}{2}\big([U^+(l)+U^-(l)]_s\big)^2, & \text{if } 0 \le [U^+(l)+U^-(l)]_s \le \chi,\\ 2k_2(l)\sqrt{W^+}+k_3(l)W^+ + \dfrac{k}{2}\chi^2, & \text{if } \chi \le [U^+(l)+U^-(l)]_s, \end{cases}$$
$$W^+ = [U^+(l)+U^-(l)]_s - \chi.$$

There is an interaction between the stone and chandelier if W^+ is slightly negative. The stone can bounce when falling without breaking the chandelier fixation device and string.

6.2. The Chandelier is Well Fixed to the Ceiling and the String is Weak

In this case, we have k_3^0 large and k_3^l small: $k_3^0 \gg k_3^l$. For instance, $k_3^0 = 10^6$ and $k_3^l = 1$ and keeping $k_2 = 1$, $m = 1$, $\rho l = 1$. We prove that the chandelier remains always fixed to the ceiling and that the string breaks in case the percussion is large.

Proposition 1. *The absolute minimum when $P > 0$ of*

$$\mathcal{G}(U,Y) = 2\big(\sqrt{U}+\sqrt{pp\{Y\}}\big) + k_3^0 U + k_3^l pp\{Y\} + \frac{U^2}{2} + \frac{(u+Y)^2}{2} - P(U+Y) + I_+(u),$$

occurs always for $U = 0$ if $k_3^0 \gg k_3^l$, in fact if

$$k_3^0 - k_3^l \ge \frac{1}{2^{1/3}}.$$

Proof. Assume that the absolute minimum occurs for $U > 0$ and $Y > 0$. We have

$$\begin{aligned} &\inf \mathcal{G}(U,Y) = \mathcal{G}(\hat{U},\hat{Y})\\ &= \big\{2\sqrt{\hat{U}} + k_3^0\hat{U} + \hat{U}^2 - P\hat{U}\big\} + 2\sqrt{\hat{Y}} + k_3^l\hat{Y} + \frac{\hat{Y}^2}{2} + \hat{U}\hat{Y} - P\hat{Y}\\ &= \mathcal{G}(\hat{U},0) + 2\sqrt{\hat{Y}} + k_3^l\hat{Y} + \frac{\hat{Y}^2}{2} + \hat{U}\hat{Y} - P\hat{Y}, \end{aligned} \tag{17}$$

and

$$\frac{\partial\mathcal{G}}{\partial Y} = \frac{1}{\sqrt{\hat{Y}}} + \hat{Y} + k_3^l + \hat{U} - P = 0,$$
$$\frac{\partial\mathcal{G}}{\partial U} = \frac{1}{\sqrt{\hat{U}}} + \hat{U} + k_3^0 + \hat{Y} - P = 0.$$

These relationships give

$$\frac{1}{\sqrt{\hat{Y}}} - \Big(\frac{1}{\sqrt{\hat{U}}} + \hat{U}\Big) = k_3^0 - k_3^l, \tag{18}$$

$$k_3^l + \hat{U} - P = -\Big(\frac{1}{\sqrt{\hat{Y}}} + \hat{Y}\Big). \tag{19}$$

Relationship (18) gives

$$\hat{Y} \leq \frac{1}{(k_3^0 - k_3^l)^2}. \tag{20}$$

Relationship (17) and (19) give

$$\begin{aligned}\inf \mathcal{G}(U,Y) = \mathcal{G}(\hat{U},\hat{Y}) &= \mathcal{G}(\hat{U},0) + \sqrt{\hat{Y}} - \frac{\hat{Y}^2}{2}\\ &\leq \inf\{\mathcal{G}(U,0)\,|\,U \in \mathbb{R}\} \leq \mathcal{G}(\hat{U},0).\end{aligned}$$

It results that

$$\sqrt{\hat{Y}} - \frac{\hat{Y}^2}{2} \leq 0.$$

Thus

$$\hat{Y} \geq 2^{2/3}.$$

We get from (20) that if the minimum occurs for $U > 0$ and $Y > 0$, we have

$$\frac{1}{(k_3^0 - k_3^l)^2} \geq 2^{2/3}.$$

This relationship is not satisfied if $k_3^0 \gg k_3^l$, for instance for $k_3^0 = 10^6$ and $k_3^l = 1$. Thus the minimum occurs for either U or Y equal to 0. Because

$$\forall X \geq 0, \;\; \mathcal{G}(X,0) \geq \mathcal{G}(0,X),$$

we have

$$\inf_U \mathcal{G}(U,0) \geq \inf_Y \mathcal{G}(0,Y).$$

Thus the minimum occurs for $U = 0$.

The chandelier is never teared off the ceiling. The string breaks if P is larger than $1 + 2^{2/3} + 2^{-1/3} = 3.3811$. For this value the velocity of the stone is $Y = 2^{2/3}$.

6.3. Mechanical Conclusions

Let us sum up the previous results in practical terms.

1. When the fixation device of the chandelier is weaker than the string:

- when percussion P is large but lower than k_0 and directed downward, the chandelier is teared off the ceiling and the string is broken. The system is broken into two pieces;

- when it is medium and directed downward, the chandelier is teared off the ceiling but the string is not broken;

- when it is small and directed downward, nothing occurs. Everything remains at rest after the collision;

- when the percussion is directed upward. Nothing occurs to the chandelier. The stone goes upward.

2. When the fixation device of the chandelier is stronger than the string:

- when the percussion is large but lower than k_0 and directed down ward, the string is broken;

- when the percussion is small and directed downward, nothing occurs. Everything remains at rest after the collision;

- when the percussion is directed upwards. Nothing occurs to the chandelier. The stone goes upward.

These properties are in agreement with experiments (or with similar experiments)!

The results show the ability of the theory to account for the basic experimental results, even though the constitutive laws are very simple. Let us note that fractures are characterized by only two quantities, k_3 which characterizes the strength of the material and the occurrence of fractures and, k_2 which characterizes the opening velocity after a fracture has occurred.

6.3.1. A Fracture Inside the Bar

If we assume there is a fracture inside the bar, the function $\mathcal{F}$ becomes

$$\mathcal{F}(U,V,W,s) = 2k_2\sqrt{U} + k_3^0 U + 2k_2\sqrt{W-U} + k_3^i(W-U)$$
$$2k_2\sqrt{pp\{V-W\}} + k_3^l pp\{V-W\} + \frac{\rho^s}{2}U^2 + \frac{\rho(l-s)}{2}W^2 + \frac{m}{2}V^2 - PV$$
$$+I_+(U) + I_+(W-U).$$

Looking for the minimum of this function with respect to s shows that: either the interior fracture is at one of the end of the bar and there is no more an interior fracture, or

$$\frac{\partial \mathcal{F}(U,V,W,s)}{\partial s} = \frac{\rho}{2}U^2 - \frac{\rho}{2}W^2 = 0,$$

which shows that $U = W$ and that there is no fracture. Thus there is no fracture within the bar, in agreement with experiments.

7. An Other Numerical Example: the Stone Falls

We have $U^-_{ext}(0) > 0$ due to relation (4), and $P = 0$. With the assumptions of the previous paragraph, function becomes $\mathcal{F}(U,V)$ becomes

$$(U,V) = 2k_2\sqrt{U} + k_3^0 U + 2k_2\sqrt{pp\{V + U^-_{ext}(0) - U\}} + k_3^l pp\{V + U^-_{ext}(0) - U\}$$
$$+\frac{\rho l}{2}U^2 + \frac{m}{2}V^2 - mU^-_{ext}(0)V + I_+(U).$$

By letting

$$Y = V + U^-_{ext}(0) - U,$$

we have

$$\mathcal{F}(U,V) = 2k_2\sqrt{U} + k_3^0 U + 2k_2\sqrt{pp\{Y\}} + k_3^l pp\{Y\}$$
$$+\frac{\rho l}{2}U^2 + \frac{m}{2}(Y+U)^2 - 2mU^-_{ext}(0)(Y+U) + 2m(U^-_{ext}(0))^2 + I_+(U)$$
$$= \mathcal{G}(U,Y) + 2m(U^-_{ext}(0))^2,$$

with $P = 2mU^-_{ext}(0)$. Thus the two problems are equivalent in terms of mathematics. The mechanical conclusions of the previous paragraph are qualitatively the same.

They are what we expect:

1. When the fixation device of the chandelier is weaker than the string:
   - when velocity $U^-_{ext}(0)$ is large but not too large ($mU^-_{ext}(0) < k_0$ to have $|\Sigma| < k_0$), the chandelier is teared off the ceiling and the string is broken. The system is broken into two pieces;
   - when it is medium, the chandelier is teared off the ceiling but the string is not broken;
   - when it is small, nothing occurs to the chandelier. The stone bounces after the collision.
2. When the fixation device of the chandelier is stronger than the string:
   - when velocity $U^-_{ext}(0)$ is large but not too large, the string is broken;
   - when it is small, nothing occurs to the chandelier, The stone bounces after the collision.

In practice, it is known also that to let the stone fall is equivalent to apply a sudden blow to the stone when at rest!

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