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Generalized Zeros of Operators and Applications

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Abstract. We define the topological degree for a class of operators and use it to find generalized zeros of operators. Using these results we get the existence of solutions for singular nonlinear elliptic equations.

1. Introduction

Let E be a dense linear subspace of a real Hilbert space H and D be a subset of E. Denote by $\langle ., . \rangle$ the scalar product in H. Let f be a mapping from D into H, then a vector u in H is said to be a *generalized zero* of f if and only if there is a sequence $\{u_n\}$ in D such that $\{u_n\}$ converges weakly to u in H and

$$\lim_{n\to\infty} \langle f(u_n), v \rangle = 0 \quad \forall v \in E.$$

Note that f may not be defined at its generalized zeros. In [4] we obtained a version of the Mountain-pass theorem and applied it to get the existence of generalized zeros of ∇g, where g is a densely defined functional on H.

In the present paper we shall use the topological degree theory to find generalized zeros of operators related to partial differential equations. First we consider the following problem.

Let N be an integer ≥ 2, and Ω be a bounded open subset of $\mathbb{R}^N$. Let $g_0, \ldots, g_N$ be real functions on Ω, and a_{ij} be a real function on $\Omega \times \mathbb{R}$ for any $i, j = 1, \ldots, N$. Assume that a_{ij} is in $L^\infty_{loc}(\Omega)$ for $1 \leq i, j \leq N$ and there exists $M > 0$ such that $\sum_{i,j=1}^N a_{i,j}(x)\xi_i\xi_j \geq M|\xi|^2$ for any $(x, (\xi_1, \ldots, \xi_N))$ in $\Omega \times \mathbb{R}^N$.

Moreover, assume that $g_i(x,t)$ is measurable in x for fixed t in $\mathbb{R}$ and continuous in t for fixed x in Ω for any $i = 0, \dots, N$.

We seek solutions in the Sobolev space $W_0^{1,2}(\Omega)$ of the following equation

$$-\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}(a_{ij}(x)\frac{\partial}{\partial x_j}u(x))+\sum_{i=1}^{N} g_i(x,u(x))\frac{\partial u}{\partial x_i}(x)+g_0(x,u(x))+a(x)=0 \ \forall x \in \Omega. \tag{1.1}$$

Under certain conditions on a_{ij}, a and g_i there exists an operator T defined on $\mathcal{D}(T)$ contained in $W_0^{1,2}(\Omega)$ such that

$$\int_\Omega \Big\{\sum_{i,j=1}^{N} a_{i,j}(x)\frac{\partial u(x)}{\partial x_i}\frac{\partial \varphi(x)}{\partial x_j}+\Big[\sum_{i=1}^{N} g_i(x,u(x))\frac{\partial u}{\partial x_i}(x)+g_0(x,u(x))+a(x)\Big]\varphi(x)\Big\}dx$$
$$= \int_\Omega \nabla T(u)(x)\nabla\varphi(x)dx \quad \forall (u,\varphi) \in \mathcal{D}(T) \times W_0^{1,2}(\Omega).$$

In this case, to solve (1.1), it is sufficient to prove the existence of solutions of the following equation

$$T(u) = 0. \tag{1.2}$$

We shall use the topological degree theory to solve (1.2). We study the problem in the following situation: the map T may only be defined on a dense subspace E of $W_0^{1,2}(\Omega)$, but T does not vanish at any u in E and we have to find generalized zeros of T in $W_0^{1,2}(\Omega)$.

In order to solve (1.2) in this case we shall define a topological degree of mappings defined in a dense subspace of $W_0^{1,2}(\Omega)$ and apply it to get a generalized zero of T in (1.2), then show that this generalized zero is a generalized solution of (1.1) in $W_0^{1,2}(\Omega)$. In the present paper, the topological degree is defined for a class of operators in Hilbert spaces. The results for the Banach spaces will appear elsewhere.

In [5, 6], Kartsatos and Skrypnik have defined the topological degree for operators defined on a dense subspace of a Banach space, but their results can only be applied to get classical solutions.

Since the functions $b_0, \dots, b_N$ in condition (3.3) in Sec. 3 are respectively in $L^{r_i}_{loc}(\Omega)$ instead of in $L^{r_i}(\Omega)$, the elliptic partial differential equations considered in this paper are more singular than those in [3].

The paper consists of two sections. In the first section we define the topological degree for mappings of class (B_+) and apply it to solve singular elliptic equations in the last section.

2. Topological Degree of Mappings in Class (B_+)

Definition 2.1. *Let $\{E_n\}_n$ be a strictly increasing sequence of finite-dimensional subspaces of a Hilbert space H such that $E \equiv \cup_{n=1}^{\infty} E_n$ is dense in H. Denote by*

P_n the orthogonal projection from H onto E_n for every integer n. Let G be an open bounded set in E. Denote by $\overline{G}^E$, G_n, $\overline{G_n}^{E_n}$ and $\partial_{E_{n_k}} G_{n_k}$ the closure of G in E, $G \cap E_n$, the closure and boundary of G_n in E_n respectively. Let f be a mapping from $\overline{G}^E$ into H. We put

$$f_n(x) = P_n(f(x)) \quad \forall n \in \mathbb{N},\ x \in \overline{G_n}^{E_n}.$$

The mapping f is said to be of class (B_+) on $\overline{G}^E$ if and only if f_n is a continuous mapping from $\overline{G_n}^{E_n}$ into E_n for any integer n and the following condition is satisfied:

(B_+) There is not any sequence $\{x_{n_k}\}_k$ in E such that the sequence $\{x_{n_k}\}_k$ is weakly convergent in H, $x_{n_k} \in \partial_{E_{n_k}} G_{n_k}$, $\langle f(x_{n_{k+1}}), x_{n_{k+1}}\rangle \le 0$ and $\langle f(x_{n_{k+1}}), v\rangle = 0$ for all $k \in \mathbb{N}$ and v in E_{n_k}.

The class (B_+) depends on the choice of $\{E_n\}_n$, and we consider only such one sequence in this section. The class (B_+) is similar to the class (S_+) in [1, 2, 8, 9].

Following the proof of Proposition 11 in [3], we have the following lemma

Lemma 2.1. *Let X_0 be a subspace of a finite-dimensional Hilbert space X, and P be the orthogonal projection from X onto X_0. Let G be an open bounded subset of X such that $G_0 \equiv G \cap X_0 \ne \emptyset$, and let f be a continuous mapping from $\overline{G}$ into X. Put*

$$f_0(x) = P \circ f(x) \quad \forall x \in \overline{G_0}^{X_0}.$$

Suppose that the Leray–Schauder topological degrees $\deg(f, G, 0)$ *and* $\deg(f_0, G_0, 0)$ *are defined but not equal. Then there exists a vector u in ∂G such that $\langle f(u), u\rangle \le 0$ and $\langle f(u), v\rangle = 0$ for all v in X_0.*

Remark. If G contains 0, then the foregoing lemma has been proved in [3].

The following lemma is the key result in order to define the topological degree for mappings of class (B_+).

Lemma 2.2. *Let G be a non-empty open bounded set in E, f be in class (B_+) on $\overline{G}^E$ and $\{f_n\}_n$ be as in Definition* 2.1. *Then there exists an integer n_0 such that the Leray–Schauder degree $deg(f_n, G_n, 0)$ is defined and*

$$deg(f_n, G_n, 0) = deg(f_{n_0}, G_{n_0}, 0) \quad \forall n \ge n_0.$$

Proof. First we note that G_n is non-empty for any sufficiently large integer n. We shall show that 0 is in $E_n \setminus f_n(\partial_{E_n} G_n)$ when n is sufficiently large. Suppose by contradiction that there exist a strictly increasing sequence of integers $\{m_l\}$ and a sequence $\{x_{m_l}\}$ such that x_{m_l} is in $\partial_{E_{m_l}} G_{m_l}$ and $f_{m_l}(x_{m_l}) = 0$ for any integer l. Since $\partial_{E_n} G_n$ is contained in $\partial_E G$ and $\overline{G}$ is a bounded subset of H, we can (and shall) suppose that $\{x_{m_l}\}$ is a weakly Cauchy sequence in $\partial_E G$, $\langle f(x_{m_{l+1}}), x_{m_{l+1}}\rangle \le 0$ and $\langle f(x_{m_{l+1}}), v\rangle = 0$ for any v in E_{m_l}. By (B_+) we get a contradiction. Thus there exists an integer m_0 such that the Leray–Schauder degree $deg(f_n, G_n, 0)$ is defined when $n \geq m_0$. In order to show the remaining part of the lemma we suppose by contradiction that there is a strictly increasing sequence of integers $\{n_k\}$ such that

$$deg(f_{n_k}, G_{n_k}, 0) \neq deg(f_{n_{k+1}}, G_{n_{k+1}}, 0) \quad \forall k \in \mathbb{N}.$$

Since $E_{n_{k+1}}$ is a finite-dimensional Hilbert space, applying Lemma 2.2 we can find a weakly Cauchy sequence $\{x_{n_k}\}_k$ in H such that x_{n_k} belongs to $\partial_{E_{n_k}} G_{n_k}$ and

$$\langle f(x_{n_{k+1}}), x_{n_{k+1}} \rangle \leq 0 \text{ and } \langle f(x_{n_{k+1}}), v \rangle = 0 \text{ for all } k \in \mathbb{N} \text{ and } v \text{ in } E_{n_k}.$$

By (B_+), we again get a contradiction, which completes the proof of the lemma. ■

Using the lemma we have the following definition.

Definition 2.2. *Let H, $\{E_n\}_n$ and E be as in Definition 2.1. Let G be an open bounded set in E, f be a mapping from $\overline{G}^E$ into H and $\{f_n\}_n$ be as in Definition 2.1. Assume that f is of class (B_+) on $\overline{G}^E$. By Lemma 2.2 we can define*

$$deg(f, G, 0) = \lim_{n \to \infty} deg(f_n, G_n, 0),$$

which is called the topological degree of f on G at 0.

The topological degree of maps in class (B_+) has following properties.

Theorem 2.1. *Let G be an open bounded set in E. We have the following assertions*

(i) *The identity map Id is of class (B_+) and $deg(Id, G, 0) = 1$ whenever 0 is in G.*

(ii) *If f is of class (B_+) and $deg(f, G, 0) \neq 0$, then f has a generalized zero in H.*

(iii) *Let h be a mapping from $[0,1] \times \overline{G}^E$ into H. Assume that*

(a) *$P_n \circ h|_{[0,1]\times\overline{G_n}^{E_n}}$ is continuous on $[0,1] \times \overline{G_n}^{E_n}$ for any integer n, and*

(b) *there is not any weakly Cauchy sequence $\{(t_{n_k}, x_{n_k})\}_k$ in $[0,1] \times H$ such that $x_{n_k} \in \partial_{E_{n_k}} G_{n_k}$, $< h(t_{n_{k+1}}, x_{n_{k+1}}), x_{n_{k+1}} > \leq 0$ and*

$$\langle h(t_{n_{k+1}}, x_{n_{k+1}}), v \rangle = 0 \quad \forall k \in \mathbb{N},\ v \in E_{n_k}.$$

Then the topological degree $\deg(h(0,.), G, 0)$ *and* $\deg(h(1,.), G, 0)$ *are defined and equal.*

Proof. By properties of the Leray-Schauder topological degree and by Definition 2.2 we get (i). Now we prove (ii). By Definition 2.2 there are a sequence $\{x_m\}_{m \geq n_0}$ in G and an integer n_0 such that $x_m \in G_m$ and $P_m \circ f(x_m) = 0$ when $m \geq n_0$. Thus we have

$$\langle f(x_m), w\rangle = 0 \quad \text{when} \quad m \geq n_0 \quad \text{and} \quad w \in E_m.$$

Since $\{E_n\}$ is a strictly increasing sequence, we get (ii).

Arguing as in the proof of Lemma 2.3 we see that $h(t,.)$ is of class (B_+) for all t in $[0,1]$ and $P_m \circ h|_{[0,1]\times\overline{G_m}^{E_n}}$ is a homotopy of compact vector fields when m is greater than some integer n_0. In this case

$$\deg\big(P_m \circ h|_{\{0\}\times\overline{G_m}^{E_n}}, G_m, 0\big) = \deg(P_m \circ h|_{\{1\}\times\overline{G_m}^{E_n}}, G_m, 0).$$

By Lemma 2.3 we can choose an integer $n \geq n_0$ such that

$$\deg(h(0,.), G, 0) = \deg(P_n \circ h|_{\{0\}\times\overline{G_n}^{E_n}}, G_n, 0)$$

and

$$\deg(h(1,.), G, 0) = \deg(P_n \circ h|_{\{1\}\times\overline{G_n}^{E_n}}, G_n, 0).$$

Combining the above equations we obtain (iii). ∎

Corollary 2.1. *Let H and E be as in Definition 2.1. Let G be an open bounded set in E and f be a mapping from $\overline{G}^E$ into H such that f_m is continuous on $\overline{G_m}^{E_m}$ for any integer m. Assume that G contains 0 and*

$$\langle f(x), x\rangle > 0 \quad \forall x \in \partial_E G.$$

Then f has a generalized zero in H.

Proof. Put

$$h(t,x) = tx + (1-t)f(x), \quad (t,x) \in [0,1] \times \overline{G}^E.$$

For any x in the boundary of G in E we have

$$\langle h(t,x), x\rangle = t\|x\|^2 + (1-t)\langle f(x), x\rangle > 0, \quad \forall t \in [0,1].$$

Thus the mapping h satisfies all conditions of (iii) in Theorem 2.1. On the other hand, $h(0,.) = f$ and $h(1,.) = Id$. Therefore by Theorem 2.1 we get the corollary.

3. A Nonlinear Singular Elliptic Equation

Let N be an integer ≥ 2, and Ω be a bounded open subset of $\mathbb{R}^N$. Let $g_0, \dots, g_N$ be real functions on Ω, and a_{ij} be a real function on $\Omega \times \mathbb{R}$ for any $i, j = 1, \dots, N$. Assume that a_{ij} is in $L^\infty_{loc}(\Omega)$ for $1 \leq i, j \leq N$ and there exists $M > 0$ such that

$$\sum_{i,j=1}^{N} a_{i,j}(x)\xi_i\xi_j \geq M|\xi|^2 \quad \forall (x, (\xi_1, \dots, \xi_N)) \in \Omega \times \mathbb{R}^N. \tag{3.1}$$

Moreover, assume that $g_i(x,t)$ is measurable in x for fixed t in $\mathbb{R}$ and continuous in t for fixed x in Ω for any $i = 0, \dots, N$, and

$$g_0(x,0) = 0 \quad \forall\ x\ \in \Omega, \tag{3.2}$$

$$|g_i(x,t)| \le b_i(x) + k_i|t|^{s_i} \quad \forall (x,t) \in \Omega\times\mathbb{R},\ \ i = 0,\dots,N, \tag{3.3}$$

and

$$-\frac{M}{2}|z|^2 - k|t|^q - c(x) \le \Big[\sum_{i=1}^N g_i(x,t)z_i + g_0(x,t) + a(x)\Big]t \ \ \forall (x,t,z) \in \Omega\times\mathbb{R}\times\mathbb{R}^N, \tag{3.4}$$

where $s_0,\dots,s_N$, k, $k_0,\dots,k_N, r_0,\dots,r_N$ and q are non-negative real numbers and $b_0,\cdots,b_N$ and c are measurable functions such that $c \in L^1(\Omega), q \in (1,2)$, $r_0 \in \big(\frac{2N}{N+2},\infty\big)$, $s_0^{-1} \in (\frac{N-2}{2N}r_0,\infty)$, $a \in L^{r_0}(\Omega)$, $r_i \in (N,\infty)$ for any $i = 1,\dots,N$, $s_i^{-1} \in \big(\frac{N-2}{2N}r_i,\infty\big)$ and $b_i \in L^{r_i}_{loc}(\Omega)$ for any $i = 0,\dots,N$.

We shall use Lemma 2.3 to find generalized solutions of the following equation

$$-\sum_{i,j=1}^N \frac{\partial}{\partial x_i}\Big(a_{i,j}(x)\frac{\partial}{\partial x_j}u(x)\Big) + \sum_{i=1}^N g_i(x,u(x))\frac{\partial u}{\partial x_i}(x) + g_0(x,u(x)) + a(x) = 0$$
$$\forall x \in \Omega. \tag{3.5}$$

First we need some notations and definitions. For any open bounded subset D of $\mathbb{R}^N$ we denote by $W_0^{1,2}(D)$ the completion of $C_c^\infty(D,\mathbb{R})$ in the following norm

$$\|u\|_D = \Big(\int_D |\nabla u|^2 dx\Big)^{1/2} \quad \forall u \in C_c^\infty(D,\mathbb{R}).$$

It is well known that $W_0^{1,2}(D)$ is a Hilbert space with the scalar product

$$\langle u,v\rangle_D = \int_D \nabla u.\nabla v dx \quad \forall u,v \in W_0^{1,2}(D).$$

Let $\{\Omega_k\}$ be an increasing sequence of open subsets of Ω such that $\overline{\Omega_k}$ is contained in Ω_{k+1} and $\Omega = \bigcup_{k=1}^\infty \Omega_k$. Choose a sequence $\{v_{1,m}\}_{m\in\mathbb{N}}$ in $C_c^\infty(\Omega_1)$ such that $\{v_{1,m}\}_{m\in\mathbb{N}}$ is a maximal orthonormal set of $W_0^{1,2}(\Omega_1)$. After that we can find a sequence $\{v_{2,m}\}_{m\in\mathbb{N}}$ in $C_c^\infty(\Omega_2)$ such that $\{v_{1,m}\}_{m\in\mathbb{N}} \cup \{v_{2,m}\}_{m\in\mathbb{N}}$ is a maximal orthonormal set of $W_0^{1,2}(\Omega_2)$. Thus by the mathematical induction we can find the set $\{v_{n,m} : n,m \in \mathbb{N}\}$ in $C_c^\infty(\Omega)$ such that $\{v_{j,m} : m \in \mathbb{N}, j = 1,\dots,k\}$ is a maximal orthonormal set of $W_0^{1,2}(\Omega_k)$ for every integer k. We rewrite $\{v_{n,m}\}_{n,m\in\mathbb{N}}$ as a sequence $\{e_k\}_{k\in\mathbb{N}}$ and denote by E_n the vector subspace spanned by $\{e_1,\dots,e_n\}$. Put $H = W_0^{1,2}(\Omega)$ and $E = \cup_n E_n$. We have the following results.

Lemma 3.1.

(i) *E is dense in H.*

(ii) *Let u be in $C_c^\infty(\Omega)$. Then there are an integer k and a sequence $\{u_n\}$ in E such that the support of u_n is contained in Ω_k for any integer n and $\{u_n\}$ converges to u in $W_0^{1,2}(\Omega)$.*

(iii) *Let m be a positive integer and u be in $W_0^{1,2}(\Omega)$. Then there exists a unique $T_m(u)$ in $W_0^{1,2}(\Omega)$ such that for every v in $W_0^{1,2}(\Omega)$*

$$\langle T_m(u), v\rangle_\Omega = \int\limits_{\Omega_m} \sum_{i,j=1}^{N} a_{ij}(x)\frac{\partial u(x)}{\partial x_i}\frac{\partial v(x)}{\partial x_j}dx$$
$$+ \int\limits_{\Omega_m} \Big[\sum_{i=1}^{N} g_i(x,u(x))\frac{\partial u}{\partial x_i}(x) + g_0(x,u(x))\Big]v(x)dx + \int\limits_{\Omega} avdx.$$

(iv) *Let $\{u_k\}$ be a sequence weakly converging to u in $W_0^{1,2}(\Omega)$. Then $\{T_m(u_k)\}$ weakly converges to $T_m(u)$ in $W_0^{1,2}(\Omega)$ for any integer m.*

(v) *Let u be in E. Then there exist an integer m_0 and a unique $T(u)$ in $W_0^{1,2}(\Omega)$ such that for any v in $W_0^{1,2}(\Omega)$*

$$\langle T(u), v\rangle_\Omega = \int\limits_{\Omega} \sum_{i,j=1}^{N} a_{ij}(x)\frac{\partial u(x)}{\partial x_i}\frac{\partial v(x)}{\partial x_j}dx$$
$$* \int\limits_{\Omega} \Big[\sum_{i=1}^{N} g_i(x,u(x))\frac{\partial u}{\partial x_i}(x) + g_0(x,u(x)) + a(x)\Big]v(x)dx,$$
$$\langle T(u), v\rangle_\Omega = \langle T_m(u), v\rangle_\Omega \quad \forall m \geq m_0.$$

(vi) *The restriction of T on E_k is a continuous mapping from E_k into $W_0^{1,2}(\Omega)$ for any integer k.*

(vii) *There is a positive real number C such that*

$$\langle T(u), u\rangle_\Omega \geq \|u\|_\Omega^2\Big(\frac{M}{2} - C\|u\|_\Omega^{q-2} - \|c\|_{L^1(\Omega)}\|u\|_\Omega^{-2}\Big) \quad \forall u \in E\setminus\{0\}.$$

Proof. Since E is dense in $C_c^\infty(\Omega)$, we get (i). Now we prove (ii). Fix u in $C_c^\infty(\Omega)$. We can choose an integer k such that the support of u is contained in Ω_k. Since $\{v_{j,m} : m \in \mathbb{N}, j = 1,\ldots,k\}$ is a maximal orthonormal set of $W_0^{1,2}(\Omega_k)$, we can find a sequence $\{u_n\}$ as in (ii).

Now we show (iii). Fix a positive integer m and put

$$G_{m,i}(u)(x) = g_i(x,u(x)) \quad \forall x \in \Omega_m,\ i = 0,\ldots,N.$$

By a result in [7, p.30] and by (3.3) we see that $G_{m,i}$ is a continuous mapping from $L^{r_i s_i}(\Omega)$ into $L^{r_i}(\Omega_m)$.

Put $p_0 = \big(1 - \frac{1}{r_0}\big)^{-1}$ and $p_i = \big(1 - \frac{1}{r_i} - \frac{1}{2}\big)^{-1}$ for any $i = 1,\ldots,N$. By conditions on r_i we see that $p_i^{-1} > \frac{N-2}{2N}$. Thus by the Sobolev embedding theorem there is a positive real number C such that

$$\Big|\int\limits_{\Omega_m} \Big[\sum_{i=1}^{N} g_i(x,u(x))\frac{\partial u}{\partial x_i}(x)\Big]v(x)dx + \int\limits_{\Omega_m} g_0(x,u(x))v(x)dx + \int\limits_{\Omega} a(x)v(x)dx\Big|$$

$$= \Big| \int_{\Omega_m} \Big[\sum_{i=1}^{N} G_{m,i}(u) \frac{\partial u}{\partial x_i} + G_{m,0}(u) \Big] v(x)dx + \int_{\Omega} avdx \Big|$$
$$\le C \Big[\sum_{i=1}^{N} \|G_{m,i}(u)\|_{r_i,m} \|u\|_{\Omega} + \|G_{m,0}(u)\|_{r_0,m} + \|a\|_{r_0} \Big] \|v\|_{\Omega},$$

where $\|w\|_r = \|w\|_{L^r(\Omega)}$ and $\|w\|_{r,m} = \|w\|_{L^r(\Omega_m)}$.

Using Holder's inequality and the assumption $a_{ij} \in L^{\infty}_{loc}(\Omega)$, we have.

$$\int_{\Omega_m} \sum_{i,j=1}^{N} a_{ij}(x) \frac{\partial u(x)}{\partial x_i} \frac{\partial v(x)}{\partial x_j} dx \le C\|u\|\|v\|. \tag{3.6}$$

Applying the Riesz theorem, we get (iii).

Now we prove (iv). Let u be in $W_0^{1,2}(\Omega)$ and $\{u_k\}$ be a sequence weakly converging to u in $W_0^{1,2}(\Omega)$. Since $r_i^{-1} s_i^{-1} > \frac{N-2}{2N}$ for any $i = 0, \dots, N$, by the Rellich–Konkrachov theorem the sequence $\{G_{m,i}(u_k)\}$ converges to $G_{m,i}(u)$ in $L^{r_i}(\Omega_m)$.

By conditions on r_i, $i = 0, \dots, N$, we can find positive real numbers q_i such that $q_0^{-1} + r_0^{-1} \le 1$, $q_i^{-1} + r_i^{-1} \le \frac{1}{2}$ for any $i = 1, \dots, N$ and $q_i^{-1} > \frac{N-2}{2N}$ for any $i = 0, \dots, N$. Since v belongs to $W_0^{1,2}(\Omega)$, by the Sobolev embedding theorem v is in L^{q_i} for any $i = 0, \dots, N$. Thus the sequence $\{vG_{m,0}(u_k)\}_k$ (respectively $\{vG_{m,i}(u_k)\}_k$) converges to $vG_{m,0}(u)$ (respectively $vG_{m,i}(u)$ for any $i = 1, \dots, N$) in $L^1(\Omega_m)$ (respectively $L^2(\Omega_m)$). Thus by (iii) and (3.6) we see that

$$\lim_{k\to\infty} \langle T_m(u_k), v\rangle_{\Omega} = \langle T_m(u), v\rangle_{\Omega}$$

and we obtain (iv).

Let u be in E, then there is an integer m_0 such that the support of u is contained in Ω_{m_0}. Therefore we obtain (v).

Since the dimension of E_k is finite, the strong topology and the weak one on E_m coincide. Therefore we get (vi) by using (iv) and (v). Fix a u in E. By (3.1), (3.4) and the Sobolev embedding theorem there is a positive real number C such that

$$\langle T(u), u\rangle_{\Omega} \ge M\|u\|_{\Omega}^2 - \int_{\Omega} \Big[\frac{M}{2}|\nabla u|^2 + k|u|^q + |c| \Big] dx$$
$$\ge \frac{M}{2}\|u\|_{\Omega}^2 - C\|u\|_{\Omega}^q - \|c\|_{L^1(\Omega)}.$$

Therefore we obtain (vii). ∎

Applying the foregoing lemma and Theorem 2.2 we get the following result.

Theorem 3.1.

(i) *Under conditions* (3.1) - (3.4) *the equation* (3.5) *has at least a generalized solution u in $W_0^{1,2}(\Omega)$, that is for any $v \in C_c^{\infty}(\Omega)$*

$$\int_{\Omega}\Big\{\sum_{i,j=1}^{N}a_{i,j}(x)\frac{\partial u(x)}{\partial x_i}\frac{\partial v(x)}{\partial x_j} + \Big[\sum_{i=1}^{N}g_i(x,u(x))\frac{\partial u}{\partial x_i}(x)+g_0(x,u(x))+a(x)\Big]v(x)\Big\}dx=0 \tag{3.7}$$

(ii) *Furthermore, this solution is a classical smooth solution if the functions* $g_0,\ldots,g_N$ *are sufficiently smooth.*

Proof. Choose a positive real number R such that

$$\frac{M}{2}-CR^{q-2}-\|c\|_{L^1(\Omega)}R^{-2}>\frac{M}{4},$$

where C and c are as in (vii) of Lemma 3.1.

Put $G=\{w\in E:||w||_{\Omega}<R\}$. We see that $\partial_E G=\{w\in E:\|w\|_{\Omega}=R\}$ and by Lemma 3.1 the mapping T satisfies the conditions of Corollary 2.1. Thus there exists a weakly Cauchy sequence $\{u_n\}$ in G such that

$$\lim_{n\to\infty}\langle T(u_n),v\rangle=0\quad\forall v\in E.$$

Let u be the weak limit of $\{u_n\}$ in $W_0^{1,2}(\Omega)$. Fix a v in E and let k be an integer such that the support of v is contained in Ω_k, we have

$$\langle T_m(u_n),v\rangle_{\Omega}=\langle T_k(u_n),v\rangle_{\Omega}\quad\forall n\in\mathbb{N},\ m\geq k.$$

Thus by (iv) of Lemma 3.1 we get

$$\langle T_m(u),v\rangle_{\Omega}=0\quad\forall m\geq k. \tag{3.8}$$

Let v be in $C_c^{\infty}(\Omega)$. By (ii) of Lemma 3.1 there are an integer k and a sequence $\{v_l\}$ in E such that the support of v_l is contained in Ω_k for any integer n and $\{v_l\}$ strongly converges to v in $W_0^{1,2}(\Omega)$. By (3.8) we have

$$\langle T_m(u),v\rangle_{\Omega}=\lim_{l\to\infty}\langle T_m(u),v_l\rangle_{\Omega}=0\quad\forall m\geq k$$

or

$$\int_{\Omega}\Big\{\sum_{i,j=1}^{N}a_{i,j}(x)\frac{\partial u(x)}{\partial x_i}\frac{\partial v(x)}{\partial x_j} + \Big[\sum_{i=1}^{N}g_i(x,u(x))\frac{\partial u}{\partial x_i}(x)+g_0(x,u(x))+a(x)\Big]v(x)\Big\}dx=0$$

for every $m\geq k$, which implies (3.7).

Now we show (ii). If v is in $W_0^{1,2}(\Omega)$ and its support is a compact subset in Ω, then, arguing as in the proof of (i), we see that (3.7) is valid for v. Therefore by the regularity theory of elliptic equations and by using convenient test functions we can get the smoothness of u if the functions g_i are of class $C^{\infty}(\Omega)$ and the sets $g_i(\Omega_k\times\mathbb{R})$ and $a(\Omega_k)$ are bounded in $\mathbb{R}$ for any integer k and any $i=0,\cdots,N$.

Example. Let N be an integer ≥ 2 and Ω be an open bounded domain in $\mathbb{R}^N$ with smooth boundary. Let s, p and q be positive real numbers such that $s > \dfrac{2N}{N+2}$, $\dfrac{1}{p} > \max\left\{1, \dfrac{s(N-2)}{(N+2)s-2N}\right\}$ and $q > \dfrac{2}{1-p}$. Let K be in $L^s_{loc}(\Omega)$ such that $K^- \equiv \max\{0, -K\}$ is in $L^q(\Omega)$. Using Theorem 3.1 we can find a generalized solution in $W_0^{1,2}(\Omega)$ to the following equation

$$-\Delta u(x) + K(x)\,\mathrm{sign}\,(u(x))|u(x)|^p + 1 = 0 \quad \forall x \in \Omega.$$

The proof of the example will be appeared elsewhwere. ∎

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