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From Power Laws to Fractional Diffusion: The Direct Way

R. Gorenflo* and Entsar A. Abdel-Rehim

First Mathematical Institute, Free University of Berlin
Arnimallee 3, D-14195 Berlin, Germany
**E-mail: gorenflo@mi.fu-berlin.de, URL:www.fracalmo.org*

Abstract. Starting from the model of continuous time random walk (Montroll and Weiss 1965) that can also be considered as a compound renewal process we focus our interest on random walks in which the probability distributions of the waiting times and jumps have fat tails characterized by power laws with exponent between 0 and 1 for the waiting times, between 0 and 2 for the jumps. By stating the relevant lemmata (of Tauber type) for the distribution functions we need not distinguish between continuous and discrete space and time. We will see that by a well-scaled passage to the diffusion limit diffusion processes fractional in time as well as in space are obtained. The corresponding equation of evolution is a linear partial pseudo-differential equation with fractional derivatives in time and in space, the orders being equal to the above exponents. Such processes are enjoying increasing popularity in applications in physics, chemistry, finance and other fields, and their behaviour can be well approximated and visualized by simulation via various types of random walks. For their explicit solutions there are available integral representations that allow to investigate their detailed structure. For ease of presentation we restrict attention to the spatially one-dimensional symmetric situation.

1. Introduction: Concepts and Notations

We consider spatially one-dimensional random walks of the following basic structure. A particle (or a wanderer) starting at the time instant $t = 0$ at the space point $x = 0$ makes jumps of random size X_k in random instants t_k, $k \in \mathbb{N} = \{1, 2, 3, \dots\}$, $0 < t_1 < t_2 \cdots \to \infty$. For convenience we set $t_0 = 0$. Then in the time interval $t_n \le t < t_{n+1}$ the particle is sitting in the point

$x = S_n := \sum_{k=1}^{n} X_k$. We assume the jumps to be *independent identically distributed* (i.i.d. for short) random variables, all having the same probability distribution as a generic real random variable X, called the jump. Likewise we assume the waiting times $T_k := t_k - t_{k-1}$ to be i.i.d. random variables, all equal in distribution to a generic non-negative random variable T, called the waiting time. So, this process is what in mathematical literature is called a *compound* (or *cumulative*) renewal process [5]. We denote the distribution functions of the waiting time T and the jump X by Φ and W, respectively, by

$$P(T \le t) = \Phi(t),\ 0 \le t < \infty;\ \ P(X \le x) = W(x),\ -\infty < x < \infty.$$

Conveniently using the language of *generalized functions* in the sense of [8] or [30] we introduce the (generalized) probability *densities* ϕ and w, so that

$$\Phi(t) = \int_0^t \phi(t')dt', 0 \le t < \infty;\ W(x) = \int_{-\infty}^{x} w(x')dx', -\infty < x < \infty.$$

Denoting for time instant t the probability density to find the particle in point x by $p(x,t)$ we then have, by conditioning on the last jump before t and using the delta function $\delta(x)$, for $0 \le t < \infty$ and $-\infty < x < \infty$ the integral equation (see [24]) of *continuous time random walk*

$$p(x,t) = \delta(x)(1-\Phi(t)) + \int_0^t \left\{ \int_{-\infty}^{\infty} w(x-x')p(x',t')dx' \right\} \phi(t-t')dt'. \qquad (1.1)$$

For the cumulative function $P(x,t) = \int_{-\infty}^{x} p(x',t)\,dx'$ we have, with the Heaviside step function $H(x)$, the equation

$$P(x,t) = H(x)(1-\Phi(t)) + \int_0^t \left\{ \int_{-\infty}^{\infty} W(x-x')dP(x',t') \right\} d\Phi(t-t'). \qquad (1.2)$$

To proceed further we use the machinery of the transforms of Laplace and Fourier. The general formulas for $s \ge 0$, $-\infty < \kappa < \infty$ are

$$\widetilde{g}(s) = \int_0^\infty e^{-st}g(t)dt = \int_0^\infty e^{-st}dG(t);\ \widehat{f}(\kappa) = \int_{-\infty}^{\infty} e^{i\kappa x}f(x)dx = \int_{-\infty}^{\infty} e^{i\kappa x}dF(x).$$

Essentially applying these formulas to probability densities with $0 \le t < \infty$ and $-\infty < x < \infty$ we can safely take the Laplace variable s as real and non-negative. Furthermore we will work with convolutions of (generalized) functions, namely with the Laplace convolution and the Fourier convolution:

$$(g_1 * g_2)(t) = \int_0^\infty g_1(t')g_2(t-t')dt';\ (f_1 * f_2)(x) = \int_{-\infty}^\infty f_1(x')f_2(x-x')dx'.$$

Then, applying the transforms of Fourier and Laplace in succession to the equation (1.1) and using the well-known operational rules, we arrive at the relation

$$\widehat{\widetilde{p}}(\kappa,s) = \frac{1-\widetilde{\phi}(s)}{s} + \widetilde{\phi}(s)\widehat{w}(\kappa)\widehat{\widetilde{p}}(\kappa,s), \tag{1.3}$$

which leads to the famous Montroll–Weiss equation, see [24],

$$\widehat{\widetilde{p}}(\kappa,s) = \frac{1-\tilde{\phi}(s)}{s}\,\frac{1}{1-\widehat{w}(\kappa)\tilde{\phi}(s)}. \tag{1.4}$$

This equation can alternatively be derived from the Cox formula, (see [5, Chapter 8, formula (4)]), describing the process as subordination of a random walk to a renewal process. By inverting the transforms one can, in principle, find the evolution $p(x,t)$ of the sojourn density for time t running from zero to infinity.

Our aim is to show that under appropriate assumptions of power laws for the distribution functions $\Phi(t)$, $t \ge 0$, and $W(x)$, $-\infty < x < \infty$, under observance of a scaling relation between the positive parameters h and τ the re-scaled random walk $S_n(h) = \sum_{k=1}^{n} hX_k$ happening at the instants $t_n(\tau) = \sum_{k=1}^{n} \tau T_k$ (with $S_0(h) = 0$, $t_0(\tau) = 0$) weakly (or in law) tends, for h and τ tending to zero, to a process obeying the space-time fractional diffusion equation. Specifically, we will show that the sojourn probability density $p_{h,\tau}(x,t)$ tends weakly to the solution $u(x,t)$ of the Cauchy problem for $t > 0$ and $x \in \mathbb{R}$

$$\underset{t\ *}{D^{\beta}}\, u(x,t) = R^{\alpha}\, u(x,t),\ u(x,0) = \delta(x). \tag{1.5}$$

Here $0 < \alpha \le 2,\ 0 < \beta \le 1$. The fractional Riesz derivative R^α (in space) is defined as follows: the Fourier transform of $R^\alpha f(x)$ is $-|\kappa|^\alpha \widehat{f}(\kappa)$ for a sufficiently well-behaved function $f(x)$. Compare [6, 25, 26]. The Caputo fractional derivative (in time) can be defined through its image in the Laplace transform domain. The Laplace transform of $\underset{t\ *}{D^{\beta}}\, g(t)$ is $s^\beta \widetilde{g}(s) - s^{\beta-1}g(0)$. We have $\underset{t\ *}{D^{\beta}}\, g(t) = \frac{dg(t)}{dt}$ for $\beta = 1$ but

$$\underset{t\ *}{D^{\beta}}\, g(t) = \frac{1}{\Gamma(1-\beta)}\left\{\frac{d}{dt}\int_0^t (t-t')^{-\beta}g(t')dt' \ -\ t^{-\beta}g(0)\right\} \quad \text{for} \quad 0<\beta<1,$$

compare [10]. In the Fourier–Laplace domain the Cauchy problem (1.5) appears in the form $s^\beta \widehat{\widetilde{u}}(\kappa,s) - s^{\beta-1} = -|\kappa|^\alpha \widehat{\widetilde{u}}(\kappa,s)$ from which we obtain

$$\widehat{\widetilde{u}}(\kappa,s) = \frac{s^{\beta-1}}{s^\beta + |\kappa|^\alpha},\ s>0,\ \ \kappa \in \mathbb{R}. \tag{1.6}$$

Let us refer to [18] for the analytical theory of representing the function $u(x,t)$, namely the fundamental solution of the space-time fractional diffusion equation, in dependence on the parameters α and β.

To carry out the passage to the diffusion limit we state in Sec. 2 two *Master Lemmata* and four simplifications relating the asymptotic behaviours of the distribution functions $W(x)$ and $\Phi(t)$ near infinity to the asymptotic behaviour of their Laplace and Fourier transforms near zero. Sec. 3 is devoted to the actual passage to the diffusion limit, in Sec. 4 some examples are presented, and a few historical comments are given in Sec. 5.

2. Six Lemmata

Definition. *As in* [3] *we call a positive measurable function* ν, *defined on some neighbourhood* $[x^*,\infty)$ *of infinity, slowly varying if* $\nu(ax)/\nu(x) \to 1$ *as* $x \to \infty$ *for every* $a > 0$. *Examples:* $(\log x)^\gamma$ *with* $\gamma \in \mathbb{R}$ *and* $\exp\left(\frac{\log x}{\log\log x}\right)$.

Master Lemma 1. *Assume* $W(x)$ *increasing,* $W(-\infty) = 0$, $W(\infty) = 1$, *symmetry* $\int\limits_{(-\infty,-x)} dW(x') = \int\limits_{(x,\infty)} dW(x')$ *for* $x \geq 0$, *let* L *be a slowly varying function and assume either* (a) *or* (b).

(a) $\sigma^2 := \int\limits_{-\infty}^{\infty} x^2 dW(x) < \infty$, *labelled as* $\alpha = 2$,

(b) $\int\limits_{(x,\infty)} dW(x) \sim b\alpha^{-1}x^{-\alpha}L(x)$ *for* $x \to \infty$, $\alpha \in (0,2)$ *and* $b > 0$.

Then, with

$$\mu = \frac{\sigma^2}{2} \text{ and } L(x) \equiv 1 \quad \text{in case (a)}, \; \mu = \frac{b\pi}{\Gamma(\alpha+1)\sin(\alpha\pi/2)} \quad \text{in case (b)}, \tag{2.1}$$

we have the asymptotics $1 - \widehat{w}(\kappa) \sim \mu|\kappa|^\alpha L(|\kappa|^{-1})$ *for* $\kappa \to 0$.

Comments. The proof can be distilled from Chapter 8 of [3]. In some sense this lemma is a partial reformulation (with a constant corrected) of Gnedenko's theorem on the domain of attraction of stable probability laws, see [9].

Master Lemma 2. *Assume* $\Phi(t)$ *increasing,* $\Phi(0) = 0$, $\Phi(\infty) = 1$, *let* M *be a slowly varying function and assume either (A) or (B).*

(A) $\rho := \int\limits_{0}^{\infty} t d\Phi(t) < \infty$, *labelled as* $\beta = 1$,

(B) $\int\limits_{(t,\infty)} d\Phi(t') \sim c\beta^{-1}t^{-\beta}M(t)$ *for* $t \to \infty$, $\beta \in (0,1)$ *and* $c > 0$.

Then, with

$$\lambda = \rho \quad \text{and } M(t) \equiv 1 \text{ in case (A)}, \; \lambda = \frac{c\Gamma(1-\beta)}{\beta} \text{ in case (B)}, \tag{2.2}$$

we have the asymptotics $1 - \widetilde{\phi}(s) \sim \lambda s^\beta M(s^{-1})$ *for* $0 < s \to 0$.

Comments. This lemma is a special case of Karamata's theorem of 1931. A proof can be found in the book [3]. In the case of existing non-generalized functions as densities the Master Lemmata imply the two following (convenient) C-Lemmata (by integration it can be shown that the assumptions of the Master Lemmata are met). See [12]. For the fully equidistant discrete case the two D-Lemmata are useful. See [15].

C-Lemma 1. (for jump densities) *Assume* $w(x) \geq 0$, $w(x) = w(-x)$ *for* $x \in \mathbb{R}$, $\int_{-\infty}^{\infty} w(x)dx = 1$ *and either* (a) *or* (b).

(a) $\sigma^2 := \int_{-\infty}^{\infty} x^2 w(x)dx < \infty$, *labelled as* $\alpha = 2$,

(b) $w(x) \sim b|x|^{-(\alpha+1)}$ *for* $|x| \to \infty$, $\alpha \in (0,2)$ *and* $b > 0$.

Then with μ *as in* (2.2), *we have the asymptotics*

$$1 - \widehat{w}(\kappa) \sim \mu|\kappa|^\alpha \text{ for } \kappa \to 0. \tag{2.3}$$

C-Lemma 2. (for waiting time densities) *Assume* $\phi(t) \geq 0$ *for* $t > 0$, $\int_0^\infty \phi(t)dt = 1$, *and either* (A) *or* (B).

(A) $\rho := \int_0^\infty t\phi(t)dt < \infty$, *labelled as* $\beta = 1$,

(B) $\phi(t) \sim ct^{-(\beta+1)}$ *for* $t \to \infty$, $\beta \in (0,1)$ *and* $c > 0$.

Then with λ *as in* (2.2), *we have the asymptotics*

$$1 - \widetilde{\phi}(s) \sim \lambda s^\beta. \tag{2.4}$$

D-Lemma 1. *Assume* $p_k \geq 0$, $\sum_{-\infty}^{\infty} p_k = 1$, *symmetry* $p_k = p_{-k}$ *for all integers* $k \in \mathbb{Z}$, *and either* (a) *or* (b).

(a) $\sigma^2 := \sum_{-\infty}^{\infty} k^2 p_k < \infty$, *labelled as* $\alpha = 2$,

(b) $p_k \sim b|k|^{-(\alpha+1)}$ *for* $|k| \to \infty$, $\alpha \in (0,2)$ *and* $b > 0$.

Then with μ *as in* (2.1) *we have the asymptotics* (2.3).

D-Lemma 2. *Assume* $c_n \geq 0$, $\sum_{n=1}^{\infty} c_n = 1$ *and either* (A) *or* (B).

(A) $\rho := \sum_1^\infty nc_n < \infty$, *labelled as* $\beta = 1$,

(B) $c_n \sim cn^{-(\beta+1)}$ *for* $n \to \infty$, $\beta \in (0,1)$ *and* $c > 0$.

Then with λ *as in* (2.2) *we have the asymptotics* (2.4).

3. Well-Scaled Passage to the Diffusion Limit

As already indicated in the introduction, we multiply the jumps X_k by a factor h, the waiting times T_k by a factor τ. So, we get a transformed random walk

$S_n(h) = \sum_{k=1}^{n} hX_k$ with jump instants $t_n(\tau) = \sum_{k=1}^{n} \tau T_k$ that we now investigate with the aim of passing to the limit $h \to 0$, $\tau \to 0$ under a scaling relation between h and τ yet to be established, assuming that the conditions of Master Lemma 1 and Master Lemma 2 are fulfilled. As it is convenient to work in the Fourier-Laplace domain we note that the density $\phi_\tau(t)$ of the reduced waiting times τT_k and the density $w_h(x)$ of the reduced jumps hX_k are $\phi_\tau(t) = \phi(t/\tau)/\tau$, $t \geq 0$; $w_h(x) = w(x/h)/h$, $-\infty < x < \infty$. The corresponding transforms are simply $\widetilde{\phi_\tau}(s) = \widetilde{\phi}(s\tau)$, $\widehat{w_h}(\kappa) = \widehat{w}(\kappa h)$. We are interested in the sojourn probability density $p_{h,\tau}(x,t)$ of the particle subject to the transformed random walk. In analogy to the Montroll-Weiss equation (1.4) we get

$$\widehat{\widetilde{p}}_{h,\tau}(\kappa,s) = \frac{1-\widetilde{\phi}_\tau(s)}{s}\,\frac{1}{1-\widehat{w}_h(\kappa)\widetilde{\phi}_\tau(s)} = \frac{1-\widetilde{\phi}(\tau s)}{s}\,\frac{1}{1-\widehat{w}(h\kappa)\widetilde{\phi}(s\tau)}. \tag{3.1}$$

Considering now s and κ *fixed* and $\neq 0$ we find for $h \to 0$, $\tau \to 0$ from the Master Lemmata (replacing there κ by κh, s by $s\tau$) by a trivial calculation, omitting asymptotically negligible terms and using the slow variation property $L(1/(\kappa h)) \sim L(1/h)$, $M(1/(s\tau)) \sim M(1/\tau)$, the asymptotics (3.2) with (3.3).

$$\widehat{\widetilde{p}}_{h,\tau}(\kappa,s) = \frac{\lambda\tau^\beta s^{\beta-1} M(1/\tau)}{\mu(h|\kappa|)^\alpha L(1/h) + \lambda(\tau s)^\beta M(1/\tau)} = \frac{s^{\beta-1}}{r(h,\tau)|\kappa|^\alpha + s^\beta}, \tag{3.2}$$

$$r(h,\tau) = \frac{\mu h^\alpha L(1/h)}{\lambda \tau^\beta M(1/\tau)}. \tag{3.3}$$

So we see that for every fixed real $\kappa \neq 0$ and positive s

$$\widehat{\widetilde{p}}_{h,\tau}(\kappa,s) \;\to\; \frac{s^{\beta-1}}{|\kappa|^\alpha + s^\beta} = \widehat{\widetilde{u}}(\kappa,s), \tag{3.4}$$

as h and τ tend to zero under the *scaling relation* $r(h,\tau) \equiv 1$. Comparing with (1.6) we recognize here $\widehat{\widetilde{u}}(\kappa,s)$ as the combined Fourier-Laplace transform of the solution to the Cauchy problem (1.5). Invoking now the continuity theorems of probability theory (compare [7]) we see that the time-parameterized sojourn probability density converges weakly (or in law) to the solution of the Cauchy problem (1.5). We state this result at the following theorem.

Theorem. *Assume the probability laws for the jumps X_k and the waiting times T_k to fulfill the conditions of the Master Lemmata* 1 *and* 2*, respectively. Replace the jumps by hX_k, the waiting times τT_k. Then for h (and consequently τ) tending to zero the solution $p_{h,\tau}(x,t)$ of the rescaled integral equation* (1.5) (*the densities there to be decorated with indices h and τ*) *converges weakly to the solution of the Cauchy problem* (1.5)*, in other words: to the fundamental solution of the space-time fractional diffusion equation $\,{}_tD_*^\beta\, u(x,t) = R^\alpha\, u(x,t)$.*

4. Examples of Random Walks

Let us first consider the *space-time fractional* diffusion equation more closely with regard to special choices of the parameters α and β. In the very particular case $\alpha = 2,\ \beta = 1$ it reduces to the *classical* diffusion equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$. In the case $\alpha = 2$, $0 < \beta < 1$ we have the *time-fractional* diffusion equation investigated in 1989 [28]. In the case $0 < \alpha < 2,\ \beta = 1$ we have the *space-fractional* diffusion equation for which the fundamental solution is a symmetric strictly stable probability density evolving in time. Whereas the time-fractional case $\alpha = 2$, $0 < \beta < 1$ exhibits *subdiffusive* behaviour, we have *superdiffusive* behaviour, if $0 < \alpha < 2$. All this can be deduced from the Fourier–Laplace representation (1.6) by observing that the variance $\langle (x(t))^2 \rangle = (\sigma(t))^2 = \int\limits_{-\infty}^{\infty} x^2 u(x,t)dx$ of the position $x(t)$ of a diffusing particle is given as $-\frac{\partial^2}{\partial \kappa^2} \widehat{u}(\kappa, t)|_{\kappa=0}$. Using the Mittag–Leffler function $E_\beta(z) = \sum\limits_{n=0}^{\infty} \frac{z^n}{\Gamma(1+n\beta)}$ (see [10]) we find by Laplace inversion the convergent series $\widehat{u}(\kappa,t) = E_\beta(-|\kappa|^\alpha t^\beta) = 1 - \frac{|\kappa|^\alpha t^\beta}{\Gamma(1+\beta)} + \frac{|\kappa|^{2\alpha} t^{2\beta}}{\Gamma(1+2\beta)} - + \cdots$ from which for $t > 0$ we get $(\sigma(t))^2 = \frac{2t^\beta}{\Gamma(1+\beta)}$ if $\alpha = 2$, $(\sigma(t))^2 = \infty$ if $0 < \alpha < 2$.

Now we present two concrete random walk models, for both assuming $0 < \alpha < 2$, $0 < \beta < 1$. *The first example* is in continuous time and continuous space. From [4] we got the idea that it is advantageous to work with functions $W(x)$ and $\Phi(t)$ that can elementarily be inverted. This is useful to produce from $[0, 1)$-uniformly distributed pseudo-random numbers the jumps and the waiting times in simulations. We take, compare [11] and [20],

$$W(x) = \frac{1}{2} + \frac{1}{2}(-1)^{\mathrm{sign}(x)} \frac{|x|^\alpha}{1 + |x|^\alpha},\ \Phi(t) = 1 - \frac{1}{1 + \Gamma(1-\beta)t^\beta}.$$

Then we have case (b) of Master Lemma 1 with $b = \alpha/2$ and $L(x) \equiv 1$, and case (B) of Master Lemma 2 with $c = 1/|\Gamma(-\beta)|$, $M(t) \equiv 1$.

In the *second example* time and space are equidistantly discretized. We take $p_0 = 0$, $p_k = b|k|^{-(\alpha+1)}$ for $0 \neq k \in \mathbb{Z}$, $c_n = cn^{-(\beta+1)}$ for $n \in \mathbb{N}$, and set $w(x) = \sum\limits_{-\infty}^{\infty} p_k \delta(x-k)$, $\phi(t) = \sum\limits_{k=1}^{\infty} c_k \delta(t-k)$ (compare with [15]). We have case (b) of D-Lemma 1, case (B) of D-Lemma 2 and identify readily (with $\zeta(z)$ denoting Riemann's zeta function) $b = \frac{1}{2\zeta(\alpha+1)}$ and $c = \frac{1}{\zeta(\beta+1)}$. The sequences $p_1, p_2, p_3, \ldots$ and $c_1, c_2, c_3, \ldots$ have the nice property of being completely monotone. The excluded border cases $\alpha = 2$ and $\beta = 1$ are singular.

To convey to the reader a feeling for fractional diffusion we present a few graphical results of approximating random walks, simulated according to the first example. They show in sequence the case of Brownian motion (classical diffusion), time-fractional diffusion, space-fractional diffusion, space-time-fractional diffusion.

Note that for $\alpha = 2$, we use the jump density $w(x) = \frac{1}{\sqrt{2\pi}} \exp(-\frac{x^2}{2})$, for $\beta = 1$ the waiting time density $\phi(t) = \exp(-t)$. For $0 < \alpha < 2$ and $0 < \beta < 1$ we take the jump distribution $W(x)$ and the waiting time distribution $\Phi(t)$, both as in the first example.

Observe some long waiting times in the case $0 < \beta < 1$ and some long jumps in the case $0 < \alpha < 2$.

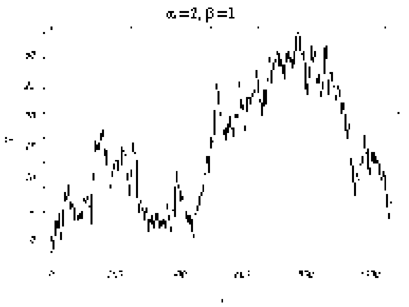

Figure 1. Normal diffusion

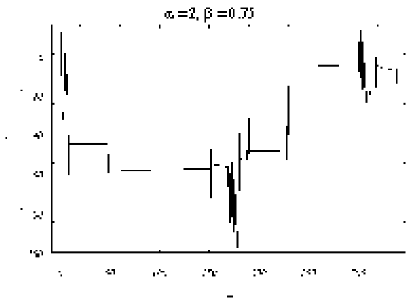

Figure 2. Time-fractional

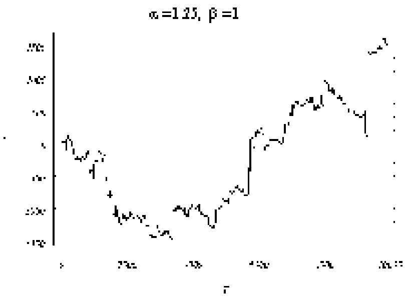

Figure 3. Space - fractional

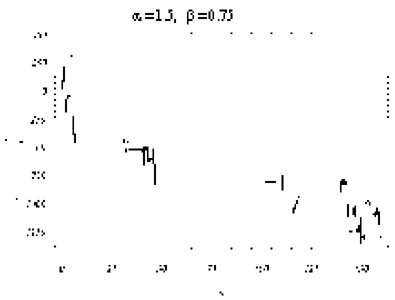

Figure 4. Space - time fractional

5. Comments, Suggestions and Conclusions

The theory of compound renewal processes, also called renewal processes with reward, in physics and other natural sciences called *continuous time random walks* (though space and time need not be continuous) began to flourish in the middle of the sixties of the past century, let us quote [23] and [24]. We cannot give here a comprehensive survey of relevant literature, so we ask all not mentioned contributors to forgive us this surely biased account. For larger lists of references and more competent appreciation of achievements and applications we recommend [2] and [23]. As an early pioneer Balakrishnan [1] deserves to be put into light. He has, in 1985, found the time-fractional diffusion equation ($\alpha = 2$, $0 < \beta < 1$) as the properly scaled diffusion limit for some random walks with power law waiting time. At that time, four years before in [28] the basic analytic theory was developed, the name *fractional diffusion* was not yet common, and so was not used in [1], hence Balakrishnan did not find the resonance he would have deserved. A decisive step forward occurred in [17] in 1995. Hilfer and Anton, roughly speaking, showed among other things that by taking the Mittag–Leffler waiting time density $-\frac{d}{dt}E_\beta(-t^\beta)$ the basic equation of continuous time random walk can be transformed to a time-fractional evolution equation for the sojourn probability density. Thus they have essentially found the time-fractional generalization of the Kolmogorov-Feller evolution equation for the compound Poisson process which e.g. is treated in [17]. However, already in [1] appears the waiting time density whose Laplace transform is $(1+s^\beta)^{-1}$ as playing a distinct role, but was not recognized as a function of Mittag–Leffler type (such functions too long having been insufficiently known). Gorenflo and Mainardi and co-authors have, beginning in 1998, published several papers on various types of approximating random walks for space-fractional and space-time fractional diffusion processes of which we quote[11 - 13] and [15], furthermore some papers (stressing the relevance of the Mittag–Leffler waiting time) motivated by applications to finance: [14, 20, 27]. In [14] the space-time fractional diffusion equation is obtained as a diffusion limit of the time-fractionalized Kolmogorov–Feller equation

$$\underset{t\ *}{D}^{\beta}\, p(x,t) = -p(x,t) + \int_{-\infty}^{\infty} w(x-x')p(x',t)dx'. \tag{5.1}$$

The publications [18] and [19] are devoted to analytic treatment via integral representations of the evolving probability densities that solve the (spatially one-dimensional) space-time fractional diffusion equation. An important concept in fractional diffusion processes is the concept of subordination (see, e.g. [22] and [21]). By our way of relating the scaling parameters in the passage to the limit we circumvent this concept. Let us mention again [12]. There we have based our scaled transition to the limit on two lemmata for the asymptotics of the transforms of the densities whereas here we work with the Master Lemmata for the distribution functions, motivated by [9]. This, of course, is more general and allows discrete and continuous probabilities and mixtures of them. However, in not so general situations the other lemmata may be simpler to apply (as we have done in [15] for the fully discrete case with regular grids). In [29], in contrast to our treatment in [12] and here, the scaling is not done via the individual steps in space and time but directly in the distributions of waiting times and jumps. However, this is equivalent to our way. Let us in this context say a few words to the essential statement of [16]. There Hilfer shows that a power law for the waiting time is not sufficient for getting in the limit a fractional diffusion process with the fractional time derivative having the same order as the power law. This seemingly negative result, however, does not hit the theory expanded here in our paper. In Hilfer's counter-example the passage to the diffusion limit is not well-scaled in our sense; in fact, in it are hidden two different scalings. Thus [16] may inspire to investigate systematically continuous time random walks that can be scaled in more than one way.

Let us, as a final statement, say that the case of non-symmetric jump distributions (bypassed in our paper) can analogously be studied.

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