Vietnam Journal of Mathematics **32:** *SI* (2004) 143–152

Vietnam Journal
of
MATHEMATICS

On a Class of Severely Ill-Posed Problems

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Abstract. The problem of calculating the value of the pseudo-differential operator $a(D,y)\varphi, y \in [0,1]$ with unbounded symbol $a(i\xi, y)$ is ill-posed. If the symbol $a(i\xi, y)$ behaves like an exponential function of ξ, then the problem is severely ill-posed. This note is devoted to the last case. The mollification method [2] is used to regularize the problem in the general L_p space setting. Error estimates of Hölder type for the regularized values and the exact values are derived. Applications of the general scheme to concrete problems from practice are presented.

1. Introduction

Let $a(i\xi, y),\ \xi \in \mathbb{R},\ y \in [0,1]$, be a given function. We define the pseudo-differential operator

$$a(D,y)\varphi(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} a(i\xi, y)\hat{\varphi}(\xi)e^{ix\xi}d\xi. \tag{1.1}$$

Here, D stands for d/dx and we use the notation $\hat{}$ for the Fourier transform. Namely, let $g \in L_1(\mathbb{R})$, we denote the Fourier transform of g by [6, p. 32]

$$\hat{g}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x)e^{-ix\xi}d\xi.$$

We shall indicate the domain of definition for $a(D,y)$ later on.

Many problems in practice lead to calculating or approximating the operator $a(D,y)$. In this note we pay attention to the case of the unbounded operators $a(D,y)$. This situation is frequently met, for example, when one is dealing with numerical differentiation, analytic continuation, parabolic equations backwards in time, the Cauchy problem for elliptic equations, etc. (see, e.g., [2]). A serious difficulty in this case is that calculating $a(D,y)$ is an ill-posed problem. It means that a small perturbation in the data φ might cause arbitrarily large errors in the value of the operator. The problem is severely ill-posed when for large $|\xi|$ the symbol $a(i\xi,y)$ behaves itself as an exponential function of $|\xi|$ (see (2.1), (3.1)).

Now let the operator $a(D,y)$ be defined for some $\varphi \in L_p(\mathbb{R}), 1 \le p \le \infty$. Let φ be approximately given by $\varphi_\varepsilon \in L_p(\mathbb{R})$ such that

$$\|\varphi - \varphi_\varepsilon\|_p \le \varepsilon. \tag{1.2}$$

Here, $\|\cdot\|_p$ denote the norm in $L_p(\mathbb{R})$. Our problem is to approximate the value

$$u(x,y) := a(D,y)\varphi(x) \tag{1.3}$$

from the data φ_ε in a stable way.

We shall use the mollification method [2] to regularize our problem. Namely, we shall mollify the measured data φ_ε by convolution with an appropriate kernel so that the problem (1.3) is well-posed with these new data. We shall show how to choose the mollification parameter so that the error estimate between the regularized value and the exact value of $a(D,y)\varphi$ is of Hölder type. To this aim we impose the following "traditional" constrain: $u(\cdot,1) \in L_p(\mathbb{R})$ and there is a positive constant M such that

$$\|u(\cdot,1)\|_p \le M. \tag{1.4}$$

We shall separate the case $p = 2$ from the other ones, since in this case we can use many nice properties of the Hilbert space $L_2(\mathbb{R})$ rather than the remained ones. We emphasize that stability results for the case when $p \ne 2$, due to its difficulties, have been very little published in the literature.

In this note we shall make use of the following notation: $\mathfrak{M}_{\nu,p}$ $(1 \le p \le \infty)$ will denote the collection of all entire functions of exponential type ν which as functions of a real $x \in \mathbb{R}$ lie in $L_p = L_p(\mathbb{R})$ [6, p. 100]. We shall denote by $E_{\nu,p}(f)$ the best approximation of f using elements of $\mathfrak{M}_{\nu,p}$ [6, p. 184], i.e.,

$$E_{\nu,p}(f) = \inf_{g\in \mathfrak{M}_{\nu,p}} \|f - g\|_{L_p(\mathbb{R})}.$$

2. The L_2-Case

Suppose that $a(\cdot,\cdot)$ is a continuous function of its variables. Furthermore, there are positive constants c_1, c_2 and τ, ρ such that

$$c_1 \exp(\tau y|\xi|^\rho) \le |a(i\xi,y)| \le c_2 \exp(\tau y|\xi|^\rho), \tag{2.1}$$

Further, at $y = 1$ we suppose that there is a positive constant c_3 such that

$$c_3 \exp(\tau|\xi|^\rho) \le |a(i\xi,1)|. \tag{2.2}$$

The condition (2.1) says that the function a behaves as an exponential function of $|\xi|$ and thus the problem (1.3) is severely ill-posed.

We assume that the operator $a(D,y)$ is defined for some $\varphi \in L_2(\mathbb{R})$ and its value $u(x,y) = a(D,y)\varphi(x)$ belongs to $L_2(\mathbb{R})$ for any $y \in [0,1]$. Because of the conditions on $a(\cdot,\cdot)$ we see that φ must be at least infinitely differentiable. Let now φ be approximately given by $\varphi_\varepsilon \in L_2(\mathbb{R})$ such that the inequality (1.2) is valid for $p = 2$.

It is clear that in general $a(D, y)$ is not applicable to φ_ε, and when it is applicable, its value may be quite different from that of $u(x, y)$. To overcome this difficulty we mollify φ_ε by convolution with the Dirichlet kernel. Namely, we define

$$\varphi_{\varepsilon,\nu}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\sin(\nu(x-z))}{x-z}\varphi_\varepsilon(z)dz.$$

Although φ_ε in general is not differentiable, the function $\varphi_{\varepsilon,\nu}$ is an entire function of exponential type ν [6, p. 316 - 318]) and it Fourier transform has compact support containing in $[-\nu, \nu]$. Furthermore,

$$\hat{\varphi}_{\varepsilon,\nu}(\xi) = \chi_{[-\nu,\nu]}(\xi)\hat{\varphi}_\varepsilon(\xi)$$

with $\chi_{[-\nu,\nu]}(\xi) = 1$ for $\xi \in [-\nu, \nu]$ and 0 otherwise.
Set

$$u_{\varepsilon,\nu}(x, y) := a(D, y)\varphi_{\varepsilon,\varphi}(x).$$

We have

$$\begin{aligned}\|u_{\varepsilon,\nu}(\cdot, y)\|_2 = \|\hat{u}_{\varepsilon,\nu}(\cdot, y)\|_2 = \|a(\cdot, y)\hat{\varphi}_{\varepsilon,\nu}\|_2 \\ \le c_2 \exp(\tau y \nu^\rho)\|\varphi_\varepsilon\|_2.\end{aligned}$$

Thus, the problem of computing $u_{\varepsilon,\nu}(x, y) = a(D, y)\varphi_{\varepsilon,\nu}$ is stable for fixed ν. Now we estimate the difference between u and $u_{\varepsilon,\nu}$. We note that the inequality (2.2) and the equalities

$$\hat{u}(\xi, y) = a(i\xi, y)\hat{\varphi}(\xi)$$

and

$$\hat{u}(\xi, 1) = a(i\xi, 1)\hat{\varphi}(\xi) = \hat{\psi}(\xi)$$

yield

$$\hat{u}(\xi, y) = \frac{a(i\xi, y)}{a(i\xi, 1)}\hat{\psi}(\xi).$$

Further, with

$$u_{0,\nu}(x, y) := \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\sin(\nu(x-z))}{x-z}u(z, y)dz,$$

we have

$$u_{\varepsilon,\nu} - u = u_{\varepsilon,\nu} - u_{0,\nu} + u_{0,\nu} - u.$$

Now, on one hand, by virtue of (2.1), (2.2) and (1.4),

$$\begin{aligned}\|u_{0,\nu}(\cdot, y) - u(\cdot, y)\|_2 &= \|\hat{u}_{0,\nu}(\cdot, y) - \hat{u}(\cdot, y)\|_2 \\ &= \Big(\int_{-\infty}^{\infty}\Big|\frac{a(i\xi, y)}{a(i\xi, 1)}\chi_{[-\nu,\nu]}(\xi)\psi(\xi) - \frac{a(i\xi, y)}{a(i\xi, 1)}(\xi)\psi(\xi)\Big|^2 d\xi\Big)^{1/2} \\ &= \Big(\int_{|\xi|\ge\nu}\Big|\frac{a(i\xi, y)}{a(i\xi, 1)}\Big|^2|\psi(\xi)|^2 d\xi\Big)^{1/2} \\ &\le \frac{c_2}{c_3}\exp(\tau(y-1)\nu^\rho)M.\end{aligned}$$

On the other hand, because of (2.1) and (1.2),

$$\begin{aligned}\|u_{\varepsilon,\nu}(\cdot,y)-u_{0,\nu}(\cdot,y)\|_2 &= \|\hat{u}_{\varepsilon,\nu}(\cdot,y)-\hat{u}_{0,\nu}(\cdot,y)\|_2\\ &= \Big(\int_{-\nu}^{\nu}|a(i\xi,y)(\hat{\varphi}(\xi)-\hat{\varphi}_\varepsilon(\xi))|^2 d\xi\Big)^{1/2}\\ &\le \max_{|\xi|\le\nu}|a(i\xi,y)|\Big(\int_{-\nu}^{\nu}|(\hat{\varphi}(\xi)-\hat{\varphi}_\varepsilon(\xi))|^2 d\xi\Big)^{1/2}\\ &\le c_2\exp(\tau y\nu^\rho)\epsilon.\end{aligned}$$

Hence

$$\|u_{\varepsilon,\nu}(\cdot,y)-u(\cdot,y)\|_2 \le c_2\exp(\tau y\nu^\rho)\epsilon+\frac{c_2}{c_3}\exp(\tau(y-1)\nu^\rho)M.$$

Taking, for example,

$$\nu=\nu^*=\Big(\frac{1}{\tau}\ln\frac{M}{\varepsilon}\Big)^{1/\rho} \tag{2.3}$$

we get

$$\|u_{\varepsilon,\nu^*}(\cdot,y)-u(\cdot,y)\|_2\le cM^y\varepsilon^{1-y} \tag{2.4}$$

with $c=c_2+c_2/c_3$.

3. The L_p-Case ($p\in[1,\infty]$)

In this section we suppose that

1. for every $y\in[0,1]$, $a(z,y)$ is an entire function with respect to $z\in\mathbb{C}$ and for all $r\in\mathbb{R}^+$ there are positive constants c_4,τ,ρ such that
$$\max_{|z|\le r}|a(z,y)|\le c_4\exp(\tau y r^\rho), \tag{3.1}$$
2. $u(x,y)$ can be represented in the form
$$u(x,y)=v(\cdot,y)*\psi(\cdot),\quad v(\cdot,y)\in L_1(\mathbb{R}),\ \psi\in L_p(\mathbb{R}), \tag{3.2}$$
3. there exist positive constants $c_5,\alpha,\beta\ge\rho$ such that
$$E_{\nu,1}(v(\cdot,y))\le c_5\exp(\alpha(y-1)\nu^\beta)\ \text{for}\ y\in[0,1). \tag{3.3}$$

We need the following result.

Lemma. (A rough generalization of Bernstein's inequality) *Let the function* $a(z,y)$ *satisfy the condition* (3.1). *Then for any* $\varphi\in\mathfrak{M}_{\nu,p}$ *and* $\nu\ge1$, *with* $c_6=c_4(3\tau\rho+3^\rho/(3^\rho-e^\rho))$, *we have the inequality*

$$\|a(D,y)\varphi\|_p\le c_6\nu^\rho\exp(\tau y\nu^\rho)\|\varphi\|_p.$$

Proof. Since $a(z,y)$ is an entire function for every $y\in[0,1]$, it can be represented in the form

$$a(z,y) := \sum_{n=0}^{\infty} a_n(y) z^n.$$

The Cauchy inequality and the inequality (3.1) yield

$$|a_n(y)| \le c_4 \frac{\exp(\tau y r^\rho)}{r^n}, \quad \forall r > 0.$$

It can be verified that the right hand side of the foregoing inequality attains its minimum at

$$r = \left(\frac{n}{\tau y \rho}\right)^{1/\rho}.$$

It follows that

$$|a_n(y)| \le c_4 \left(\frac{e\tau y \rho}{n}\right)^{n/\rho}.$$

Thus, following Bernstein's inequality [6, p. 116]), we get

$$\begin{aligned}
\|a(D,y)\varphi\|_p &= \|\sum_{n=0}^{\infty} a_n(y) D^n \varphi\|_p \\
&\le \sum_{n=0}^{\infty} |a_n(y)| \|D^n \varphi\|_p \\
&\le \sum_{n=0}^{\infty} |a_n(y)| \nu^n \|\varphi\|_p \\
&\le c_4 \|\varphi\|_p \sum_{n=0}^{\infty} \left(\frac{e\tau y \rho \nu^\rho}{n}\right)^{n/\rho} \\
&:= c_4 \|\varphi\|_p \sum_{n=0}^{\infty} \left(\frac{q}{n}\right)^{n/\rho}.
\end{aligned}$$

Here $q = e\tau y \rho \nu^\rho$.

The function of $p > 0$

$$g(p) = \left(\frac{q}{p}\right)^{p/\rho}$$

attains its maximum at $p = q/e$. Hence

$$\begin{aligned}
\sum_{n=0}^{\infty} \left(\frac{q}{n}\right)^{n/\rho} &= \sum_{n \le 3q/e} \left(\frac{q}{n}\right)^{n/\rho} + \sum_{n > 3q/e} \left(\frac{q}{n}\right)^{n/\rho} \\
&\le 3\frac{q}{e} \exp\left(\frac{q}{e\rho}\right) + \sum_{n > 3q/e} \left(\frac{e}{3}\right)^{n/\rho} \\
&< 3\frac{q}{e} \exp\left(\frac{q}{e\rho}\right) + \frac{3^\rho}{3^\rho - e^\rho}.
\end{aligned}$$

Thus,

$$\|a(D,y)\varphi\|_p \le c_4\|\varphi\|_p\Big(3\tau\rho\nu^\rho \exp(\tau\nu^\rho) + \frac{3^\rho}{3^\rho - e^\rho}\Big).$$

The last yields the inequality in the lemma for $\nu \ge 1$.

We note that the result of this lemma is rather weak. If we impose some more conditions on the function a, then we may get better estimates. For example, if

$$a(z,y) = \sum_{n=0}^{\infty} a_n(y) z^n$$

with $a_n(y) \ge 0$, then for $\varphi \in \mathfrak{M}_{\nu,p}$,

$$\|a(D,y)\varphi\|_p \le a(\nu,y)\|\varphi\|_p.$$

Or with $p = 2$,

$$\|a(D,y)\varphi\|_2 \le \max_{|\xi|\le\nu} |a(i\xi,y)|\|\varphi\|_2.$$

Now we introduce the de la Vallée Poussin kernel [6, p. 304] which is defined by

$$k_\nu(x) := \frac{1}{\pi\nu}\frac{\cos(\nu x) - \cos(2\nu x)}{\nu^2}\ , \quad \nu > 0.$$

This kernel belongs to $\mathfrak{M}_{2\nu,1}$ and has many nice properties. In particular, the convolution of a function $\varphi \in L_p(\mathbb{R})$ with this kernel belongs to $\mathfrak{M}_{2\nu,p}$ [6, p. 304-306]:

$$\varphi_\nu(x) := \int_{-\infty}^{\infty} k_\nu(x-z)\varphi(z)dz = k_\nu * \varphi \in \mathfrak{M}_{2\nu,p}$$

and

$$\|\varphi_\nu - \varphi\|_p \le (1 + 2\sqrt{3})E_{\nu,p}(\varphi).$$

Further,

$$\|\varphi_\nu\|_p \le 2\sqrt{3}\|\varphi\|_p.$$

We are now in a position to regularize our problem. To do that first we mollify φ_ε by convolution with the de la Vallée Poussin kernel:

$$\varphi_{\varepsilon,\nu}(x) := \int_{-\infty}^{\infty} k_\nu(x-z)\varphi_\varepsilon(z)dz := k_\nu * \varphi_\varepsilon.$$

From Lemma we see that the problem of calculating

$$u_{\varepsilon,\nu} := a(D,y)\varphi_{\varepsilon,\nu}(x)$$

is stable in the L_p-norm for fixed $\nu > 0$. With

$$u_{0,\nu} = k_\nu * u,$$

we have

$$u_{\varepsilon,\nu} - u = u_{\varepsilon,\nu} - u_{0,\nu} + u_{0,\nu} - u.$$

For $\nu \geq 1$, from Lemma and (1.2),

$$\|u_{\varepsilon,\nu}(\cdot,y) - u_{0,\nu}(\cdot,y)\|_p \leq c_6 2\sqrt{3}(2\nu)^\rho \exp(\tau y(2\nu)^\rho)\varepsilon.$$

On the other hand, by virtue of (3.2), (3.3) and the properties of the de la Vallée Poussin kernel,

$$\begin{aligned}\|u_{0,\nu}(\cdot,y) - u(\cdot,y)\|_p &= \|k_\nu * (v(\cdot,y) * \psi(\cdot)) - v(\cdot,y) * \psi(\cdot)\|_p \\ &= \|(k_\nu * v(\cdot,y)) * \psi(\cdot) - v(\cdot,y) * \psi\|_p \\ &= \|(k_\nu * v(\cdot,y) - v(\cdot,y)) * \psi(\cdot)\|_p \\ &\leq \|k_\nu * v(\cdot,y) - v(\cdot,y)\|_1 \|\psi\|_p \\ &\leq c_5 \exp(\alpha(y-1)\nu^\beta)M.\end{aligned}$$

Thus,

$$\|u_{\varepsilon,\nu}(\cdot,y) - u(\cdot,y)\|_p \leq c_6 2\sqrt{3}(2\nu)^\rho \exp(\tau y(2\nu)^\rho)\varepsilon + c_5 \exp(\alpha(y-1)\nu^\beta)M.$$

Since $\rho \leq \beta$,

$$\|u_{\varepsilon,\nu}(\cdot,y) - u(\cdot,y)\|_p \leq c_6 2\sqrt{3}(2\nu)^\rho \exp(\tau y(2\nu)^\rho)\varepsilon + c_5 \exp(\alpha(y-1)\nu^\rho)M.$$

For ε small enough, taking, for example,

$$\nu = \nu^{**} = \Big(\frac{1}{\tau y 2^\rho + \alpha(1-y)} \ln \frac{M}{\varepsilon}\Big)^{1/\rho}, \tag{3.4}$$

we arrive at the inequality

$$\begin{aligned}&\|u_{\varepsilon,\nu^{**}}(\cdot,y) - u(\cdot,y)\|_p \\ &\leq \Big(c_6 \frac{2\sqrt{3}}{\tau y 2^\rho + \alpha(1-y)} \ln \frac{M}{\varepsilon} + c_5\Big) M^{1-\frac{\tau y 2^\rho}{\tau 2^\rho y + \alpha(1-y)}} \varepsilon^{\frac{\alpha(1-y)}{\tau 2^\rho y + \alpha(1-y)}},\end{aligned} \tag{3.5}$$

which is of Hölder type.

4. Applications

4.1. Analytic Continuation

1. Problem A. Let $\sigma > 0$ be a given number. We denote the strip $z = x+iy \in \mathbb{C}$ with $|y| < \sigma$ by Ω. Let $f(z)$ be analytic in Ω. Suppose that f is given only on the real axis, we have to extend f analytically from this data to the strip Ω.

2. Problem B. Let R be a given positive constant greater than 1. Set

$$x(\rho) = \frac{1}{2}\Big(\rho + \frac{1}{\rho}\Big)\cos\varphi, \;\; y(\rho) = \frac{1}{2}\Big(\rho - \frac{1}{\rho}\Big)\sin\varphi, \quad 1 \leq \rho \leq R, \;\; 0 \leq \varphi < 2\pi.$$

We see that

$$\frac{x^2(\rho)}{a^2(\rho)} + \frac{y^2(\rho)}{b^2(\rho)} = 1, \ a(\rho) = \frac{1}{2}\Big(\rho + \frac{1}{\rho}\Big), \ b(\rho) = \frac{1}{2}\Big(\rho - \frac{1}{\rho}\Big)$$

describes an ellipse (denoting by $\Omega(\rho)$) in the complex plane with the foci at ± 1 and the sum of the semi-axes equalling ρ. Let $f(z)$ be analytic in the interior of the ellipse with $\Omega(R)$, and given only in the interval $[-1, 1]$. The problem is to compute $f(z)$ in $\Omega(R) \setminus [-1, 1]$.

We note that Problem B can be easily transformed to Problem A.

Suppose that for any $y \in [-\sigma, \sigma]$ the function $f(\cdot + iy) \in L_p(\mathbb{R})$ with $1 \leq p \leq \infty$. Further, we suppose that $f(x)$ is real-valued and there is a positive constant M such that $\|f(\cdot \pm i\sigma)\|_p \leq M$. Instead of the exact $f(x)$ it is supposed that $f_\varepsilon(x)$ is given, where $f_\varepsilon \in L_p(\mathbb{R})$, and

$$\|f - f_\varepsilon\|_p \leq \varepsilon.$$

Here, $\varepsilon > 0$ is the noise level.
Since f is analytic in Ω,

$$f(z) = f(x + iy) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x)}{n!}(iy)^n = \sum_{n=0}^{\infty} \frac{(iy)^n}{n!} D^n f(x).$$

Set

$$a(D, y) := \sum_{n=0}^{\infty} \frac{(iy)^n}{n!} D^n = e^{iyD}.$$

We can say that the problem of analytic continuation is in fact an application of the "complex" translation operator to the function under consideration. The differential operator of infinite order $a(D, y)$ has not always a meaning for general functions f and when it has a meaning, it might be unbounded. It is proved that [4] the problem of analytic continuation is severely ill-possed and belongs to the class (1.3).

4.2. The Cauchy Problem for Laplace's Equation

The Cauchy problem

$$\begin{aligned} u_{xx}(x, y) + u_{yy}(x, y) = 0, &\quad -\infty < x < \infty, \ 0 < y < 1, \\ u(x, 0) = \varphi(x), &\quad -\infty < x < \infty, \\ u_y(x, 0) = 0, &\quad -\infty < x < \infty \end{aligned}$$

is well-known to be severely ill-posed. Formally, the solution of this problem can be represented in the form [2]

$$u(x, y) = \sum_{n=0}^{\infty} \frac{y^{2n} D^{2n}}{(2n)!} \varphi(x) := \cosh(yD)\varphi(x).$$

To guarantee the stability of the solution we impose the condition

$$\|u(\cdot,1)\|_p \le M.$$

Consider the problem

$$v_{xx} + v_{yy} = 0, \quad -\infty < x < \infty,\ 0 < y < 1, \tag{4.1}$$
$$v_y(x,0) = 0, \quad -\infty < x < \infty, \tag{4.2}$$
$$v(x,1) = \delta(x), \quad -\infty < x < \infty. \tag{4.3}$$

Here δ is the Dirac function (see, e.g., [6]). It is proved that [3] $v(\cdot,y) \in L_1(\mathbb{R})$ for $y \in [0,1)$ and for $\sigma \in (0, 1-y)$ there is a constant c_7 such that

$$E_{\nu,1}\Big(\frac{\partial^{n+m} v(\cdot,y)}{\partial x^n \partial y^m}\Big) \le c_7 \frac{(n+m)!}{(1-y-\sigma)^{n+m}}, \quad n,m = 0,1,2,\dots .$$

Furthermore, the solution of the boundary value problem

$$u_{xx} + u_{yy} = 0,\ \ -\infty < x < \infty,\ \ 0 < y < 1,$$
$$u_y(x,0) = 0,\ \ u(x,1) = \psi(x),\ \ -\infty < x < \infty,$$

with $\psi \in L_p(\mathbb{R})$ can be represented in the form of the convolution

$$u(x,y) = v(\cdot,y) * \psi(\cdot).$$

Thus, the Cauchy problem for the Laplace equation can be considered as a special case of the problem (1.3). For more details, the reader is referred to [3].

4.3. The Heat Equation Backwards in Time

Consider the heat equation backwards in time

$$u_t(x,t) = u_{xx}(x,t),\ -\infty < x < \infty,\ 0 \le t \le T, \tag{4.4}$$
$$u(x,T) = u_T(x), -\infty < x < \infty. \tag{4.5}$$

Suppose that, the solution $u(\cdot,t)$ of the problem (4.4) - (4.5) and the Cauchy data $u_T(\cdot)$ are to be in $L_p(\mathbb{R})$, $1 < p \le \infty$, and instead of the exact $u_T(\cdot)$ we have only the measured data $u_T^\epsilon \in L_p(\mathbb{R})$ such that

$$\|u_T^\epsilon - u_T\|_p \le \epsilon. \tag{4.6}$$

Formally,

$$u(x,t) = \exp(-tD^2)u_T(x).$$

And to guarantee the stability of the solution, we suppose that

$$\|u(\cdot,0)\|_p \le M.$$

It is well known that

$$u(x,t) = \int_{-\infty}^{\infty} k(x-y,t)u(y,0)dy, \quad \text{for } p > 1,$$

where

$$k(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-x^2/(4t)}.$$

And in [2] it has been proved that

$$E_{\nu,1}(k(\cdot,t)) \le \frac{4}{\pi} e^{-t\nu^2}, \quad \text{if } \nu \ge \sqrt{3/(2t)}. \tag{4.7}$$

Thus, this problem belongs to the class (1.3).

4.4. Inversion of the Laplace Transform

Denote by

$$\mathcal{L}(f)(s) = \int_0^\infty e^{-sx} f(x) dx := F(s)$$

the Laplace transform of the function f. We are interested in inverting $\mathcal{L}$ from F, when it is given only for real s. Following [1] we set

$$Vf(x) = e^{x/2} f(e^x).$$

It has been proved in [1] that if $F \in H^2 \cup L_2(0,\infty)$ and $V\mathcal{L}F(x)$ has an analytic extension in the strip $\Omega := \{|\Im z| \le \pi\}$, then

$$\mathcal{L}^{-1}F(x) = V^{-1}(V\mathcal{L}F(x+i\pi) + V\mathcal{L}F(x-i\pi))/(2\pi)).$$

Here, H^2 stands for the Hardy space [5].

We see immediately from the last formula that the general scheme (1.3) is applicable to this particular case of inversion of the Laplace transform.

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