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Periodic Solutions of Forced Dissipative p-Liénard Equations with Singularities

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Abstract. Using a recent continuation theorem for periodic solutions of quasilinear equations of p-Laplacian type, we extend some results of Habets and Sanchez for positive periodic solutions to equations of the form

$$(|x'|^{p-2}x')' + f(x)x' + g(x) = h(t)$$

with $p > 1$, $|f(x)| \geq c > 0$ and g singular at 0.

1. Introduction

In the last fourty years, periodic solutions of forced second order nonlinear differential equations or systems with singular restoring forces have been considered by a number of authors, whose contributions are described and listed in [2]. Such nonlinearities occur naturally in the description of particules submitted to Newtonian type forces or to restoring forces caused by compressed gases.

Recently, various extensions of fundamental topological degree techniques to quasilinear equations or systems of the form

$$(|x'|^{p-1}x')' = f(t, x, x') \quad (p > 1),$$

in particular the upper and lower solutions method and continuation theorems, have been used in [2] to generalize the results of Lazer-Solimini [3] for the T-periodic boundary value problem with g singular at 0

$$x'' + g(x) = h(t), \quad x(0) = x(T), \quad x'(0) = x'(T),$$

to the T-periodic boundary value problem for the *p-Liénard equation*

$$(|x'|^{p-1}x')' + f(x)x' + g(x) = h(t), \quad x(0) = x(T), \quad x'(0) = x'(T). \tag{1}$$

Here $p > 1$, $f : [0, \infty[\to \mathbb{R}$ is an arbitrary continuous function, $g :]0, \infty[\to \mathbb{R}$ is continuous, $g(x) \to \pm\infty$ when $x \to 0+$ and satisfies some further conditions near $+\infty$ and/or zero. For $h \in L^r(0,T)$ $r \geq 1$, we write

$$\overline{h} = \frac{1}{T}\int_0^T h(t)\,dt, \quad \|h\|_r = \Big(\int_0^T |h(t)|^r\,dt\Big)^{1/r}.$$

We list here the main theorems of [2], in a slightly more general form which easily follows from the arguments of this paper.

Theorem 1. *Assume that $h^+ \in L^\infty(0,T)$, $h^- \in L^1(0,T)$ and that g satisfies the condition*

$$\limsup_{x\to+\infty} g(x) < \overline{h} \leq \sup_{[0,T]} h < \liminf_{x\to 0+} g(x). \tag{2}$$

Then problem (1) *has at least one positive solution.*

This theorem, proved in [2] using upper and lower solutions, implies the following result, first obtained by Lazer–Solimini [3] when $p = 2$, $f \equiv 0$, and h is continuous.

Corollary 1. *Equation*

$$(|x'|^{p-2}x')' + f(x)x' + \frac{\beta}{x^\alpha} = h(t)$$

with $f : [0, +\infty[\to \mathbb{R}$ continuous, $h^+ \in L^\infty(0,T)$, $h^- \in L^1(0,T)$, $\alpha > 0$, and $\beta > 0$, has a positive T-periodic solution if and only if $\overline{h} > 0$.

Theorem 2. *Assume that $h \in L^1(0,T)$, and that g satisfies the conditions*

$$\limsup_{x\to 0+} g(x) < \overline{h} < \liminf_{x\to\infty} g(x), \tag{3}$$

$$\int_0^1 g(u)du = -\infty, \tag{4}$$

$$\limsup_{x\to+\infty} g(x) < \infty. \tag{5}$$

Then Problem (1) *has at least one positive solution.*

This theorem, proved in [2] using a continuation theorem from [4], implies the following result, first obtained by Lazer–Solimini [3] when $p = 2$, $f \equiv 0$, and h is continuous.

Corollary 2. *Equation*

$$(|x'|^{p-2}x')' + f(x)x' - \frac{\beta}{x^\alpha} = h(t)$$

with $f : [0, +\infty[\to \mathbb{R}$ *continuous,* $\alpha \geq 1$, $\beta > 0$ *and* $h \in L^1(0, T)$ *has a positive* T*-periodic solution if and only if* $\overline{h} < 0$.

In this paper, we use the same continuation approach to show that, in the dissipative case, condition (5) can be dropped, proving the following result, first obtained by Habets-Sanchez [1] when $p = 2$, and g satisfies slightly stronger conditions.

Theorem 3. *Assume that* $h \in L^2(0, T)$,

$$f(x) \geq c > 0 \quad (\textit{resp.} \quad f(x) \leq -c < 0) \quad \textit{for all} \quad x \in [0, \infty[, \tag{6}$$

and that conditions (3) *and* (4) *hold. Then Problem* (1) *has at least one positive solution.*

2. Proof of Theorem 3

The proof is based upon the following continuation result, an easy consequence of Theorem 3.1 in [4].

Lemma 1. *Assume that there exist constants* $R > r > 0$, $R' > 0$ *such that the following conditions hold.*

1. *For each* $\lambda \in]0, 1]$, *each possible* T*-periodic solution* x *of the equation*

$$(|x'|^{p-2}x')' + \lambda f(x)x' + \lambda g(x) = \lambda h(t) \tag{7}$$

   satisfies the inequalities $r < x(t) < R$ *and* $|x'(t)| < R'$ *for all* $t \in [0, T]$.
2. *Each possible solution* c *of the equation* $g(c) = \overline{h}$ *satisfies* $r < c < R$.
3. $(g(r) - \bar{h})(g(R) - \bar{h}) < 0$.

Then problem (1) *has at least one solution* x *such that* $r < x(t) < R$ *for all* $t \in [0, T]$.

To apply Lemma 1 to problem (1), we need some a priori estimates on the possible solutions.

Lemma 2. *Assume that condition* (3) *holds. Then there exist* $R_1 > R_0 > 0$ *such that:*

(i) $g(x) < \overline{h}$ *for all* $x \in]0, R_0]$ *and* $g(x) > \overline{h}$ *for all* $x \in [R_1, +\infty[$.

(ii) *For each possible* T*-periodic solution* x *of* (7) *there exist* $t_0, t_1 \in [0, T]$ *such that* $x(t_0) > R_0$ *and* $x(t_1) < R_1$.

Proof. If x is a T-periodic solution of (7), we have, by integrating (7) over $[0, T]$,

$$\frac{1}{T} \int_0^T g(x(t))\, dt = \overline{h}. \tag{8}$$

Assumption (3) implies the existence of some $R_0 > 0$ such that $g(u) < \overline{h}$ whenever $0 < u \leq R_0$. Consequently, if $0 < x(t) \leq R_0$ for all $t \in [0, T]$, we get

$g(x(t)) < \overline{h}$ for all $t \in [0,T]$ and hence

$$\frac{1}{T}\int_0^T g(x(t))\,dt < \overline{h}.$$

This and (8) imply that $x(t_0) > R_0$ for some $t_0 \in [0,T]$. On the other hand, assumption (3) implies the existence of some $R_1 > R_0$ such that $g(u) > \overline{h}$ whenever $u \geq R_1$. Thus, if $x(t) \geq R_1$ for all $t \in [0,T]$, we deduce

$$\frac{1}{T}\int_0^T g(x(t))\,dt > \overline{h}.$$

This and (8) imply the existence of some $t_1 \in [0,T]$ such that $x(t_1) < R_1$. ■

Lemma 3. *If x is any solution of equation* (7)*, then, for a.e. $t \in]0,T[$,*

$$\frac{d}{dt}\Big(\frac{|x'(t)|^p}{q} + \lambda G(x(t))\Big) = -\lambda f(x(t))[x'(t)]^2 + \lambda h(t)x'(t), \tag{9}$$

where $G(x) = \int_a^x g(s)\,ds$, for some arbitrary $a > 0$.

Proof. Letting $u = |x'|^{p-2}x'$, so that $x' = |u|^{q-2}u$, and $|u|^q = |x'|^p$, where $\frac{1}{p} + \frac{1}{q} = 1$, we see that equation (7) is equivalent to the system

$$x' = |u|^{q-2}u, \quad u' = -\lambda f(x)x' - \lambda g(x) + \lambda h(t). \tag{10}$$

Multiplying the first equation by u' and the second one by x' we obtain the identity

$$\frac{d}{dt}\left(\frac{|u(t)|^q}{q} + \lambda G(x(t))\right) = -\lambda f(x(t))[x'(t)]^2 + \lambda h(t)x'(t),$$

which gives (9). ■

Lemma 4. *Assume that conditions* (3) *and* (6) *hold. Then there exist $R_2 > 0$, $R_3 > R_1$ and $R_4 > 0$ such that each possible T-periodic solution x of* (7) *satisfies the inequalities*

$$\int_0^T |x'(t)|^2\,dt \leq R_2^2, \quad x(t) < R_3, \quad |x'(t)| < \lambda^{\frac{1}{p-1}} R_4 \quad (t \in [0,T]). \tag{11}$$

Proof. If x is a possible T-periodic solution of (7), then integrating both members of (9) over $[0,T]$ and using the boundary conditions, we get

$$\int_0^T f(x)|x'|^2 dt = \int_0^T hx' dt,$$

and hence, using assumption (6) and Cauchy–Schwarz inequality

$$\|x'\|_2 \leq c^{-1}\|h\|_2 = R_2.$$

Using Lemma 2 (ii) and Cauchy–Schwarz inequality, we get

$$x(t) = x(t_1) + \int_{t_1}^{t} x'(s)\,ds \le R_1 + \sqrt{T}\|x'\|_2 < R_1 + \sqrt{T}R_2 + 1 = R_3 \quad (t \in [0,T]).$$

Now, x' having mean value zero, $x'(t_2) = 0$ for some $t_2 \in [0,T]$. Integrating both members of (7) beween t_2 and t, we get, with $F(x) = \int_0^x f(s)\,ds$,

$$|x'(t)|^{p-2}x'(t) = -\lambda[F(x(t) - F(x(t_2))] - \lambda\int_{t_2}^{t} g(x(s))\,ds + \lambda\int_{t_2}^{t} h(s)\,ds,$$

and hence

$$|x'(t)|^{p-1} \le \lambda|F(x(t) - F(x(t_2))| + \lambda\Big|\int_{t_2}^{t} |g(x(s))|\,ds\Big| + \lambda\Big|\int_{t_2}^{t} |h(s)|\,ds\Big|. \quad (12)$$

Now, from Lemma 2 (i) and $R_0 < R_1 < R_3$, it follows that

$$\sup_{0<x\le R_3} g(x) = \max_{R_0\le x\le R_3} g(x) := b,$$

so that $g(x(t)) \le b$ for each possible T-periodic solution of (7) and all $t \in [0,T]$. Consequently,

$$|g(x(t))| = |b - g(x(t)) - b| \le |b - g(x(t))| + |b| = b - g(x(t)) + |b| = 2b^+ - g(x(t)),$$

hence, using (8)

$$\Big|\int_{t_2}^{t} |g(x(s))|\,ds\Big| \le \int_0^T |g(x(s))|\,ds \le (2b^+ - \overline{h})T,$$

which, introduced in (12), gives

$$|x'(t)|^{p-1} \le 2\lambda \max_{[0,R_3]} |F| + \lambda(2b^+ - \overline{h})T + \lambda\int_0^T |h(s)|\,ds = \lambda R_4^{p-1}.$$

■

Lemma 5. *Assume that conditions* (3)*,* (4) *and* (6) *hold. Then there exists* $r \in]0, R_0[$ *such that each possible T-periodic solution of* (7) *is such that* $x(t) > r$ *for all* $t \in [0,T]$.

Proof. If x is a T-periodic solution of (7) and t_0 is given by Lemma 2 (ii), then it follows from (9) that

$$\frac{|x'(t)|^p}{q} - \frac{|x'(t_0)|^p}{q} + \lambda \int_{t_0}^{t} f(x)(x')^2 ds = \lambda \int_{t}^{t_0} g(x)x' ds + \lambda \int_{t_0}^{t} hx' ds$$

yielding the estimate

$$\lambda \int_{x(t)}^{x(t_0)} g(u) du \geq -\frac{|x'(t_0)|^p}{q} + \lambda \int_{t_0}^{t} f(x)(x')^2 ds - \lambda \int_{t_0}^{t} hx' ds,$$

and hence

$$\begin{aligned} \lambda \int_{x(t)}^{x(t_0)} [g(u) - \overline{h}] du \geq & -\frac{|x'(t_0)|^p}{q} + \lambda \int_{t_0}^{t} f(x)(x')^2 ds - \lambda \int_{t_0}^{t} hx' ds - \lambda \int_{x(t)}^{x(t_0)} \overline{h} ds \\ = & -\frac{|x'(t_0)|^p}{q} + \lambda \int_{t_0}^{t} f(x)(x')^2 ds - \lambda \int_{t_0}^{t} (h - \overline{h}) x' ds. \end{aligned}$$

From Lemma 4 we get, if $\widetilde{h} = h - \overline{h}$,

$$\lambda \int_{x(t)}^{x(t_0)} [g(u) - \overline{h}] du \geq -\frac{\lambda^{\frac{p}{p-1}} R_4^p}{q} - \lambda \max_{[0,R_3]} |f| R_2^2 - \lambda \|\widetilde{h}\|_2 R_2,$$

which gives

$$\int_{x(t)}^{x(t_0)} [g(u) - \overline{h}]\, du \geq -\frac{R_4^p}{q} - \max_{[0,R_3]} |f| R_2^2 - \|\widetilde{h}\|_2 R_2 := -R_5. \tag{13}$$

If t is such that $x(t) \leq R_0$, we get from (13)

$$\int_{x(t)}^{R_0} [g(u) - \overline{h}]\, du + \int_{R_0}^{x(t_0)} [g(u) - \overline{h}]\, du \geq -R_5.$$

i.e., by Lemma 4,

$$\int_{x(t)}^{R_0} [g(u) - \overline{h}]\, du \geq -R_5 - \int_{R_0}^{R_3} |g(u) - \overline{h}|\, du := -R_6.$$

Now, by (4), there exists $r \in]0, R_0[$ such that

$$\int_{r}^{R_0} [g(u) - \overline{h}]\, du < -R_6.$$

As the function $s \mapsto \int_s^{R_0} [g(u) - \overline{h}]\, du$ is increasing on $]0, R_0]$ (recall, by Lemma 2 that $g - \overline{h}$ is negative on $]0, R_0]$), it follows from the inequalities

$$\int_{x(t)}^{R_0} [g(u) - \overline{h}]\, du \geq -R_6 > \int_{r}^{R_0} [g(u) - \overline{h}]\, du$$

that $x(t) > r$. Thus either $x(t) > R_0$ or $x(t) > r$, i.e. $x(t) > r$ for all $t \in [0, T]$. ■

Proof of Theorem 3. We show that the results of Lemmas 2, 4 and 5 imply that the assumptions of Lemma 1 hold. This is clear for assumption 1 with $R = R_3$, $R' = R_4$. Assumptions 2 and 3 follow from Lemma 2, as we have $r < R_0$ and $R = R_3 > R_1$. ■

Corollary 3. *Assume that the continuous function* $f : [0, +\infty[\to \mathbb{R}$ *satisfies condition* (6)*, the continuous function* $g :]0, +\infty[\to \mathbb{R}$ *satisfies condition* (4)*, and that* $g(x) \to -\infty$ *as* $x \to 0+$ *and* $g(x) \to +\infty$ *as* $x \to +\infty$*. Then Problem* (1) *has at least a positive solution for each* $h \in L^2(0, T)$*.*

For example, the problems

$$(|x'|^{p-1}x')' + cx' + a \exp x - \frac{\beta}{x^\alpha} = h(t), \quad x(0) = x(T), \quad x'(0) = x'(T),$$
$$(|x'|^{p-1}x')' + cx' + ax^\gamma - \frac{\beta}{x^\alpha} = h(t), \quad x(0) = x(T), \quad x'(0) = x'(T),$$

with $a > 0$, $c \neq 0$, $\alpha \geq 1$, $\beta > 0$, $\gamma > 0$, have at least one positive solution for each $h \in L^2(0, T)$.

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