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Inverse Analysis of Magnetic Charge Densities using Discrete Fourier Transform with Tikhonov Regularization

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Abstract. This paper is concerned with a computational method for recovering induced magnetic fields due to the existence of surface or subsurface cracks. The inversion formula can be simply represented by an integral equation of the first kind. The problem is transformed into a well-posed difference equation using output least squares approach with a Tikhonov regularization. A fast computational algorithm is also proposed using discrete Fourier convolution techniques.

1. Introduction

Crack identification problems arising in eddy current inspections are essential parts for assessing the structural integrity of nuclear power plants (i.e., [1, 2]). It is well-known that the existence of crack can be detected as a sets of magnetic dipoles. In this paper, we deal with the reconstruction of magnetic charge densities from the magnetic measurements. Taking in remind that there are severe singularities of such distributions at the neighborhood of cracks, our inverse analysis is essentially ill-posed. We consider a thin plate $V = (0; h) \times \Omega$ with h being the thickness of the plate, and a two-dimensional planar domain. Let $\phi(x)$ be the magnetic scalar potential in air region at the field point x and let $m(x')$ is the magnetization vector at the source point x'. Then, at the field point outside the volume V, the magnetic scalar potential induced by the magnetization at a source point is represented as

$$\phi(x) = \frac{1}{4\pi\mu_0} \int\limits_V m(x')\nabla' \frac{1}{|x - x'|} dx' \tag{1}$$

(see [4] for more details). Let us define the magnetic charge density by $\rho(x') = \nabla' m(x')$ and we suppose that the only vertical component of flux measurements are available from the practical points of views. Then the problem is reduced to the following inverse analysis for the magnetic charge density ρ such that

$$\int\limits_\Omega k(x - x')\rho(x')dx' = y_d(x) \quad x \in \Omega \tag{2}$$

where y_d denotes a measured flux data. The kernel function of $k(x)$ is given by

$$k(x) = \frac{1}{4\pi\mu_0\sqrt{|x|^2 + d^2}}$$

where d implies the so-called lift-off distance. The severe ill-posedness due to the fluctuations of this parameter d occurs in the inversion of (2).

2. Inversion Technique with Tikhonov Regularization

Our major interest of the paper is to recover the nonsmooth parameter function $\rho(x)$ in such a way that

$$\min_{\rho \in H_0^1(\Omega)} J(\rho) = \frac{1}{2}\|K\rho - y_d\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|\nabla\rho\|^2_{L^2(\Omega)} \tag{3}$$

where the operator $K : L^2(\Omega) \to L^2(\Omega)$ with the kernel $k(x - x')$. For the minimizer $\rho^* \in H_0^1$, the necessary and sufficient conditions of the optimality for (3) is represented by the variational inequality (see [3])

$$\langle K\rho^*, K(\rho^* - \rho)\rangle + \alpha\langle\nabla\rho^*, \nabla(\rho^* - \rho)\rangle \le \langle y_d, K(\rho^* - \rho)\rangle \tag{4}$$

where $\langle\cdot,\cdot\rangle$ denotes the L^2-inner product. To solve the corresponding linear system of (4), our iterative procedures can be performed by solving

$$(K^*K - \alpha\nabla^2)\rho = K^*y_d. \tag{5}$$

Noting that, for any $\phi \in H^2(\Omega)$

$$K\phi = \phi * k(x) = \int \lim t_\Omega k(x - x')\phi(x')dx',$$

the Fourier transform has the representation,

$$\mathcal{F}(\phi * k) = \hat{k}^*(\omega)\hat{\phi}(\omega).$$

Hence the major computational savings can be achieved by applying the Fourier transform to the convolution in (5).

3. Numerical Scheme and Experimental Results

To implement the iterative solver (5), we use the finite difference scheme. We divide the domain Ω into N^2 subsquares Ω_{ij} with side having equal length h. Let $x_{ij} = (x_1^i, x_2^j)$ be the centroid of each Ω_{ij}. Then we use the central difference approximation of ∇^2 at x_{ij}:

$$-\nabla_h^2 \rho := \frac{4\rho_{i,j} - \rho_{j+1,j} - \rho_{i-1,j} - \rho_{i,j+1} - \rho_{i,j-1}}{h^2}.$$

The two-dimensional discrete Fourier transform of the kernel k is preliminary calculated and stored in the memory storage. The linear system can be solved by using the conjugate gradient step with the preconditioning. Thus, our numerical scheme is summarized as follows:

[Estimation Algorithm]

Step 0: Compute $\tilde{b} = (\hat{y}_d * \hat{k}^*)\hat{}$ where $(\cdot)\hat{}$ denotes the inverse of the discrete Fourier transform. Given an initial guess $\rho^{(0)}$, compute

$$\rho^{(0)\tilde{}} = \left\{(\hat{\rho}^{(0)} * k)\hat{} * k^*\right\}\hat{} - \tilde{b}.$$

Evaluate the residual

$$r^{(0)} = \rho^{(0)\tilde{}} - \nabla_h^2 \rho^{(0)} - \tilde{b}.$$

and solve $-\nabla_h^2 \rho^{(0)} = r^{(0)}$. Set $i = 0$.

Repeat the following steps {

Step 1: Compute

$$s = \left\{(\hat{p}^{(i)} * k)\hat{} * k^*\right\}\hat{} - \alpha \nabla_h^2 p^{(i)}$$

and evaluate

$$\delta = \frac{\langle p^{(i)} \quad r^{(i)}\rangle}{\langle p^{(i)} \quad s\rangle}.$$

Step 2: Update $\rho^{(i)}$ and $p^{(i)}$,

$$\rho^{(i+1)} = \rho^{(i)} + \delta p^{(i)}$$
$$r^{(i+1)} = r^{(i)} - \delta s$$

Step 3: If the convergence is accepted, then stop

Step 4: Solve $\nabla_h^2 q = r^{(r+1)}$, evaluate $\varepsilon = \dfrac{\langle q, s\rangle}{\langle p^{(i)}, s\rangle}$ and update

$$p^{(i+1)} = q + \varepsilon p^{(i)}$$

and $i = i + 1$

}

In our numerical experiments, we take Ω to be the unit square, i.e., $\Omega = (0,1) \times (0,1)$ and Ω was divided into $N^2 = 64 \times 64$ subsquares with $h = 1/64$. We assume that there exists a single crack at the center of the thin plate. Then the original magnetic source ρ_{true} was given by a negative and a positive Gaussian functions. Fig. 1 illustrates the contour map of the magnetic source function. The distance between both peaks shown in Fig. 1 corresponds to the crack length. In the example shown below, the observation data was calculated through the convolution and add a random noise to the data,

$$y_d(x_{ij}) = K\rho_{true} + \sigma \ \mathrm{rand}(x_{ij})$$

where rand(x_{ij}) is a uniformly distributed random generator. The lift-off distance and the Tikhonov parameter were preassigned as $h = 0.05$ and $\alpha = 1.0 \times 10^{-6}$, respectively. Figures 2 and 3 demonstrate the observation data with 20% noise and its recovering result in the numerical experiment.

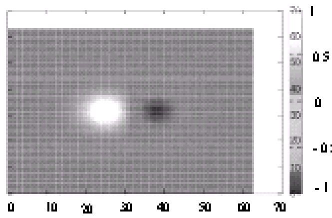

Fig. 1. The contour map of the original magnetic source.

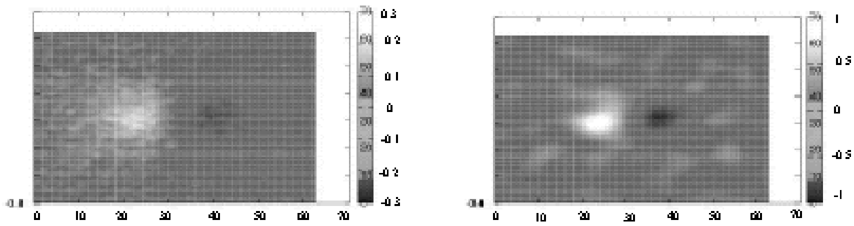

Fig. 2. The observation data with 20% noise and the recovering magnetic source.

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