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Lagrange Multiplier Rule for Nonlinear Inequalities

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Abstract. The paper is about an alternative proof for the existence of nontrivial solutions (Lagrange multiplier rule) of nonlinear eigenvalue inequalities. An application to obstacle problems is given.

1. Introduction

We are concerned here with an eigenvalue inequality of the form

$$\langle Au - \lambda Bu, v - u\rangle \geq 0, \quad \forall v \in K, \tag{1}$$

(K is a closed convex set in a Banach space) and some of its extensions. In the particular case where $K \equiv X$, (1) becomes the equation

$$A(u) - \lambda B(u) = 0. \tag{2}$$

An important result concerning the eigensolution is the Lagrange multiplier rule: "Let X be a Banach space and $F, G : X \to \mathbb{R}$ be two Fréchet differential real functionals. If F achieves its minimum restricted to the set $M = \{v \in X : G(v) = G(u)\}$ at the point u and if $G'(u) \neq 0$, then there exists $\lambda \in \mathbb{R}$ such that $F'(u) = \lambda G'(u)$".

We now consider the more general unconstrained problem given by the inequality

$$\langle Au - \lambda Bu, v - u\rangle + \psi(v) - \psi(u) \geq 0, \quad \forall v \in X, \tag{3}$$

where $\psi : X \to \mathbb{R} \cup \infty$ is an extended real proper and convex functional.

2. The Main Result

Theorem 2.1. *Let F, G be two Fréchet differential functionals on a normed linear space X with their Fréchet derivatives $A, B : X \to X^*$. Let $\psi : X \to \mathbb{R} \cup \infty$*

be a proper, convex, extended real functional. Assume that:

(i) *$F+\psi$ achieves its minimum on a non-empty set $M=\{v\in X : G(v)=r\}$ for some real number r at $u\in D(\psi)$, where $D(\psi)$ is the effective domain of ψ.*

(ii) *There are $h,k\in D(\psi)$ such that $\langle Bu,h-u\rangle>0$ and $\langle Bu,k-u\rangle<0$. Then there exists at least one $\lambda\in\mathbb{R}$ such that*

$$\langle Au-\lambda Bu, v-u\rangle+\psi(v)-\psi(u)\geq 0,\quad \forall v\in X.$$

Remark 1.

(i) This theorem generalizes the Lagrange multiplier rule. By letting $\psi=0$ we get $A(u)=\lambda B(u)$.

(ii) Let K be a convex set in X and let ψ be the indicator function of K, it follows from (3) that

$$\langle Au-\lambda Bu, v-u\rangle\geq 0,\qquad \forall v\in K.$$

(iii) The theorem was first proved by Kubrusly in [4] using an indirect arguement and was applied in [2] and [3] to solve the eigenvalue problems in Orlicz-Sobolev spaces.

Proof. It suffices to prove (4) for any $v\in D(\psi)$. From that observation we define

$$\begin{aligned}\Omega^+&=\{v\in D(\psi):\langle Bu,v-u\rangle>0\}\\ \Omega^-&=\{v\in D(\psi):\langle Bu,v-u\rangle<0\}\\ \Omega^0&=\{v\in D(\psi):\langle Bu,v-u\rangle=0\}.\end{aligned}$$

Then $D(\psi)=\Omega^+\cup\Omega^-\cup\Omega^0$.

By the assumption $\Omega^+\neq\emptyset$ and $\Omega^-\neq\emptyset$. Now let $v\in\Omega^+$ and $w\in\Omega^-$.

Given $\epsilon>0$, we define for $t,\alpha,\beta\geq 0$

$$f(t,\alpha,\beta)=u+t(v-u)+tb(w-u)+\alpha(v-u)+\beta(w-u),$$

where $b=-\dfrac{\langle Bu,v-u\rangle}{\langle Bu,w-u\rangle}>0$.

Then

$$\begin{aligned}G(f(t,\alpha,\beta))&=G(u)+t\langle Bu,v-u\rangle+tb\langle Bu,w-u\rangle+\alpha\langle Bu,v-u\rangle\\ &\quad+\beta\langle Bu,w-u\rangle+o(|t|+|\alpha|+|\beta|)\\ &=G(u)+\alpha\langle Bu,v-u\rangle+\beta\langle Bu,w-u\rangle+o(|t|+|\alpha|+|\beta|).\end{aligned}$$

Let $\alpha,\beta\in[0,\epsilon t]$ then $o(|t|+|\alpha|+|\beta|)=o(|t|)$.

Furthermore, for $t>0$ sufficiently small,

$$\begin{cases}\text{if } \alpha=\epsilon t,\beta=0: G(f)>G(u),\\ \text{if } \alpha=0,\beta=\epsilon t: G(f)<G(u).\end{cases}$$

Fix $\beta=0$, let α vary from ϵt to 0 then either $G(f)-G(u)$ changes sign or $G(f)>G(u)$ for all $0\leq\alpha\leq\epsilon t$.

For the first case there is $\alpha = \alpha(t) \in [0, \epsilon t]$ such that $G(f) = G(u)$. And for the second case, we fix $\alpha = 0$ and let β vary from 0 to ϵt, there is $\beta = \beta(t) \in [0, \epsilon t]$ such that $G(f) = G(u)$.
In general, for each $t > 0$ there exist $\alpha, \beta \in [0, \epsilon t]$ such that $f(t, \alpha, \beta) \in M$ and thus for any $t > 0$ sufficiently small

$$\begin{aligned} 0 &\le F(f) + \psi(f) - F(u) - \psi(u) \\ &= F(f) - F(u) + \psi\left((1 - t - tb - \alpha - \beta)u + (t + \alpha)v + (tb + \beta)w\right) - \psi(u) \\ &\le \langle Au, t(v - u) + tb(w - u) + \alpha(v_u) + \beta(w - u)\rangle + o(|t|) \\ &\quad + (1 - t - tb - \alpha - \beta)\psi(u) + (t + \alpha)\psi(v) + (tb + \beta)\psi(w) - \psi(u) \\ &= t\left[\langle Au, v - u\rangle + \psi(v) - \psi(u) + b\left(\langle Au, w - u\rangle + \psi(w) - \psi(u)\right)\right] \\ &\quad + \alpha\langle Au, v - u\rangle + \beta\langle Au, w - u\rangle - (\alpha + \beta)\psi(u) + \alpha\psi(v) + \beta\psi(w) + o(|t|) \\ &\le tc + 3(\alpha + \beta)M + o(|t|), \end{aligned}$$

where

$$\begin{aligned} c &= \langle Au, v - u\rangle + \psi(v) - \psi(u) + b\left(\langle Au, w - u\rangle + \psi(w) - \psi(u)\right) \\ &= \langle Bu, v - u\rangle \left[\frac{\langle Au, v - u\rangle + \psi(v) - \psi(u)}{\langle Bu, v - u\rangle} - \frac{\langle Au, w - u\rangle + \psi(w) - \psi(u)}{\langle Bu, w - u\rangle}\right], \end{aligned}$$

and $M = \max\{|\psi(u)|, |\psi(v)|, |\psi(w)|, |\langle Au, v - u\rangle|, |\langle Au, w - u\rangle|\}$.

Since $\alpha, \beta \in [0, \epsilon t]$,

$$0 \le tc + 6\epsilon M t + o(|t|).$$

Dividing the above inequality by t and then let $t \to 0^+$ we obtain

$$0 \le c + 6\epsilon M.$$

Since $\epsilon > 0$ is chosen arbitrarily, it follows that $c \ge 0$. Hence, for any $v \in \Omega^+$ and $w \in \Omega^-$ we have shown that:

$$\frac{\langle Au, v - u\rangle + \psi(v) - \psi(u)}{\langle Bu, v - u\rangle} - \frac{\langle Au, w - u\rangle + \psi(w) - \psi(u)}{\langle Bu, w - u\rangle} \ge 0.$$

Therefore,

$$\lambda_2 = \inf_{v \in \Omega^+} \frac{\langle Au, v - u\rangle + \psi(v) - \psi(u)}{\langle Bu, v - u\rangle} \ge \sup_{w \in \Omega^-} \frac{\langle Au, w - u\rangle + \psi(w) - \psi(u)}{\langle Bu, w - u\rangle} = \lambda_1.$$

Let λ be in $[\lambda_1, \lambda_2]$, it is easy to verify that

$$\langle Au - \lambda Bu, v - u\rangle + \psi(v) - \psi(u) \ge 0, \quad \forall v \in \Omega^+ \cup \Omega^-. \tag{5}$$

If $v \in \Omega^0$, define a sequence $v_n = \frac{1}{n}h + \left(1 - \frac{1}{n}\right)v$. Since $D(\psi)$ is convex and $\langle Bu, v_n - u\rangle = \frac{1}{n}\langle Bu, h - u\rangle > 0$, $v_n \in \Omega^+$. By (5),

$$\begin{aligned} 0 &\le \langle Au - \lambda Bu, v_n - u\rangle + \psi(v_n) - \psi(u) \\ &\le \langle Au - \lambda Bu, v_n - u\rangle + \frac{1}{n}\psi(h) + (1 - \frac{1}{n})\psi(v) - \psi(u). \end{aligned}$$

Let $n \to \infty$ we obtain

$$\langle Au - \lambda Bu, v - u\rangle + \psi(v) - \psi(u) \geq 0, \quad \forall v \in \Omega^0. \tag{6}$$

Combining (5), (6) we have the theorem for any $\lambda \in [\lambda_1, \lambda_2]$. ■

3. An Application to Obstacle Problems

Let Ω be an open bounded domain in $\mathbb{R}^N$, $p > 1$ and q be such that the Sobolev the compact imbedding $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ holds. Let $f \in W_0^{1,p}(\Omega)$, $f \geq 0$. We will find $(u, \lambda) \in K \times \mathbb{R}$ such that

$$\int_\Omega |\nabla u|^{p-2}\nabla u(\nabla v - \nabla u) - \lambda \int_\Omega |u|^{q-2}u(v-u) \geq 0, \forall v \in K, \tag{7}$$

where $K = \{u \in W_0^{1,p}(\Omega) : u \geq f \text{ on } \Omega\}$ is a closed convex set of $W^{1,p}(\Omega)$.

Let ψ be the indicator functional of K; that is

$$\begin{cases} \psi(u) = 0, \text{ if } u \in K, \\ \psi(u) = \infty, \text{ otherwise.} \end{cases}$$

Now, let us define on $W_0^{1,p}(\Omega)$:

$$F(u) = \frac{1}{p}\int_\Omega |\nabla u|^p dx,$$
$$G(u) = \frac{1}{q}\int_\Omega u^q dx.$$

Then the differential operators A, B of F, G (respectively) are given by

$$\langle Au, v\rangle = \int_\Omega |\nabla u|^{p-2}\nabla u \nabla v dx,$$
$$\langle Bu, v\rangle = \int_\Omega |u|^{q-2}uv dx.$$

Let $r > 0$ be a real number such that $r > G(f)$. By the compact imbedding $W^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$, G is weakly continuous and thus the level set $G_r = \{u \in W_0^{1,p}(\Omega) : G(u) = r\}$ is weakly closed.

We observe that F is a norm in $W_0^{1,p}(\Omega)$ and ψ is convex lower semi-continuous and positive. Thus $F + \psi$ is weak lower semi-continuous and coercive on $W_0^{1,p}(\Omega)$. Hence $F + \psi$ achieves its minimum on G_r at some $u_r \in G_r$.

Since $(F + \psi)(u) = (F + \psi)(|u|)$, we can assume $u_r \geq 0$. Moreover, by the choice of r, $u_r \neq f$.

To apply the main theorem it remains to verify (ii). To see this, let $v_1 = f$ and $v_2 = 2u_r$ be in K, then

$$\begin{cases} \int_\Omega |u_r|^{q-2}u_r(v_1 - u_r) < 0, \\ \int_\Omega |u_r|^{q-2}u_r(v_2 - u_r) > 0. \end{cases}$$

Therefore, there is $\lambda \in \mathbb{R}$ such that (u_r, λ) solves (7).

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