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Temperature Determination from Interior Measurements: the Case of Temperature Nonlinearly Dependent Heat Source*

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Abstract. We consider the problem of recovering the temperature $u(x, y)$ in a body represented by the half-plane $\mathbb{R}\times\mathbb{R}^+$ from measurements performed at interior points of the body. The function $u(x, y)$ satisfies the nonlinear elliptic equation

$$\Delta u = f(x, y, u(x, y)), \quad x \in \mathbb{R},\ y > 0.$$

The problem is ill-posed. Using the method of Fourier transforms and the method of truncated, we shall prove the uniqueness and give a regularization result. Error estimate is given.

We consider the problem of determining the temperature $u(x, y)$ in a body represented by the half-plane $\mathbb{R} \times \mathbb{R}^+$ from measurements performed at interior points of the body. The temperature $u(x, y)$ satisfies the following equation

$$\Delta u = f(x, y, u(x, y)), \quad x \in \mathbb{R},\ y > 0 \tag{1}$$

subject to the conditions

$$u(x, 1) = \varphi(x) \ \ x \in \mathbb{R} \tag{2}$$

and

$$u(x, y) \to 0 \quad \text{when} \quad |x|, y \to \infty. \tag{3}$$

The paper consists of two parts. In Part I, we determine $u(x, y)$ in the half plane $x \in \mathbb{R}$, $y > 1$. In Part II, we determine $u(x, y)$ in the strip $x \in \mathbb{R}$, $0 \le y < 1$.

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Part I

Put

$$G(x,y,\xi,\eta)=-\frac{1}{4\pi}\ln\frac{(x-\xi)^2+(y-\eta)^2}{(x-\xi)^2+(y+\eta-2)^2}. \tag{4}$$

For $y>1,\ x\in\mathbb{R}$, integrating the identity

$$\frac{\partial}{\partial\xi}(uG_\xi-Gu_\xi)+\frac{\partial}{\partial\eta}(uG_\eta-Gu_\eta)=-Gf \tag{5}$$

over the domain $(-m,m)\times(1,n)\backslash B((x,y),\varepsilon)$, where $B((x,y),\varepsilon)$ is the ball with center at (x,y) and radius $\varepsilon>0$ and letting $n\to\infty$, $m\to\infty$, $\varepsilon\to 0$, we get, after some rearrangements,

$$u(x,y)=Au(x,y), \tag{6}$$

where

$$Au(x,y)=\int_{-\infty}^{+\infty}G_\eta(x,y;\xi,1)\varphi(\xi)d\xi-\int_{-\infty}^{+\infty}\int_{1}^{+\infty}G(x,y;\xi,\eta)f(\xi,\eta,u(\xi,\eta))d\xi d\eta. \tag{7}$$

Then, we readily get the following result:

Theorem 1. *Suppose that for all* $(\xi,\eta,\zeta)\in\mathbb{R}\times\mathbb{R}^+\times\mathbb{R}$

$$\left|f_\zeta^{'}(\xi,\eta,\zeta)\right|\le p(\xi,\eta), \tag{8}$$

where $p(\xi,\eta)\in L^1(\mathbb{R}\times(1,+\infty))$, $p\ge 0$ *satisfies*

$$K\equiv\sup_{(x,y)\in\mathbb{R}\times(1,+\infty)}\left|\int_{-\infty}^{+\infty}\int_{1}^{+\infty}G(x,y;\xi,\eta)p(\xi,\eta)d\xi d\eta\right|<1. \tag{9}$$

Put

$$J=\Big\{u\in C(\mathbb{R}\times(1,+\infty))\Big|\lim_{\sqrt{x^2+y^2}\to+\infty}u(x,y)=0\Big\}. \tag{10}$$

Then $A:J\to J$ *is a contraction and hence u is uniquely determined and can be found by successive approximation.*

Part II

In Part I (Theorem 1), we found $u(x,y), x\in\mathbb{R},\ y\ge 1$. Therefore, $\frac{\partial u}{\partial y}(x,1)$ is determined. Consider the equation

$$\Delta u=f\left(x,y,u\left(x,y\right)\right),\quad x\in\mathbb{R},\ y\in(0,1), \tag{11}$$

subject to the conditions (12)-(13) below

$$u(x,1)=\varphi(x),\quad x\in\mathbb{R}, \tag{12}$$

$$\frac{\partial u}{\partial y}(x,1)=\psi(x),\quad x\in\mathbb{R}. \tag{13}$$

Let $C_\infty(\mathbb{R}\times[0,1))$ denote the Banach space of bounded complex valued continuous functions w on $\mathbb{R}\times[0,1)$ with the norm

$$\|w\|_\infty = \sup_{(x,y)\in\mathbb{R}\times[0,1)} |w(x,y)|$$

1. Uniqueness of Solution

We first consider the uniqueness problem for (11)–(13)

Theorem 2. *We write* $u_{(y)}(x) = u(x,y)$.
Let $b > 0$.
Let $\mathcal{I}$ *be the set* $\big\{u \in C_\infty(\mathbb{R}\times[0,1)) | u_{(y)} \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ *and supp* $\hat{u}_{(y)} \subset [-b,b]\ \forall y \in [0,1)\big\}$ *where*

$$\hat{u}_{(y)}(\zeta) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} u_{(y)}(x) e^{-i\zeta x} dx \tag{14}$$

Suppose that f *satisfies*

$$\big|f(\xi,\eta,u_1(\xi,\eta)) - f(\xi,\eta,u_2(\xi,\eta))\big| \le p(\xi,\eta)\big|u_1(\xi,\eta) - u_2(\xi,\eta)\big|, \tag{15}$$

for all $(\xi,\eta) \in \mathbb{R}\times[0,1)$ *and* $u_1,\ u_2 \in \mathcal{I}$, *where* p *is a nonnegative function satisfying*

$$\frac{2be^b}{\pi}\int_0^1\int_{-\infty}^{+\infty} p(\xi,\eta) d\xi d\eta < 1. \tag{16}$$

Then (11)–(13) *has at most one solution in* $\mathcal{I}$.

Proof. Let u_1, u_2 be two solutions of (11)–(13). Putting $v = u_1 - u_2$, we have

$$\Delta v = f(x,y,u_1(x,y)) - f(x,y,u_2(x,y)), \quad x\in\mathbb{R},\ y\in(0,1), \tag{17}$$

$$v(x,1) = 0, \quad x\in\mathbb{R}, \tag{18}$$

$$\frac{\partial v}{\partial y}(x,1) = 0, \quad x \in \mathbb{R}. \tag{19}$$

Let

$$\Gamma(x,y;\xi,\eta) = -\frac{1}{4\pi}\ln\left[(x-\xi)^2 + (y-\eta)^2\right],$$
$$G(x,y;\xi,\eta) = \Gamma(x,y;\xi,\eta) - \Gamma(x,-y;\xi,\eta).$$

For $x\in\mathbb{R},\ 0<y<1$, integrating the identity

$$\frac{\partial}{\partial\xi}(-vG_\xi + Gv_\xi) + \frac{\partial}{\partial\eta}(-vG_\eta + Gv_\eta) = G\big[f(\xi,\eta,u_1(\xi,\eta)) - f(\xi,\eta,u_2(\xi,\eta))\big] \tag{20}$$

over the domain $(-n,n)\times(0,1)\backslash B((x,y),\varepsilon)$ and letting $n\to\infty, \varepsilon\to 0$, we get, after some rearrangements,

$$v(x,y) = \frac{1}{\pi}\int\limits_{-\infty}^{+\infty} v(\xi,0)\frac{y}{(x-\xi)^2+y^2}d\xi$$

$$+\frac{1}{4\pi}\int\limits_0^1\int\limits_{-\infty}^{+\infty}\ln\frac{(x-\xi)^2+(y-\eta)^2}{(x-\xi)^2+(y+\eta)^2}\left[f(\xi,\eta,u_1(\xi,\eta))-f(\xi,\eta,u_2(\xi,\eta))\right]d\xi d\eta. \quad (21)$$

Letting $y \to 1$, we have

$$\int\limits_{-\infty}^{+\infty} v(\xi,0)\frac{1}{(x-\xi)^2+1}d\xi$$

$$+\frac{1}{4}\int\limits_{-\infty}^{+\infty}\int\limits_0^1\ln\frac{(x-\xi)^2+(1-\eta)^2}{(x-\xi)^2+(1+\eta)^2}.\left[f(\xi,\eta,u_1(\xi,\eta))-f(\xi,\eta,u_2(\xi,\eta))\right]d\xi d\eta = 0. \quad (22)$$

Put

$$F_{(y)}(x) = \frac{y}{x^2+y^2}, L_{(\eta,y)}(x) = \ln\frac{x^2+(y-\eta)^2}{x^2+(y+\eta)^2} \quad (0<y,\eta<1,\ x\in R).$$

Then,

$$\hat{F}_{(y)}(\zeta) = \sqrt{\frac{\pi}{2}}e^{-y|\zeta|}$$

and

$$\hat{L}_{(\eta,y)}(\zeta) = \sqrt{2\pi}\frac{1}{|\zeta|}\left[e^{-(y+\eta)|\zeta|} - e^{-|y-\eta|\ |\zeta|}\right]. \quad (23)$$

We write $v_{(y)}(x) = v(x,y)$. Put

$$H_{(\eta,u_1-u_2)}(\xi) = f(\xi,\eta,u_1(\xi,\eta)) - f(\xi,\eta,u_2(\xi,\eta)). \quad (24)$$

By (23), Eq. (22) can be rewritten as

$$v_{(0)}(.) * F_{(1)}(x) + \frac{1}{4}\int\limits_0^1 L_{(\eta,1)} * H_{(\eta,u_1-u_2)}(x)d\eta = 0$$

where $\varphi * \psi(x) = \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{+\infty}\varphi(x-\xi)\psi(\xi)d\xi$ with $\varphi\in L^1(\mathbb{R}), \psi\in L^2(\mathbb{R})$.

Taking the Fourier transform, we have

$$\hat{v}_{(0)}(\zeta)\hat{F}_{(1)}(\zeta) + \frac{1}{4}\int\limits_0^1 \hat{L}_{(\eta,1)}(\zeta)\hat{H}_{(\eta,u_1-u_2)}(\zeta)d\eta = 0$$

and by (23)

$$\hat{v}_{(0)}(\zeta) = -\frac{1}{2}\int_0^1 \frac{1}{|\zeta|}\left[e^{-\eta|\zeta|} - e^{\eta|\zeta|}\right]\hat{H}_{(\eta,u_1-u_2)}(\zeta)d\eta. \tag{25}$$

From (21), we have

$$v_{(y)}(x) = \frac{\sqrt{2}}{\sqrt{\pi}}v_{(0)}(.) * F_{(y)}(x) + \frac{1}{2\sqrt{2\pi}}\int_0^1 L_{(\eta,y)} * H_{(\eta,u_1-u_2)}(x)d\eta. \tag{26}$$

Taking the Fourier transform, we have

$$\hat{v}_{(y)}(\zeta) = \frac{\sqrt{2}}{\sqrt{\pi}}\hat{v}_{(0)}(\zeta)\hat{F}_{(y)}(\zeta) + \frac{1}{2\sqrt{2\pi}}\int_0^1 \hat{L}_{(\eta,y)}(\zeta)\hat{H}_{(\eta,u_1-u_2)}(\zeta)d\eta. \tag{27}$$

By (23) and (25), Eq. (27) takes the form

$$\hat{v}_{(y)}(\zeta) = \frac{1}{2}\int_0^1 \frac{1}{|\zeta|}\left[e^{(\eta-y)|\zeta|} - e^{-|y-\eta|\ |\zeta|}\right]\hat{H}_{(\eta,u_1-u_2)}(\zeta)d\eta. \tag{28}$$

for $\zeta \in [-b,b]$. By (15), we get

$$\begin{aligned}
\left|\hat{H}_{(\eta,u_1-u_2)}(\zeta)\right| &= \frac{1}{\sqrt{2\pi}}\left|\int_{-\infty}^{+\infty}\left[f(\xi,\eta,u_1(\xi,\eta)) - f(\xi,\eta,u_2(\xi,\eta))\right]e^{-i\zeta\xi}d\xi\right| \\
&\le \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\left|f(\xi,\eta,u_1(\xi,\eta)) - f(\xi,\eta,u_2(\xi,\eta))\right|d\xi \\
&\le \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} p(\xi,\eta)\left|u_1(\xi,\eta) - u_2(\xi,\eta)\right|d\xi \\
&\le \frac{1}{\sqrt{2\pi}}\|v\|_\infty\int_{-\infty}^{+\infty} p(\xi,\eta)d\xi.
\end{aligned} \tag{29}$$

Using the inequalities $\left|\frac{e^{\eta|\zeta|}-e^{-\eta|\zeta|}}{|\zeta|}\right| \le 2e^{|\zeta|} \le 2e^b$ for $\zeta \in [-b,b]$ and $\eta \in [0,1)$ and $\left|\frac{e^{-\alpha|\zeta|}-e^{-\beta|\zeta|}}{|\zeta|}\right| \le 2e^{|\zeta|} \le 2e^b$ for $\alpha,\beta \in [0,2]$, we can prove that

$$|\hat{v}_{(y)}(\zeta)| \le \frac{2}{\sqrt{2\pi}}e^b\|v\|_\infty\int_0^1\int_{-\infty}^{+\infty} p(\xi,\eta)d\xi d\eta$$

for all $\zeta \in \mathbb{R}, y \in [0,1)$. Hence

$$\|\hat{v}_{(.)}(.)\|_\infty \le \frac{2}{\sqrt{2\pi}} e^b \|v\|_\infty \int_0^1 \int_{-\infty}^{+\infty} p(\xi,\eta) d\xi d\eta. \tag{30}$$

Likewise

$$|v_{(y)}(x)| = \frac{1}{\sqrt{2\pi}} \int_{-b}^{b} \hat{v}_{(y)}(\zeta) e^{ix\zeta} d\zeta \le \frac{1}{\sqrt{2\pi}} 2b \|\hat{v}_{(.)}(.)\|_\infty \quad \forall x \in \mathbb{R},\ y \in [0,1).$$

Thus

$$\|v\|_\infty \le \frac{1}{\sqrt{2\pi}} 2b \left\|\hat{v}_{(.)}(.)\right\|_\infty. \tag{31}$$

The inequalities (30)-(31) imply

$$\|v\|_\infty \le \frac{2}{\pi} b e^b \|v\|_\infty \int_0^1 \int_{-\infty}^{+\infty} p(\xi,\eta) d\xi d\eta.$$

Hence $u_1 = u_2$ and the proof is complete. ■

2. Nonlinear Approximation and Regularization

In this section, we determine an approximation of the solution of (11)–(13) in the form

$$u_\varepsilon = v_\varepsilon + w_\varepsilon,$$

where v_ε is an approximation to the solution v of the problem

$$\Delta v = 0, \quad x \in \mathbb{R},\ y \in (0,1), \tag{32}$$

$$v(x,1) = \varphi(x), \quad x \in \mathbb{R}, \tag{33}$$

$$\frac{\partial v}{\partial y}(x,1) = \psi(x), \quad x \in \mathbb{R}, \tag{34}$$

and w_ε is an approximation to the solution w of the problem

$$\Delta w = g(x,y,w), \quad x \in \mathbb{R},\ y \in (0,1), \tag{35}$$

$$w(x,1) = 0, \quad x \in \mathbb{R}, \tag{36}$$

$$\frac{\partial w}{\partial y}(x,1) = 0, \quad x \in \mathbb{R}, \tag{37}$$

in which $g(x,y,w) = f(x,y,w+v_0)$ with v_0 being the exact solution of Problem (36)–(38).

2.1. Regularization of Problem (32)–(34)

For $x \in \mathbb{R},\ 0 < y < 1$, integrating the identity

$$\frac{\partial}{\partial \xi}(-vG_\xi + Gv_\xi) + \frac{\partial}{\partial \eta}(-vG_\eta + Gv_\eta) = 0$$

over the domain $(-n,n) \times (0,1)\backslash B((x,y),\varepsilon)$ and letting $n \to \infty, \varepsilon \to 0$, we get, after some rearrangements,

$$v(x,y) = -\int_{-\infty}^{+\infty} \big[\varphi(\xi)G_\eta(x,y,\xi,1) - G(x,y,\xi,1)\psi(\xi)\big]d\xi + \int_{-\infty}^{+\infty} G_\eta(x,y,\xi,0)v(\xi,0)d\xi. \tag{38}$$

Letting $y \to 1$ in (38), we have

$$\frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{1}{(x-\xi)^2+1} v_{(0)}(\xi)d\xi + \int_{-\infty}^{+\infty} \big[-\varphi(\xi)G_\eta(x,1,\xi,1) + G(x,1,\xi,1)\psi(\xi)\big]d\xi = \varphi(x). \tag{39}$$

This equation can be rewritten in operator form as follows

$$F_{(1)} * v_{(0)}(.)(x) = \pi K_{(1)}(x) + \frac{\sqrt{\pi}}{\sqrt{2}}\varphi(x), \tag{40}$$

where

$$K_{(y)}(x) = -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \big[-\varphi(\xi)G_\eta(x,y,\xi,1) + G(x,y,\xi,1)\psi(\xi)\big]d\xi. \tag{41}$$

Put

$$M_{(y,1)}(x) = \frac{1-y}{x^2+(y-1)^2} - \frac{1+y}{x^2+(y+1)^2}.$$

We have

$$\hat{M}_{(y,1)}(\zeta) = \frac{1}{2}\pi\big[e^{(y-1)|\zeta|} - e^{-(y+1)|\zeta|}\big]. \tag{42}$$

On the other hand, in view of (41), (42) and (23), we get

$$K_{(y)}(x) = -\frac{1}{4\pi}\big[2\varphi * M_{(y,1)}(x) + \psi * L_{(1,y)}(x)\big]. \tag{43}$$

By (23) and (40), we have

$$\hat{v}_{(0)}(\zeta) = e^{|\zeta|}\big(\sqrt{2\pi}\hat{K}_{(1)}(\zeta) + \hat{\varphi}(\zeta)\big). \tag{44}$$

Applying the Fourier transform with respect to the variable x in the relation (38), we get, in view of (44),

$$\hat{v}_{(y)}(\zeta) = N(\zeta, y) \tag{45}$$

where

$$N(\zeta,y) = e^{|\zeta|}\hat{F}_{(y)}(\zeta)\Big(2\hat{K}_{(1)}(\zeta) + \frac{\sqrt{2}}{\sqrt{\pi}}\hat{\varphi}(\zeta)\Big) - \sqrt{2\pi}\hat{K}_{(y)}(\zeta).$$

We get the following result, the proof of which is immediate (and is not reproduced here).

Proposition 1. *Suppose* $v_{(y)}(.) \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$ *and* $\hat{v}_{(y)}(.) \in L^1(\mathbb{R})$ *for* $y \in [0,1)$. *Let* v_0 *be an exact solution of* (45) *with exact data* N_0 *in the right hand side and let* N *be measured data such that* $\|N - N_0\|_2 < \varepsilon, \|.\|_2$ *the norm in* $L^2(\mathbb{R} \times (0,1))$. *Then there exists a regularized solution* v_ε *such that*

$$\|v_0 - v_\varepsilon\|_2 < \varepsilon.$$

2.2. Regularization of Problem (35)–(37)

Let $v_0 \in L^2(\mathbb{R} \times (0,1))$ be an exact solution of (45) and let $v_\varepsilon \in L^2(\mathbb{R} \times (0,1))$ be a regularized solution.

For $x \in \mathbb{R},\ 0 < y < 1$, integrating the identity

$$\frac{\partial}{\partial \xi}(-uG_\xi + Gu_\xi) + \frac{\partial}{\partial \eta}(-uG_\eta + Gu_\eta) = Gg$$

over the domain $[(-n,n) \times (0,1)] \backslash B((x,y),\varepsilon)$ and letting $n \to \infty, \varepsilon \to 0$, we get, after some rearrangements,

$$\begin{aligned} w(x,y) = &\int_{-\infty}^{+\infty} w(\xi,0) G_\eta(x,y,\xi,0) d\xi \\ &- \int_{-\infty}^{+\infty} \int_0^1 G(x,y,\xi,\eta) g(\xi,\eta,w(\xi,\eta)) d\xi d\eta. \end{aligned} \tag{46}$$

This gives

$$\begin{aligned} w(x,y) = &\int_{-\infty}^{+\infty} w(\xi,0) G_\eta(x,y,\xi,0) d\xi \\ &- \int_{-\infty}^{+\infty} \int_0^1 G(x,y,\xi,\eta) f(\xi,\eta,v_0(\xi,\eta) + w(\xi,\eta)) d\xi d\eta. \end{aligned} \tag{47}$$

We write $w_{(y)}(x) = w(x,y)$.

Suppose that f satisfies the following conditions

$$f(\xi,\eta,0) = 0, \tag{48}$$

$$\left|f(\xi,\eta,\zeta_1) - f(\xi,\eta,\zeta_2)\right| \le \left|p(\xi,\eta)\right| \, |\zeta_1 - \zeta_2| \tag{49}$$

for all $(\xi,\eta,\zeta) \in \mathbb{R} \times [0,1) \times \mathbb{R}$, where $p \in L^2(\mathbb{R} \times (0,1))$.

Under the foregoing condition on f, we state (and prove) the following Lemma 1, which will be used in the proof of Theorem 3.

Lemma 1. *Suppose that* f *satisfies the conditions* (48)–(49) *and that* $v_\varepsilon \in L^2(\mathbb{R} \times (0,1))$.

Let $T_{(v_\varepsilon)} : L^2(\mathbb{R}\times(0,1)) \to L^2(\mathbb{R}\times(0,1))$ be defined by

$$T_{(v_\varepsilon)}w(x,y) = \frac{1}{4\pi}\int_{-b}^{b}\int_{0}^{1}\int_{-\infty}^{+\infty}\frac{1}{|\zeta|}\left[e^{(\eta-y)|\zeta|} - e^{-|y-\eta|\,|\zeta|}\right]f_{(\eta,w,v_\varepsilon)}(\xi)e^{-i\xi\zeta}e^{i\zeta x}d\xi d\eta d\zeta \tag{50}$$

where

$$f_{(\eta,w,v_\varepsilon)}(\xi) \equiv f(\xi,\eta,v_\varepsilon(\xi,\eta)+w(\xi,\eta))$$

and b is a fixed positive number such that

$$\alpha \equiv \frac{4}{\pi}be^{2b}\|p\|_2^2 < 1.$$

Then $T_{(v_\varepsilon)}$ is a contraction.

Proof. Put

$$Q_{(y)}(\zeta) = Q(y,\zeta) = \frac{1}{2\sqrt{2\pi}}\int_{0}^{1}\int_{-\infty}^{+\infty}\frac{1}{|\zeta|}\left[e^{(\eta-y)|\zeta|} - e^{-|y-\eta|\,|\zeta|}\right]f_{(\eta,w,v_\varepsilon)}(\xi)e^{-i\xi\zeta}d\xi d\eta,$$

$$\zeta\in[-b,b], y\in[0,1)$$

and

$$Q_{(y)}(\zeta) = 0, \quad \zeta\notin[-b,b], y\in[0,1).$$

Using the inequalities

$$\frac{1}{|\zeta|}|e^{(\eta-y)|\zeta|} - e^{-|y-\eta|\,|\zeta|}| \le 4e^b \quad \forall\eta,y\in[0,1),\zeta\in[-b,b]$$

and

$$|f_{(\eta,w,v_\varepsilon)}(\xi)| \le |p(\xi,\eta)|\big(|v_\varepsilon(\xi,\eta)| + |w(\xi,\eta)|\big) \quad \forall(\xi,\eta)\in\mathbb{R}\times[0,1), w\in L^2(\mathbb{R}\times[0,1)),$$

we get $Q_{(y)}\in L^2(\mathbb{R})$ and $\widehat{T_{(v_\varepsilon)}w}_{(y)}(\zeta) = Q_{(y)}(\zeta)$.

It follows that

$$T_{(v_\varepsilon)}w \in L^2(\mathbb{R}\times(0,1)),$$

$$\begin{aligned}
&\|T_{(v_\varepsilon)}w_1 - T_{(v_\varepsilon)}w_2\|_2^2 = \|\widehat{T_{(v_\varepsilon)}w}_{1(.)}(.) - \widehat{T_{(v_\varepsilon)}w}_{2(.)}(.)\|_2^2\\
&\le \frac{1}{8\pi}\int_{-b}^{b}\int_{0}^{1}\int_{0}^{1}\int_{-\infty}^{+\infty}\frac{1}{|\zeta|^2 2}\left[e^{\left(\eta-y\right)|\zeta|} - e^{-|\eta-y|\,|\zeta|}\right]^{2}p^2(\xi,\eta)d\xi d\eta d\zeta dy\|w_1-w_2\|_2^2 \qquad (51)\\
&\le \frac{4}{\pi}be^{2b}\|p\|_2^2\|w_1-w_2\|_2^2\\
&\le \alpha\|w_1-w_2\|_2^2
\end{aligned}$$

Hence $T_{(v_\varepsilon)}$ is a contraction. This completes the proof. ∎

Theorem 3. *Let $v_0 \in L^2(\mathbb{R}\times(0,1))$ be an exact solution of* (45) *and let $v_\varepsilon \in L^2(\mathbb{R}\times(0,1))$ be a regularized solution.*
Suppose that f satisfies the conditions (48)–(49).
Assume the exact solution $w_0 \in L^2(\mathbb{R}\times(0,1))$ of (35)–(37) *satisfies*

$$\widehat{w}_{0(\eta)}(\zeta)e^{|\zeta|}\sqrt{|\zeta|} \in L^2(\mathbb{R}\times(0,1)). \tag{52}$$

Then there exists a function w_ε such that

$$\|w_\varepsilon - w_0\|_2 \le \sqrt{\frac{18E^2\|p\|_2^2}{\pi} + 2\|v_\varepsilon - v_0\|_2^2},$$

where

$$E = \left\|\widehat{w}_{0(\eta)}(\zeta)e^{|\zeta|}\sqrt{|\zeta|}\right\|_2. \tag{53}$$

Proof. Let b be the positive solution of the equation

$$\frac{8}{\pi}be^{2b}\|p\|_2^2 = \frac{1}{3}. \tag{54}$$

Let $T_{(v_\varepsilon)} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be defined by

$$T_{(v_\varepsilon)}w(x,y) = \frac{1}{4\pi}\int_{-b}^{b}\int_0^1\int_{-\infty}^{+\infty}\frac{1}{|\zeta|}\left[e^{(\eta-y)|\zeta|} - e^{-|y-\eta|\,|\zeta|}\right]f_{(\eta,w,v_\varepsilon)}(\xi)e^{-i\xi\zeta}e^{i\zeta x}d\xi d\eta d\zeta. \tag{55}$$

Since $T_{(v_\varepsilon)}$ is a contraction, there exists a unique $w_\varepsilon \in L^2(\mathbb{R})$ such that

$$T_{(v_\varepsilon)}w_\varepsilon = w_\varepsilon$$

and w_ε can be obtained by successive approximation.

As in the proof of Theorem 2, we have

$$\hat{w}_{0(y)}(\zeta) = \frac{1}{2\sqrt{2\pi}}\int_0^1\int_{-\infty}^{+\infty}\frac{1}{|\zeta|}\left[e^{(\eta-y)|\zeta|} - e^{-|y-\eta|\,|\zeta|}\right]f_{(\eta,w_0,v_0)}(\xi)e^{-i\xi\zeta}d\xi d\eta.$$

Furthermore

$$
\begin{aligned}
&\|w_0 - w_\varepsilon\|_2^2 = \|\widehat{w}_{0(.)} - \widehat{w}_{\varepsilon(.)}\|_2^2 \\
\le &\int_{-\infty}^{+\infty}\int_0^1 \left|\widehat{w}_{0(y)}(\zeta) - \widehat{T_{(v_\varepsilon)} w}_{\varepsilon(y)}(\zeta)\right|^2 dy d\zeta \\
\le 2 &\int_{|\zeta|>b}\int_0^1 (\widehat{w}_{0(y)}(\zeta))^2 dy d\zeta \\
&+ \frac{1}{4\pi}\int_{-b}^{b}\int_0^1 \Big|\int_0^1\int_{-\infty}^{+\infty} \frac{1}{|\zeta|}\left[e^{(\eta-y)|\zeta|} - e^{-|y-\eta|\,|\zeta|}\right] \\
&\times \left[f_{(\eta,w_0,v_0)}(\xi) - f_{(\eta,w_\varepsilon,v_\varepsilon)}(\xi)\right] e^{-i\xi\zeta} d\xi d\eta\Big|^2 dy d\zeta \\
\le 2 &\int_{|\zeta|>b}\int_0^1 \frac{(\widehat{w}_{0(y)}(\zeta)\sqrt{|\zeta|}e^{|\zeta|})^2}{be^{2b}} dy d\zeta \\
&+ \frac{1}{4\pi}\int_{-b}^{b}\int_0^1 \Big|\int_0^1\int_{-\infty}^{+\infty} \frac{1}{|\zeta|}\left[e^{(\eta-y)|\zeta|} - e^{-|y-\eta|\,|\zeta|}\right] \\
&\times \left[f_{(\eta,w_0,v_0)}(\xi) - f_{(\eta,w_\varepsilon,v_\varepsilon)}(\xi)\right] e^{-i\xi\zeta} d\xi d\eta\Big|^2 dy d\zeta \\
\le &\frac{2E^2}{be^{2b}} + \frac{1}{3}(\|w_0 - w_\varepsilon + v_0 - v_\varepsilon\|_2)^2,
\end{aligned}
$$

therefore

$$\frac{1}{3}\|w_0 - w_\varepsilon\|_2^2 \le \frac{16E^2\|p\|_2^2}{\pi} + \frac{2}{3}\|v_0 - v_\varepsilon\|_2^2$$

Hence

$$\|w_0 - w_\varepsilon\|_2 \le \sqrt{\frac{48E^2\|p\|_2^2}{\pi} + 2\|v_0 - v_\varepsilon\|_2^2}.$$

This completes the proof. ■

Regularization of Problem (11)–(13)

From Proposition 1 and Theorem 3 we readily get the following result:

Theorem 4. *Let* $N(\zeta, y)$ *be defined in* (45).
Let u_0 *be an exact solution of* (11)–(13) *corresponding to the exact data* $N_0(\zeta, y)$ *and let* N *be a measured data such that*

$$\|N - N_0\|_2 < \varepsilon.$$

Under assumptions in Proposition 1 *and Theorem* 3*, there exists a regularized solution*

$$u_\varepsilon = v_\varepsilon + w_\varepsilon$$

such that

$$\|u_\varepsilon - u_0\|_2 \leq \varepsilon + \sqrt{\frac{48E^2\|p\|_2^2}{\pi} + 2\varepsilon^2}.$$

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