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Global and Perturbative Approach to Some Equations of Fluid Dynamics

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Abstract. By adding linearly a noise to the equations of classical dynamical systems such as Burgers, Navier–Stokes equations etc., one can express the systems as measures which can be studied globally and/or through perturbative expansions.

By adding a suitable random noise to the classical equations of fluid dynamics, i.e. by transforming them into Stochastic Partial Differential Equations (SPDE), it is possible to express them in a form similar to the one obtained from Quantum Field Theories (QFT). Both techniques: the ones of SPDE and that of QFT can be used to study these equations.

1. Examples of Use of SPDE in Physics

1.1. The Brownian Motion

The Brownian motion can be defined by

$$\underline{X} = (X_1, X_2, X_3),$$
$$\frac{dX_i(t)}{dt} = F_i(\underline{X}(t)) + \dot{w}_i(t) \quad \text{or} \quad dX_i(t) = F_i(\underline{X}(t))dt + dw_i(t).$$

It is a Gaussian measure characterized by

$$E[w_i(t)] = 0 \quad \text{and} \quad E[w_i(t),\ w_j(s)] = \delta_{ij} t \wedge s,$$

where

$$\underline{w}(t) = \int_0^t \underline{\dot{w}}(s)ds,$$

and

$$E[f(\underline{w}(t))] = \int dP(\underline{w})f(\underline{w}(t))$$

with $dP(\underline{w}) = e^{\frac{-1}{2}} \int \sum_i w_i^2(s)ds$.

1.2. Fluid Equations with a Random Force

Examples of such equations are

- Burgers equation $(n = 1)$

$$du = v\partial_{xx}^2 udt - u\partial_x udt - dw.$$

- Navier–Stokes equation $(n = 3)$

$$du = v\nabla^2 udt - u\nabla udt - \frac{1}{\rho}\nabla Pdt - dw.$$

- Euler equation $(n = 3)$

$$du = -u\nabla udt - \frac{1}{\rho}\nabla Pdt - dw,$$

where u is a function of (x, t) and $x \in \mathbb{R}^n$.

1.3. Euclidean Quantum Field Theory (EQFT)

Example of scalar field ϕ.

Let $x \in \mathbb{R}^n$ and $S(\phi)$ be an action, the corresponding SPDE is

$$d\phi(\tau, x) = -\frac{1}{2}\frac{\delta S}{\delta \phi}(\tau, x)d\tau + dw(\tau, x), \tag{1}$$

where $w(\tau, x) = w_r(x)$ and $E(w_r(x), w_\lambda(y)) = t \wedge \lambda D(x, y)$ and with the functional derivative

$$\frac{\delta S}{\delta \phi}$$

defined by

$$\frac{\delta}{\delta \phi(\tau, x)} \int S(\phi(\sigma, y), \partial_\mu \phi(\sigma, y))d\sigma d^n y$$

with

$$S(\phi) = \int S(\phi(\sigma, y), \partial_\mu \phi(\sigma, y))d\sigma d^n y.$$

Remark. In EQFT, the non-stochastic forces are the gradient of some function, this is not the case for the fluid equations.

2. SPDE and EQFT

In EQFT, the quantities of interest are the vacuum expectation values:

$$\langle \phi(x_1)\phi(x_2)\ldots\phi(x_m)\rangle,$$

which are the moments of a measure $d\mu \simeq e^{-S(\phi)} \prod d\phi$.
This measure μ is in fact the equilibrium measure associated to the equation (1).

$$\lim_{T\to\infty} E_{\phi_0}(F(\phi_T)) = \int F(\phi)d\mu.$$

Remark that the time t appearing in equation (1) is not the physical time. In fact, when $n = 4$, the euclidean quantum field $\phi(x) = \phi(x_0, x_1, x_2, x_3)$ is the analytic continuation of the relativistic one with $x_0 = it$, t being the physical relativistic time. A typical action is of the form: $S = S_0 + S_I$ with

$$S_0 = \frac{1}{2}\int_{R^n} (\phi(x), (-\Delta + a^2)\phi(x))d^n x,$$
$$S_I = \lambda \int_{R^n} P(\phi(x))d^n x,$$

$P(\phi(x))$ being a polynomial with constant coefficients in the variable $\phi(x)$. To study the associated measure, one starts in field theory from the formal expression:

$$\begin{aligned} E(\phi(x_1)\phi(x_2)\dots\phi(x_m)) &= \langle \phi(x_1)\phi(x_2)\dots\phi(x_m)\rangle \\ &= \int \phi(x_1)\phi(x_2)\dots\phi(x_m)d\mu \\ &= \frac{\int \phi(x_1)\phi(x_2)\dots\phi(x_m)e^{-S_0-S_I}\prod d\phi}{\int e^{-S_0-S_I}\prod d\phi} \qquad (2)\\ &= \frac{\int \phi(x_1)\phi(x_2)\dots\phi(x_m)e^{-S_0-S_I}d\mu_C}{\int e^{-S_I}\prod d\phi}, \end{aligned}$$

where it has been given, through Minlos' theorem, a rigorous meaning to the formal expression:

$$d\mu_C \simeq \frac{e^{-S_0}\prod d\phi}{\int e^{-S_0}\prod d\phi}.$$

μ_C is a Gaussian measure of mean 0 and covariance $C = (-\Delta + a^2)^{-1}$, the inverse of the quadratic form defined by S_0.

Remark. For $a \neq 0$, μ_C is less singular than the Wiener measure (of covariance $\left(-\frac{d^2}{dt^2}\right)^{-1}$).

To give a meaning to expressions like (2), one has to perform some regularization. We give some examples.
a) The covariance $C(x, y)$ is singular for a dimension $n \geq 2$ as can be seen from

$$C(x, x) = \frac{1}{(2\pi)^n}\int \frac{1}{p^2 + a^2}d^n p.$$

One controls this singularity by introducing an ultraviolet momentum cutoff K such that $|p| \leq K$. One introduces subscripts K to cutoff expressions.

b) Usual integrals in (2) are over the whole space $\mathbb{R}^n$, thus if a is 0, there is a singularity when $p = 0$. One thus introduces an infrared cutoff restricting the space integration to a finite volume element Λ.

For the existence of the measure μ, a minimum requirement is that e^{-S_I} does not grow too much. Experience shows that the cutoff S_I has to satisfy:

$$S_I^{(K,\Lambda)} \geq -C(\Lambda)K^\alpha$$

with $\alpha \leq n$. This corresponds technically to the so-called positivity of the energy.

A powerful tool to study expression like (2) is the integration by part formula:

$$\int (\phi(x))^n F(\phi) d\mu_C = n \int \int (\phi(x))^{n-1} C(x,y) \frac{\delta F(\phi)}{\delta \phi(y)} d^n y d\mu_C.$$

3. Application to Fluid Dynamics

One generally looks at systems of the form

$$\begin{aligned} d\widehat{\phi}_i(t) &= V_i(\widehat{\phi}(t))dt + dw_i(t), \\ \widehat{\phi}_i(t)|_{t=0} &= \phi_0. \end{aligned} \tag{3}$$

If V_i can be written as

$$V_i(\widehat{\phi}(t)) = \sum_j a_{ij} \widehat{\phi}_j(t) + \widetilde{V}_i(\widehat{\phi}(t)),$$

where $\widetilde{V}$ has no linear terms, then one can compare (3) to

$$\begin{aligned} d\phi_i(t) &= \sum_j a_{ij} \phi_j(t) dt + dw_i(t), \\ \phi_i(t)|_{t=0} &= \phi_0 \end{aligned} \tag{4}$$

by the so called Girsanov formula

$$E_{\phi_0}^{(W)} \big[F(\widehat{\phi}(T)) \big] = E_{\phi_0}^{(W)} \big[F(\phi(T)) e^{\xi T} \big], \tag{5}$$

where

$$\xi_T = \sum_i \int_0^T \widetilde{V}_i(\phi(s)) dw_i(s) - \frac{1}{2} \sum_i \int_0^T \widetilde{V}_i^2(\phi(s)) ds.$$

The first integrals in ξ_T are Ito integrals. They are defined by

$$\int_0^T f(s) dw(s) = \lim_{n \to \infty} \sum_{j=0}^{n-1} f(s_j)(w(s_{j+1}) - w(s_j))$$

with

$$0 = s_0 < s_1 < \cdots < s_{n-1} < s_n = T.$$

Remark that the right hand side of formula (5) is similar, up to a normalization factor, to the final term of the formula (2).

Rewriting equation (4) in the form

$$\begin{aligned} d\phi(t) &= A\phi(t) + dw(t), \\ \phi(0) &= \phi_0, \end{aligned} \tag{6}$$

the solution of this equation is the Gaussian variable

$$\phi(t) = e^{At}\phi_0 + \int_0^t e^{A(t-s)} dw(s)$$

if the spectrum $\sigma(-A) \geq 0$.

Remark that in scalar EQFT: $-A = C^{-1} \geq a^2$.

4. The Specificity of SPDE Associated to EQFT

There are two types:

a) In EQFT, the deterministic terms are gradient terms:

As a consequence, the Ito integral can be expressed as an usual integral. In fact:

$$\int_0^T \widetilde{V}_i(\phi(s)) dw_i(s) = \int_0^T \widetilde{V}_i(\phi(s)) \bullet dw_i(s) - \frac{1}{2}\int_0^T \frac{\delta}{\delta\phi(s)} \widetilde{V}_i ds$$

and

$$\begin{aligned} \sum_i \int_0^T \widetilde{V}_i(\phi(s)) dw_i(s) &= \sum_i \int_0^T \widetilde{V}_i(\phi(s)) \bullet \Big(-\sum_i a_{ij}\phi_j(s) ds + d\phi_i(s) \Big) \\ &= -\sum_i \int_0^T \widetilde{V}_i(\phi(s)) \bullet \Big(\sum_i a_{ij}\phi_j(s) ds + S(\phi(T)) - S(\phi_0), \end{aligned}$$

since in the gradient case $V_i = \dfrac{\delta}{\phi_i} V$.

The last equality shows that all the quantities are only functions of ϕ and no more of the Brownian motion. This shows therefore that in

$$E^W_{\phi_0}\big[E(\widetilde{\phi}(T))\big] = E^W_{\phi_0}\big[F(\phi(T)) e^{\xi_T}\big],$$

the last expectation can be taken relatively to ϕ. In the proof, a use has been made of the Stratanovitch integral whose definition is in this case equivalent to the usual Riemann integral definition.

b) The SPDE is a way to study the invariant measure for which one has a reasonable guess. On the contrary, for fluid equations one is both interested in the finite time correlations and in the existence of invariant measures. The main difference is that for EQFT, the time is not a real time but a parameter measuring how to converge from the model to the physical reality.

5. Study of Stochastic Dynamical Systems Using SPDE

The idea is to study the second term of

$$E^W_{\phi_0}\big[E(\widetilde{\phi}(T))\big] = E^W_{\phi_0}\big[F(\phi(T))e^{\xi_T}\big]$$

by using the methods of field theory. The expression $E^W_{\phi_0}\big[E(\widetilde{\phi}(T))e^{\xi_T}\big]$ can be studied both globally and pertubatively (by expanding the exponent).

To give a meaning to the right hand side of (5) one has to seriously cutoff the expression in the bracket. One has to face three different difficulties:

1. If one takes as the linear part terms like the one in the Navier-Stokes equation, i.e. $\nu\Delta u$, one sees that they generate an infrared singularity. The corresponding Gaussian measure is the Fourier transform of $1/p^2$. It is therefore necessary to introduce a space cutoff Λ.

2. Experience from EQFT shows that one cannot hope to give a meaning to the measure without giving also a meaning to the perturbative expansion. It is easy to see, expanding the exponent that as soon as the space dimensions is equal to 2, there are ultraviolet singularities to appear. One has therefore to introduce a momentum cutoff K and also renormalization counterterms (the fact one needs to renormalize the theory was pointed out by Yakots and Orzag [3]).

3. As for the measure in field theory, one has to be able to found a lower bound for ξ_T. The difficulty, in contradistinction to the EQFT case, is due to existence of the Ito integral part which requires a bound covering the path history (remark that since the fluid equations are macroscopic equations, they are of the reaction-diffusion type and do not exhibit a property like the positivity of the energy).

6. Some Results

Postponing the difficulties inherent to the first 2 points, a proposal has been given to solve the third point. The systems which are studied are 2-dimensional (one for time and one for space), without infrared singularity and of the polynomial type. Then one can show that if one can split $\widetilde{V}$ into two parts

$$\widetilde{V} = \widetilde{V}_{NG} + \widetilde{V}_G, \tag{7}$$

where $\widetilde{V}_G$ is the functional derivative of a positive functional and if in some way, the gradient part dominates the non-gradient part, then we can bound the exponent.

Just to give an idea, Jona-Lasinio and myself [1] derive the following bounding condition

$$-\|\widetilde{V}_G C^{(1-\rho)/2}\|^2 + (p-1)\big[\|\widetilde{V}_{NG} C^{(1-\rho)/2}\|^2 + 2(:\widetilde{V}_{NG};, C^{(1-\rho)} : \widetilde{V}_G :)\big] < M,$$

where ρ, $0<\rho<1$, is given and there should exist M and $p>1$.

We also prove for the toy model (space dimension = 0) [2]:

$$d\phi_t = \;= -\frac{1}{2}(\phi_t + \lambda : \phi_t^3 :)dt + dw_t,$$
$$\phi_0 = \psi.$$

Theorem 1. *For ψ finite and λ small enough, there exists*

$$\langle F(\phi)\rangle = \lim_{T\to\infty} E_\psi(F(\phi_T)e^{\xi_T})$$

the limit being independent of ψ.

One shows in the same way that this result can be extended to the case of a vector random variable $\phi = (\phi_1, \dots, \phi_n)$, in which case $-(1/2)\phi_t$ is replaced by $-(1/2)A\phi_t$ where A can be an arbitrary $n \times n$ matrix with positive spectrum.

The same theorem proves the existence of the time correlations and their exponential decrease if $\inf \sigma(-A) > 0$.

In the general case (say Navier–Stokes), renormalization counterterms and therefore momentum cutoff K need to be introduced and it is expected that higher order terms like

$$f(K)\frac{\delta}{\delta u}(u^2)^2$$

will appear and restore the missing positivity, $f(K) \to 0$ when $K \to \infty$.

References

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