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On Friction-Induced Instabilities and Vibrations

Nguyen Quoc Son

Laboratoire de Mécanique des Solides, CNRS-umr7649,
Ecole Polytechnique, 91128-Palaiseau, France

Abstract. Some problems of friction-induced instabilities, vibrations and noise emissions are considered in this paper. The presentation is focussed on the possibility of flutter instability of the steady sliding equilibrium of an elastic solid in unilateral contact with a moving obstacle. The system of governing equations is given to obtain the steady sliding equilibrium and to discuss its stability. It is shown that the steady sliding equilibrium is generically unstable by flutter. This instability leads to a dynamic response which can be periodic or chaotic. Principal results of the literature are reported here. In the spirit of Hopf bifurcation, the existence problem of periodic solutions has been discussed in my research team for an academic problem of coaxial cylinders. It has been shown by a semi-analytical approach that an interesting family of these periodic solutions are stick-slip-separation waves propagating on the contact surface in a similar way as Schallamach waves in statics. The transition from a given position near equilibrium to the propagation of waves has been illustrated by a numerical approach using finite element simulations. Some phenomena of nonlinear vibration and noise emission in the daily life, such as brake squeal for example, can be discussed in this spirit.

1. Elastic Solids in Frictional Contact

1.1. Coulomb's Law of Dry Friction

At a contact point of a solid with an obstacle, the relative velocity $v = v_s - v_o$ is by definition the difference between the velocities of the material points of the solid and of the obstacle in contact. The relative velocity v and reaction R can be decomposed into normal and tangential components

$$v = v_T + v_N n, \;\; R = T + Nn, \tag{1}$$

where n denotes the external normal vector to the obstacle at a contact point,

v_T is the sliding velocity vector, T the tangential reaction vector and N the normal reaction, N and v_N are scalars. The unilateral condition of contact implies that the normal reaction must be non-negative $N \geq 0$. When there is contact, Coulomb's law of dry friction states that the friction criterion must be satisfied and that the friction force must have the opposite direction to the sliding velocity,

$$\phi = \|T\| - fN \leq 0, \text{ and } \|T\| = fN, \; T = -a v_T, \; a \geq 0 \text{ if } v_T \neq 0, \tag{2}$$

where f denotes the friction coefficient. In particular, the dissipation by friction is $-T v_T = fN\|v_T\|$. Coulomb's law has often been interpreted in the literature as a non-associated law since the velocity (v_T, v_N) is not a normal to the domain of admissible forces. A second interpretation consists of saying that Coulomb's friction is a standard dissipative law with a state-dependent dissipation potential, cf. [24, 28]. Indeed, normality law is satisfied by the flux v_T and the force T since T and v_T are related through a state-dependent dissipation potential $D(v_T, N)$

$$T = -D_{,v_T} \text{ with } D = fN\|v_T\|, \tag{3}$$

where $D_{,v_T}$ is understood in the sense of sub-gradient. The set of admissible forces, which is a sphere of radius fN, depends on the present state through the present value of N.

1.2. Governing Equations

The simple case of an elastic solid occupying a volume V in the undeformed position is considered. The solid is submitted to given forces and displacements $r^d = r^d(\lambda(t))$, $u^d = u^d(\lambda(t))$ respectively on the portions S_r, S_u of the boundary S, and $\lambda(t)$ denotes a control parameter defining the loading history. On the complementary part S_R, the solid may enter into contact with a moving obstacle $h(m,t) < 0$ and the non-penetration condition is

$$h(x + u(x,t), t) \geq 0 \; \forall x \in S_R. \tag{4}$$

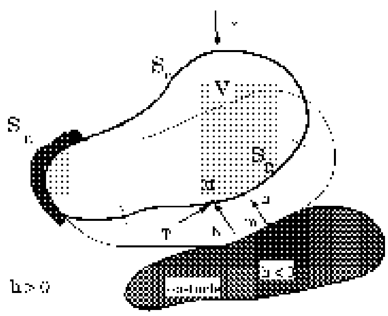

Figure 1. A solid in unilateral contact with an obstacle

The mechanical response of the solid are governed by unilateral contact conditions, Coulomb's law and classical equations of elastodynamics. In Lagrange description, the governing equations are

$$\begin{aligned}&\text{Div } b = \rho\ddot{u},\ b = W,_{\nabla u}\ \ \forall x \in V,\\ &b n_s = r^d\ \forall x \in S_r,\ b n_s = N n + T\ \forall x \in S_R,\ u = u^d\ \forall x \in S_u,\end{aligned} \tag{5}$$

where $W(\nabla u)$ and b denote respectively the elastic energy per unit volume and the unsymmetric Piola–Lagrange's stress. In particular, the unknowns (u, N) must satisfy the local equations

$$u = u^d\ \forall x \in S_u,\ h \geq 0,\ N \geq 0,\ Nh = 0\ \forall x \in S_R \tag{6}$$

and the variational inequality

$$\begin{aligned}&\int_V (\nabla u^* - \nabla\dot{u}) : W,_{\nabla u}(\nabla u)dV - \int_{S_r} r^d \cdot (u^* - \dot{u})da + \int_V \rho\ddot{u} \cdot (u^* - \dot{u})dV\\ &\quad - \int_{S_R} (v_N^* - v_N)N da + \int_{S_R} fN(\|v_T^*\| - \|v_T\|)dS \geq 0\ \ \forall u^* \in U_{ca},\end{aligned} \tag{7}$$

where U_{ca} denotes the set of kinematically admissible rates

$$U_{ca} = \{u^* | u^* = \dot{u}^d \text{ on } S_u\} \tag{8}$$

and v^* and v are the relative rates

$$v_N = n\,\dot{u} - v_{oN},\ v_T = \dot{u} - (n\,\dot{u})n - v_{oT},\ v_N^* = n\,u^* - v_{oN},\ v_T^* = u^* - (n\,u^*)n - v_{oT} \tag{9}$$

with $n = h,_m/\|h,_m\|$ and $v_{oN} = -h,_t/\|h,_m\|$. The system of solid in frictional contact under loads is associated with an energy potential and a dissipation potential:

$$\begin{aligned}&\mathbf{E}(u, \lambda, \mu) = \int_V W(\nabla u)dV - \int_{S_r} r^d(\lambda) \cdot u da - \int_{S_R} \mu h(x + u)da,\\ &\mathbf{D}(N, v_T) = \int_{S_R} fN\|v_T\|da,\end{aligned} \tag{10}$$

where μ denotes the Lagrange multipler associated with the constraints (4) and $N = \mu\|h,_m\|$. The variational inequality (7) can be condensed as

$$(\mathbf{J} + \mathbf{E},_u)(u^* - \dot{u}) + \mathbf{D}(N, v_T^*) - \mathbf{D}(N, v_T) \geq 0\ \forall u^* \in U_{ca}, \tag{11}$$

where J denotes the inertial terms. Some regularizations of the frictional contact problem have been proposed in the literature:

- Non-local Coulomb's law, proposed by Duvaut [11], in which the local normal reaction N is replaced by its mean value $\bar{N}$ on an elementary representative surface S_{re}

$$\bar{N} = \frac{1}{S_{re}} \int_{S_{re}} N(x)\ ds.$$

- Normal compliance law, discussed by Oden and Martins [29], Kikuchi and Oden [13], Andersson [4], Klarbring et al. [16], ... It consists of replacing Signorini's relations of unilateral contact by a relationship giving the normal reaction as a function of the gap $h(x+u(x))$. For example, nonlinear springs of energy $\varphi(h)$ per unit surface may be added to the system where $\varphi(h)$ is a regular function permitting an approximation of Signorini's conditions. In particular, with $\varphi(h) = \frac{k}{2}\langle h\rangle_-^2$, then the normal compliance law consists of writing that

$$N = k\|h,_m\|\langle h\rangle_-.$$

General discussions on the existence of a solution of the quasi-static problem in small deformation have been given in the literature, when there is regularization by normal compliance cf. [4, 16] and by nonlocal Coulomb's law cf. [8]. In these cases, it has been proved that the existence of at least one solution is ensured when the friction coefficient is small enough.

2. Divergence and Flutter Instabilities of an Equilibrium

There is flutter (or divergence) instability if, under disturbances, the system will leave the equilibrium position with (or without) growing oscillations. As usual, the possibility of divergence instability can be discussed in a purely static approach for the case of fixed obstacles in the same spirit as in plasticity [7, 12, 28, 32]. The static approach consists of defining the stability as the absence of additional displacement when the load does not vary $\dot{\lambda} = 0$. A stability criterion has been discussed in the literature [7, 15, 19, 25, 27]. The condition of positivity

$$\begin{aligned} \mathbf{I}(u^*) = \int_V \nabla u^* : \mathcal{L} : \nabla u^* dV + \int_{S_{Rc}} N u_T^* \cdot C \cdot u_T^* da \\ + \int_{S_{Rc}} f\dot{N}(u^*)\|u_T^*\| da > 0 \quad \text{for all} \quad u^* \neq 0 \in V_{ad}^o, \end{aligned} \tag{12}$$

where $\dot{N} = \dot{N}(u^*)$ is obtained from the expression $N = n_s b n_s$, is a criterion of stability since it ensures the static stability of the considered equilibrium.

However, for moving obstacles, the possibility of flutter instabilities are the principal action induced by friction. In particular, the instability by flutter of the steady sliding equilibrium has been recently discussed. For the sake of clarity, the instability of the steady sliding equilibrium of an elastic solid in contact with a moving rigid half-space, in translation motion at a constant velocity w parallel to the free surface, is considered here in small deformation.

2.1. Steady Sliding Equilibrium

The steady sliding equilibrium u of the solid must satisfy

$$\int_V \nabla\delta u : L : \nabla u dV - \int_{S_r} r^d \delta u \ da - \int_{S_R} (\delta u_N N + f N \tau \delta u_T) da = 0.$$

This equation leads formally to a system of reduced equations of the form

$$N = k_{NN}[u_N] + k_{NT}[u_T] + N^d, \ T = fN\tau = k_{TN}[u_N] + k_{TT}[u_T] + T^d.$$

The principal unknown u_N must satisfy

$$u_N = \mathbf{A}[N] + \mathbf{B}, \ N \geq 0, \ u_N \geq 0, \ N u_N = 0 \tag{13}$$

with

$$\begin{aligned} \mathbf{A} &= (k_{NN} - k_{NT} k_{TT}^{-1} k_{TN})^{-1} (I - f \ k_{NT} h_{TN}), \\ h_{TN}[N] &= k_{TT}^{-1}[N\tau], \\ \mathbf{B} &= (k_{NN} - k_{NT} k_{TT}^{-1} k_{TN})^{-1} \left[N^d - k_{NT} k_{TT}^{-1}[T^d]\right]. \end{aligned} \tag{14}$$

It is clear that the linear operator $\mathbf{A}$ is not symmetric if $f \neq 0$:

$$(N^*, \mathbf{A}[N]) = \int_{S_R} N^*(x) \mathbf{A}[N](x) \ dS \neq (N, \mathbf{A}[N^*]). \tag{15}$$

Thus, a linear complementary problem (LCP) must be considered. In particular, the existence and uniqueness of a steady sliding solution are ensured if $\mathbf{A}$ is positive-definite or P-positive [9, 14, 28].

2.2. Flutter Instability of the Steady Sliding Equilibrium

The stability of the steady sliding position can be obtained from the study of small perturbed motions. However, the equations of motion cannot be linearized without the assumption of effective contact. Indeed, in the presence of a loose contact, a small perturbed motion is not necessarily governed by linear equations because of the possibility of separation. Under the assumption of an effective contact, if the sliding speed is never zero, the dynamic equations can be written as

$$\begin{aligned} &\int_V \delta u \cdot \rho \ddot{u} dV + \int_V \nabla\delta u : L : \nabla u dV + \int_{S_R} N \delta u_N dS \\ &+ \int_{S_R} f N \frac{\dot{u}_T - w}{\| \dot{u}_T - w \|} \cdot \delta u_T dS = 0 \ \ \forall \delta u, \delta N. \end{aligned} \tag{16}$$

The linearization is then possible for sliding motions. The nature of this particular problem can be better understood in the discretized form. After discretization, the linearized equations for sliding motions are

$$\begin{aligned} U_N^* &= 0, \\ M_{YY}\ddot{Y}^* + K_{YY}Y^* &= f\Phi \ N^* + f\Phi_{\dot{Y}}\dot{Y}^*, \\ M_{NY}\ddot{Y}^* + K_{NY}Y^* &= N^* \end{aligned} \tag{17}$$

where $u = (U_N, Y)$ and $\Phi(\dot{Y})$ is a matrix dependent on the direction of sliding. The general expression $u^* = e^{st}\ U$ with $U = (U_N = 0, X)$ then leads to a generalized eigenvalue problem

$$s^2(M_{YY} - f\Phi M_{NY})X - sf\Phi_{\dot{Y}}X + (K_{YY} - f\Phi K_{NY})X = 0. \tag{18}$$

Thus, the considered equilibrium is asymptotically stable (with respect to sliding motions) if $\Re(s) < 0$ for all s and unstable if there exists at least one value s such that $\Re(s) > 0$. This generalized eigenvalue problem can be written as $(s^2\bar{M} + s\bar{C} + \bar{K})\ X = 0$ with non-symmetric matrices $\bar{M}, \bar{K}$ and complex eigenvalues and eigenvectors. This analysis leads to the definition of a critical value $f_d \geq 0$ such that the considered equilibrium is unstable when $f > f_d$.

2.3. Example on the Sliding Contact of Two Elastic Layers

The simple example of the frictional contact of two elastic infinite layers is considered here as an illustrating example. This problem was discussed analytically by Adams [1], by Martins *et al* [18] and by Renardy [37].

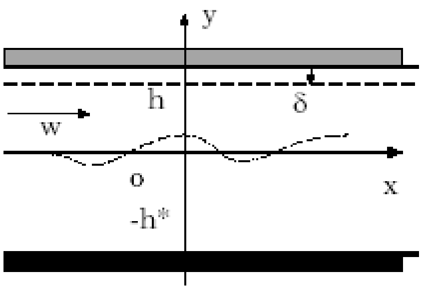

Figure 2. Sliding contact of two elastic layers

The contact in plane strain with friction of two infinite elastic layers, of thickness h and h^* respectively as shown in Figure 2, is considered. The lower face of the bottom layer is maintained fixed in the axes $Oxyz$. The upper face of the top layer, assumed to be in translation of velocity w in the direction Ox, is compressed to the bottom layer by an implied displacement $\delta < 0$. At the interface $y = 0$, the contact is assumed to obey Coulomb's law of friction with a constant friction coefficient. The celerities of dilatation and shear waves are first introduced:

$$c_1 = \sqrt{(\lambda + 2\mu)/\rho},\ \ c_2 = \sqrt{\mu/\rho},\ \ \tau = c_2/c_1,$$

for the top layer and for the bottom layer (superscript *), to write the governing equations for the displacements of the top layer $wte_1 + u(x - wt, y, t)$ and of the bottom layer $u^*(x, y, t)$ under the form:

$$\begin{cases} (1-\tau^2(\frac{w}{c_2})^2)u_{x,xx}+\tau^2 u_{x,yy}+(1-\tau^2)u_{y,xy}=\tau^2(u_{x,tt}-2\frac{w}{c_2}u_{x,tx}), \\ u_{y,yy}+\tau^2(1-(\frac{w}{c_2})^2)u_{y,xx}+(1-\tau^2)u_{x,xy}=\tau^2(u_{y,tt}-2\frac{w}{c_2}u_{y,tx}), \\ u^*_{x,xx}+\tau^{*2}u^*_{x,yy}+(1-\tau^{*2})u^*_{y,xy}=\tau^{*2}u^*_{x,tt}, \\ u^*_{y,yy}+\tau^{*2}u^*_{y,xx}+(1-\tau^{*2})u^*_{x,xy}=\tau^{*2}u^*_{y,tt}. \end{cases} \tag{19}$$

Boundary and interface conditions are

$$u(x-wt,h,t)=0,\ \ u^*(x,-h^*,t)=0,\ \ u_y(x-wt,0,t)=u^*_y(x,0,t),$$
$$\sigma_{yy}(x,0,t)=\sigma^*_{yy}(x,0,t),\ \sigma_{xy}(x,0,t)=\sigma^*_{xy}(x,0,t),\ f\sigma_{yy}(x,0,t)=-\sigma_{xy}(x,0,t).$$

The stability of the steady sliding solution can be obtained by a linearization of the dynamic equation under the assumption of sliding perturbed motions near the steady sliding state. These motions are searched for under the form of slip waves of wave-length $L=1/k$:

$$u(x-wt,y,t)=e^{2\pi kct}e^{2ik\pi(x-wt)}X(y),\ \ u^*(x,y,t)=e^{2\pi kct}e^{2ik\pi x}X^*(y).$$

The condition of existence of non null displacement modes (X,X^*) requires that the pair c and k must be a root of the following equation:

$$\begin{aligned}\mathcal{F}(c,k)=\rho c_2^2(A(p,q,kh))(iB(p^*,q^*,kh^*)+fC(p^*,q^*,kh^*))\\ +\rho^* {c_2^*}^2(A(p^*,q^*,kh^*))(iB(p,q,kh)-fC(p,q,kh))=0,\end{aligned} \tag{20}$$

where p,q,A,B,C are appropriate functions, [23]

$$\begin{aligned}
p^2 &= 1+(\frac{c-iw}{c_2})^2,\\
q^2 &= 1+\tau^2(\frac{c-iw}{c_2})^2,\\
A(p,q,kh) &= -4pq(1+p^2)+pq(4+(1+p^2)^2)\cosh(2\pi pkh)\cosh(2\pi qkh)\\
&\quad -((1+p^2)^2+4p^2q^2)\sinh(2\pi pkh)\sinh(2\pi qkh),\\
B(p,q,kh) &= q(1-p^2)(\sinh(2\pi pkh)\cosh(2\pi qkh)\\
&\quad -pq\cosh(2\pi pkh)\sinh(2\pi qkh)),\\
C(p,q,kh) &= pq(3+p^2)-pq(3+p^2)\cosh(2\pi pkh)\cosh(2\pi qkh)\\
&\quad +(2p^2q^2+(1+p^2))\sinh(2\pi pkh)\sinh(2\pi qkh).
\end{aligned}$$

The case of a rigid top layer is obtained when $c_2\to+\infty$

$$\mathcal{F}(c,k)=iB(p^*q^*,kh^*)+fC(p^*,q^*,kh^*)=0. \tag{21}$$

For an elastic half-plane compressed into a moving rigid half-plane, cf. Martins *et al.* [18], the results are:

$$\mathcal{F}(c)=iq^*(1-{p^*}^2)+f(1+{p^*}^2-2p^*q^*)=0.$$

In the case of two elastic half-planes, cf. Adams [1], this equation can be written as:

$$\mathcal{F}(c) = \rho c_2^2((1+p^2)^2 - 4pq)(iq^*(1-p^{*2}) + f(1+p^{*2} - 2p^*q^*)) \\ + \rho^* c_2^{*2}((1+p^{*2})^2 - 4p^*q^*)(iq(1-p^2) - f(1+p^2-2pq)) = 0. \quad (23)$$

It has been established in each case that there exists a critical value $f_d \geq 0$ such that the steady sliding solution is unstable for $f \geq f_d$. For example, $f_d = 0$ occurs for the system of two layers of finite depths while the possibility $f_d > 0$ may happen in the sliding contact of half-spaces, cf. [2, 18 - 20, 34].

3. Stick-Slip-Separation Waves

The fact that the steady sliding solution is unstable leads to the study of possible dynamic bifurcations of the sliding contact of solids. In the spirit of Hopf bifurcation [38], a periodic response can be expected as an alternative stable response. This possibility has been explored in the example of two coaxial cylinders, [20, 22, 31].

The mechanical response in plane strain of a brake-like system composed of an elastic tube, of internal radius R and external radius R^*, in frictional contact on its inner surface with a rotating rigid cylinder of radius $R + \Delta$ and of angular rotation Ω has been discussed, cf. Fig. 3. The mismatch $\Delta \geq 0$ is a load parameter controlling the normal contact pressures. This model problem enables us to exhibit the existence of nontrivial periodic solutions in the form of stick-slip or stick-slip-separation waves propagating on the contact surface.

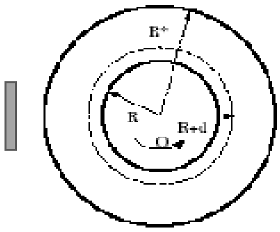

Figure 3. The problem of coaxial cylinders in frictional contact.

The governing equations of the system follow from the kinetic relations, the fundamental law, the linear elastic constitutive equations, and the boundary unilateral contact conditions with Coulomb's friction:

$$\begin{cases} \bar{\epsilon} = (\nabla \bar{u})_s, \\ \mathrm{Div}\bar{\sigma} = \gamma \ddot{\bar{u}}, \\ \bar{\sigma} = \dfrac{\nu}{(1+\nu)(1-2\nu)} Tr(\bar{\epsilon}) I + \dfrac{1}{1+\nu}\bar{\epsilon}, \\ \bar{u}_r(\xi,\theta,t) = \bar{u}_\theta(\xi,\theta,t) = 0, \\ \bar{\sigma}_{rr}(1,\theta,t) = -p(\theta,t),\ \bar{\sigma}_{r\theta}(1,\theta,t) = -q(\theta,t), \\ \bar{u}_r \geq \delta,\ p \geq 0,\ p(\bar{u}_r - \delta) = 0, \\ |q| \leq fp,\ q(1-\dot{\bar{u}}_\theta) - fp|1-\dot{\bar{u}}_\theta| = 0, \end{cases} \quad (24)$$

where non-dimensional variables are introduced

$$\bar{u} = \frac{u}{R},\ \bar{\sigma} = \frac{\sigma}{E},\ \bar{r} = \frac{r}{R},\ \gamma = \frac{\rho R^2 \Omega^2}{E},\ \xi = \frac{R^*}{R},\ \delta = \frac{\Delta}{R},\ \bar{t} = \Omega t,\ \dot{\bar{u}} = \frac{d\bar{u}}{d\bar{t}}.$$

The steady sliding solution is given by

$$\begin{cases} \bar{u}_{er} = \delta \dfrac{1}{\xi^2 - 1}\Big(\dfrac{\xi^2}{\bar{r}} - \bar{r}\Big),\ \bar{u}_{e\theta} = \delta f \dfrac{1}{\xi^2 - 1}\Big(\dfrac{\xi^2}{\bar{r}} - \bar{r}\Big)\Big(1 + \dfrac{1}{\xi^2(1-2\nu)}\Big), \\ p_e = \delta \dfrac{1}{\xi^2 - 1}\dfrac{1}{1+\nu}\Big(\xi^2 + \dfrac{1}{1-2\nu}\Big) > 0,\ q_e = f p_e. \end{cases} \tag{25}$$

Since closed form dynamical solutions cannot be generated, two complementary approaches has been followed. The first approach is semi-analytical after a reduction to a simpler system of equations. The second approach consists of a numerical simulation by the finite element method and appropriate time-integrations.

An interesting simplification to the problem is obtained when the displacement is sought in the form

$$\bar{u}_r = U(\theta, \bar{t})F(\bar{r}),\ \bar{u}_\theta = V(\theta, \bar{t})F(\bar{r}),\ F(\bar{r}) = \frac{1}{\xi^2 - 1}\Big(\frac{\xi^2}{\bar{r}} - \bar{r}\Big). \tag{26}$$

In this approximation, the following local equations are obtained from the virtual work equation

$$\begin{cases} \ddot{U} - bU'' - dV' + gU = P, \\ \ddot{V} - aV'' + dU' + hV = Q, \\ P \geq 0,\ U - \delta \geq 0,\ P(U - \delta) = 0, \\ |Q| \leq fP,\ Q(1 - \dot{V}) - fP|1 - \dot{V}| = 0, \end{cases} \tag{27}$$

where $'$ denotes the derivative with respect to θ and a, b, g, h, d are material and geometry constants. All of them are positive except for the coupling coefficient d. Finally, only the non-dimensional displacements on the contact surface $U(\theta, t)$ and $V(\theta, t)$ and the non-dimensional reactions $P(\theta, t)$ and $Q(\theta, t)$ remain as unknowns in the reduced equations.

The steady sliding solution, given by $U_e = \delta,\ V_e = \delta f g/h,\ P = P_e$ and $Q_e = fP_e$, is unstable for the reduced system. Indeed, under the assumption of sliding motions, a small perturbed motion is described by $U = U_e$, $V = V_e + V_*$, $P = P_e + P_*$ and $Q = Q_e + Q_*$. It follows that

$$\ddot{V}_* - aV_*'' + fdV_*' + hV_* = 0. \tag{28}$$

If a general solution is sought in the form $V_* = e^{s\bar{t}}e^{ik\theta}$, then $-s^2 = ak^2 + h + ikfd$. When $f = 0$, it follows that $s = \pm i\omega_k$ with $\omega_k^2 = ak^2 + h$. Thus two harmonic waves propagating in opposite senses of the form $\cos(k\theta \pm \omega_k \bar{t} + \varphi)$ are obtained as in classical elasticity. When $f > 0$ and $d > 0$, then $s = \pm(s_{rk} + is_{ik})$, $s_{rk} > 0$, $s_{ik} < 0$, thus a general solution of the difference V_* of the form $V_* = e^{\pm s_{rk}\bar{t}} \cos(k\theta \pm s_{ik}\bar{t} + \varphi)$ is obtained and represents two waves propagating in opposite senses: an exploding wave in the sense of the implied rotation, and a damping wave propagating in the opposite direction. If $f > 0$ and $d < 0$, the exploding wave propagates in the opposite sense since the previous expression of s is still valid with $s_{rk} > 0$ and $s_{ik} > 0$.

It is expected that in some particular situations, there is a dynamic bifurcation of Poincaré-Andronov-Hopf's type. This means that the perturbed motion may evolve to a periodic response. This transition has been observed numerically in many examples of the literature, cf. for example [30] or [40]. To explore this idea, a periodic solution has been sought in the form of a wave propagating at constant velocity:

$$U = U(\phi),\ V = V(\phi),\ \phi = \theta - \bar{c}\bar{t}, \tag{29}$$

where $\bar{c}$ is the non-dimensional wave velocity, U and V are periodic functions of period $T = 2\pi/k$. The physical velocity of the wave is thus $c = |\bar{c}|R\Omega$ and the associated dynamic response is periodic of frequency $|\bar{c}|k\Omega$. The propagation occurs in the sense of the rotation when $c > 0$. According to the regime of contact, a slip wave, a stick-slip wave, a slip-separation wave or a stick-slip-separation wave can be discussed. The governing equations of such a wave follow from (24):

$$\begin{cases} (\bar{c}^2 - b)U'' - dV' + gU = P, \\ (\bar{c}^2 - a)V'' + dU' + hV = Q, \\ P \geq 0,\ U \geq \delta,\ P(U - \delta) = 0, \\ |Q| \leq fP,\ Q(1 - \dot{V}) - fP|1 - \dot{V}| = 0. \end{cases} \tag{30}$$

The existence of stick-slip waves is obtained when the load is sufficiently strong or when the rotation is slow. For example, for $\xi = 1.25$ and $f = 1$, stick-positive slip solutions are obtained for $8 \leq k \leq 12$. It is found that c must have the sign of d. These waves propagate in the sense of the previous exploding perturbed motions, thus opposite to the rotation of the cylinder when $d < 0$, with a frequency and a celerity independent of the rotation velocity Ω. The celerity is close to the celerities of dilatation and shear waves in the solid while the frequency is inversely proportional to the radius R. For example, for $\sqrt{E/\rho}$ $= 1000\,\text{m/s}$, $\xi = 1.25$, $f = 1$, $R = 1\,\text{m}$ and $\Omega = 100\,\text{rad/s}$, the results obtained concerning the mode-8 wave are $\Psi = 0.839$, $c = 1255\,\text{m/s}$ and the associated frequency is 10045 Hz. If $\xi = 2$, $f = 0.3$, $R = 0.5\,\text{m}$, $\Omega = 10\,\text{rad/s}$, a frequency 8240 Hz and a celerity 1030 m/s are obtained for $k = 4$ as shown in Fig. 4. For $\xi =$ 1.15 for example, d is positive and the propagation goes in the rotation direction. The limiting case $\xi \to 1$ can be interpreted as the sliding motion of a rigid plate on an elastic layer or of a rigid half-space on an elastic half-space [1, 18]. The obtained solution is a wave with an oscillation about the steady sliding response. The amplitude of the wave is linearly proportional to the rotation Ω. It also increases with the friction coefficient f and decreases with the mismatch. Thus, for vanishing rotations, the steady sliding solution is recovered as the limit of the dynamic response. The stick-slip solution can no longer be available if the rotation is strong enough since the associated pressure may become negative. In the same spirit, for a small mismatch, the pressure may become negative under the assumption of a stick-slip regime everywhere. This means that the possibility of separation is not excluded when the mismatch is not strong enough or if the rotation or the friction coefficient is sufficiently high.

A numerical simulation with an explicit scheme using Lagrange multipliers [5, 6] has been performed. The case $\xi = 2$ and $f = 0.3$ has been considered. Starting from a motionless initial state, the mismatch displacement is then increased linearly from 0 to its final value. A cyclic limit response is then obtained for large time. The numerical simulation leads to a stick-slip wave in mode 4 without forcing and the obtained response is close to the analytical solution of the reduced approach.

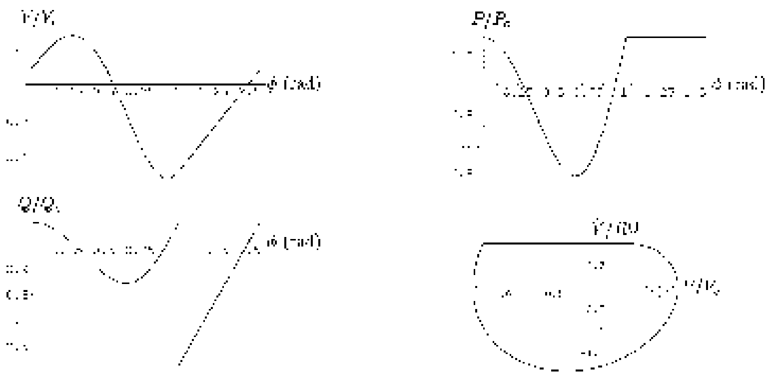

Figure 4. Semi-analytical approach: an example of stick-slip wave in mode 4 with $\xi = 2$, $f = 0.3$, $\Omega = 10$ rad/s, $R = 0.25$ m and $\delta = 0.005$. It is found that $\Psi = 0.644$, $c = 1030$ m/s. Phase diagram and variations of V/V_e, P/P_e and Q/Q_e in $[0, 2\pi/k]$.

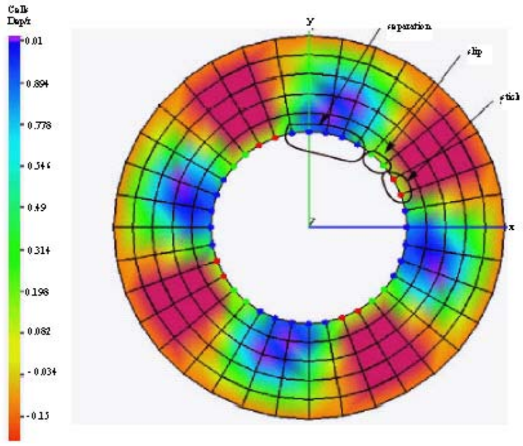

Figure 5. A mode-4 stick-slip-separation wave, obtained by numerical simulations for $\xi = 2$, $\Omega = 50$ rad/s, $f = 0.7$, $\delta = 0.001$. The isovalues of the radial displacement (mm): separation, slip and stick nodes on the contact surface are given respectively in red, green, blue.

It was also checked that a stick-slip-separation wave is effectively obtained when the mismatch is small enough or when the friction is high enough. For example, when $\Omega = 50\,\mathrm{rad/s}$, $\delta = 0.001$ and $f = 0.7$, the limit cycle results as a stick-slip-separation wave. The result for radial displacements is shown in Fig. 5.

In fact, it is well known that a periodic response does not systematically result from the flutter instability of the steady sliding solution. It has been observed in various examples of discrete or continuous systems that the response may be quasi-periodic or non-periodic or chaotic, [28, 29, 33]. Periodic responses prevail in the example of coaxial cylinders because of the special geometry of the system. Periodic solutions under the form of stick-slip waves have been also obtained by Adams in the sliding contact of two elastic half-spaces [3]. The generation of dynamic stick-slip-separation waves on the contact surface can be compared to Shallamach waves in the sliding contact of rubber in statics, cf. [3, 5, 17, 19, 20, 31, 35, 36, 39 - 41].

4. Friction-Induced Vibrations and Noises

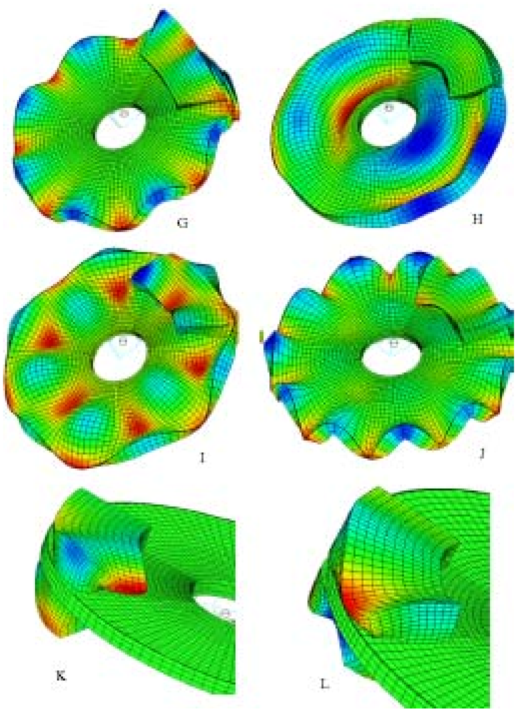

Figure 6. Some unstable modes of the system pad-disk in an automotive disk brake [20, 23]

It is well known that the presence of friction induces mechanical vibrations and noise emittences in the sliding contact of solids. For example, the creaking noise of a door, the unsteady motion with fits and starts of a windscreenwiper can be interpreted as stick-slip motions resulting from the instability of the steady sliding solution. In particular, the phenomena of squeal [10] have been interpreted in the literature following this approach. The squeals of band brakes in washing machines have been discussed in [26], the squeals of automotive disk brakes has been examined in [20, 21]. cf. Fig. 6. In the same spirit, the squeal of a system glass-rubber in finite deformation is considered in [40].

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