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# Cesàro-One Summability and Uniform Convergence of Solutions of a Sturm–Liouville System

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**Abstract.** In this paper a Galerkin method is used to construct series solutions of a homogeneous Sturm–Liouville problem defined on $[0, \pi]$. The series constructed are shown to converge to a specified du Bois-Reymond function $f$ in $L^2[0, \pi]$. It is then shown that the series solutions can be made to converge uniformly to the specified du Bois-Reymond function when averaged by the Cesaro-one summability method. Therefore, in the Cesaro-one sense, every continuous function $f$ on $[0, \pi]$ is the uniform limit of solutions of non-homogeneous Sturm–Liouville problems.

## 1. Introduction

Galerkin methods are used in separable Hilbert spaces to construct and compute $L^2[0, \pi]$ solutions to large classes of differential equations. In this note a Galerkin method is used to construct series solutions of a nonhomogeneous Sturm-Liouville problem defined on $[0, \pi]$. The series constructed are shown to converge to a specified du Bois-Reymond $f$ function in $L^2[0, \pi]$. It is then shown that the series solutions can be made to converge uniformly to the specified du Bois-Reymond function when averaged by the Cesro-one summability method. Therefore, in the Cesàro-one sense, every continuous function $f$ on $[0, \pi]$ is the uniform limit of solutions of nonhomogeneous Sturm–Liouville problems.

## 2. Cesàro-One Summability and the Solutions of a Sturm–Liouville System

Consider the regular Sturm–Liouville problem:

$$\begin{aligned} y'' + y &= 0, \\ y(0) = y(\pi) &= 0 \end{aligned} \tag{1}$$

defined on $[0, \pi]$. This problem has an associated complete orthonormal system of eigenvalues and eigenfunctions $(\lambda_n, \varphi_n)$ in $L^2[0, \pi]$, where

$$\begin{aligned} \lambda_n &= 1 - n^2, \\ \varphi_n &= \sin nx. \end{aligned} \tag{2}$$

Also, consider a du Bois-Reymond function $f$ on $[0, \pi]$. This means that $f$ is continuous, and $f$ has a Fourier series $f \sim \sum_{n=1}^{\infty} \alpha_n \varphi_n$, where $\alpha_n = \langle f, \varphi_n \rangle$ such that the Fourier series $\sum_{n=1}^{\infty} \alpha_n \varphi_n$ diverges on a dense $G_\delta$ subset of $[0, \pi]$, du Bois-Reymond [4]. Assume further that $f(0) = f(\pi) = 0$. From a theorem of Fejér [3], it is well known that the Fourier series of $f$ *converges uniformly* after an application of Cesàro-one summability ($(C, 1)$ summability). A modern outline of this result may be found in Bruckner et al. [1], while a detailed development of Cesàro summability is given in Hardy [2].

For each positive integer $N$, set $g_N = \sum_{n=1}^{N} \lambda_n \alpha_n \varphi_n$, then there exists a solution $y_N$ to the nonhomogeneous Sturm–Liouville problem:

$$\begin{aligned} y'' + y &= g_N, \\ y(0) = y(\pi) &= 0, \end{aligned} \tag{3}$$

namely

$$y_N = \sum_{n=1}^{N} \frac{\lambda_n \alpha_n}{\lambda_n} \varphi_n = \sum_{n=1}^{N} \alpha_n \varphi_n. \tag{4}$$

The functions $\{y_N\}$ diverge on a dense $G_\delta$ subset of $[0, \pi]$, but converge uniformly to $f$ when averaged by the $(C, 1)$ summability method. Thus, in the sense described here $f$ is a uniform limit of solutions to the problem

$$y'' + y = \sum_{n=1}^{N} \lambda_n \alpha_n \varphi_n, \tag{5}$$

where the right-hand side is quite badly behaved as $N$ becomes large. In case the Fourier series for $f$ actually does converge uniformly, the arguments still apply and $f$ is a uniform limit of solutions to the Sturm–Liouville problem in the same sense.

Moreover, notice that for any fixed value of $N$, the solution $y_N$ of the Sturm–Liouville problem (3) appears to be well behaved. It is only as $N$ increases without bound that $\{y_N\}$ becomes pathological. The $(C, 1)$ summability method applied here provides a "canonical regularization" of the sequence of solutions $\{y_n\}$ by assigning it to an averaged sequence where the limiting value converges uniformly to $f$. From this example, it is suspected that more general summability methods may have significant application in regularizing solutions of differential equations modeling complex multiscale physical phenomena, such as turbulent flow. In the next section, the Sturm–Liouville problem (1) is derived from a simple heat transfer problem with continuous initial data.

## 3. The Sturm–Liouville System via a Heat Equation

To motivate the Sturm–Liouville system (1), consider the following initial-boundary value problem from the theory of heat transfer:

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2} + u, \\ u(t,0) &= u(t,\pi) = 0, \\ u(0,x) &= f(x), \end{aligned} \tag{6}$$

where $f$ is a du Bois-Reymond function. Separating variables

$$u(t,x) = \tau(t)\varphi(x) \tag{7}$$

one obtains

$$\frac{\tau'}{\tau} = \frac{\varphi''}{\varphi} + 1 = \lambda, \tag{8}$$

where

$$\begin{aligned} \lambda_n &= 1 - n^2, \\ \varphi_n &= \sin nx. \end{aligned} \tag{9}$$

Equation (8) also implies that

$$\tau_n = \tau_n(0)\exp[\lambda_n t] \tag{10}$$

so that $u$ becomes

$$u(t,x) = \sum_{n=1}^{\infty} \tau_n(0)\exp[\lambda_n t]\sin nx. \tag{11}$$

Notice that evaluating equation (11) at $t = 0$ yields

$$u(0,x) = f(x) = \sum_{n=1}^{\infty} \tau_n(0)\sin nx, \tag{12}$$

where $\tau_n(0) = \alpha_n = \langle f, \varphi_n\rangle$, then since $f$ is a du Bois-Reymond function, the series $\sum\limits_{n=1}^{\infty} \alpha_n \sin nx$ converges in $L^2[0,\pi]$ to $f$. However, $\sum\limits_{n=1}^{\infty} \alpha_n \sin nx$ diverges on a dense $G_\delta$ subset of $[0,\pi]$ and after the application of $(C,1)$ summability, converges uniformly to $u(0,x)$.

Next set

$$u_N(t,x) = \sum_{n=1}^{N} \alpha_n \exp[\lambda_n t]\sin nx$$

so that

$$\frac{\partial u_N}{\partial t} = \sum_{n=1}^{N} \alpha_n \lambda_n \exp[\lambda_n t]\sin nx,$$

where $\left.\frac{\partial u_N}{\partial t}\right|_{t=0} = \sum_{n=1}^{N} \alpha_n \lambda_n \sin nx = g_N$ is the $g_N$ defined in the previous section. Furthermore, $\left.\frac{\partial u_N}{\partial t}\right|_{t=0} = \left.\frac{\partial^2 u_N}{\partial x^2}\right|_{t=0} + u_N|_{t=0} = g_N$ so that $y_N$ of equation (4) is the solution to the ordinary differential equation $\left.\frac{\partial^2 u_N}{\partial x^2}\right|_{t=0} + u_N|_{t=0} = g_N'' + y_N = g_N$ of equation (5).

Finally notice that while the series defining $\frac{\partial u_N}{\partial t}$ is pathological at $t = 0$, it converges absolutely and uniformly for all $x \in [0, \pi]$ for each $t > 0$. This can be shown as follows. Since $\{\alpha_n\} \in l^2$, the $\{\alpha_n\}$ are bounded by some constant $M$, then

$$\begin{aligned}\sum_{n=1}^{\infty} \left|\alpha_n (1 - n^2) \exp\left[(1 - n^2)t\right] \sin nx\right| &\le \sum_{n=1}^{\infty} |\alpha_n| \, |(1 - n^2)| \exp[(1 - n^2)t] \\ &\le M \sum_{n=1}^{\infty} |(1 - n^2)| \exp[1 - n^2)t].\end{aligned} \tag{13}$$

Since $t$ is fixed, the integral test $\int_1^\infty (1 - n^2) \exp[(1 - x^2)t] dx < \infty$ implies that the right hand side of equation (13) is finite, and thus the series for $\frac{\partial u_N}{\partial t}$ converges absolutely and uniformly for all $x \in [0, \pi]$ for each $t > 0$.

## 4. Conclusions

The following conclusions may be drawn from the above example:

C1. In the $(C, 1)$ sense, every continuous function $g$ on $[0, \pi]$ can be mapped to an $f$ satisfying $f(0) = f(\pi) = 0$ by $f(t) = g(t) - \left[\frac{g(\pi) - g(0)}{\pi}\right] t - g(0)$, which is a $(C, 1)$ limit of solutions to the Sturm–Liouville problem given by system (3).

C2. The method used for approximating the solution is of the Galerkin type. The solutions $y_N$ are projections into the finite dimensional subspaces of the Hilbert space $L^2[0, \pi]$, which have as basis elements $\{\varphi_n\}_{n=1}^N$.

C3. The Galerkin method fails to work in the sense that the resulting approximate solutions do not converge pointwise to a solution on $[0, \pi]$ (although $L^2$ convergence is assured). Furthermore, the set of functions $f \in C[0, \pi]$, whose Fourier series diverge pointwise on a dense $G_\delta$ subset of $[0, \pi]$, is the complement of a Baire first category set in $C[0, \pi]$ (see [1]).

## 5. Open Research Problems

In situations in which numerical values are required and Galerkin methods are used, ordinary $L^2$ convergence is of limited utility. It is with this in mind that we pose the following problems:

Suppose $\{\varphi_n\}_{n\in\eta}$ is an orthonormal basis for the Hilbert space $H \equiv L^2[a,b]$. Let $f \in H$, then $f \cong \{\alpha_n\} \in l_n^2$, where $\alpha_n = \int_a^b f(t)\varphi_n(t)dt$ and by the Riesz–Fischer theorem [1], $\cong$ is an isometric isomorphism. The following research problems are raised by consideration of the above discussion:

P1. Find necessary and sufficient conditions on $\{\alpha_n\}$ such that $\sum_{n=1}^{\infty} \alpha_n\varphi_n$ converges uniformly to $f$ on $[a,b]$.

P2. Find summability methods $S$ such that $\{S(\alpha_n)\}$ satisfies the conditions of P1.

P3. Do the results of P1 and P2 extend to arbitrary $H$ and $l_\eta^2$?.

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