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Lavrentiev Regularization of Nonlinear Ill-Posed Problems

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Abstract. In this paper we study the method of Lavrentiev regularization to reconstruct solutions $x^\dagger$ of nonlinear ill-posed problems $F(x) = y$ where instead of y noisy data $y^\delta \in X$ with $\|y - y^\delta\| \le \delta$ are given and $F : D(F) \subset X \to X$ is a *monotone* nonlinear operator. In this method regularized approximations are obtained by solving the singularly perturbed nonlinear operator equation $F(x)+\alpha(x-\overline{x}) = y^\delta$ with some initial guess $\overline{x}$. Assuming certain conditions concerning the nonlinear operator F and the smoothness of the element $\overline{x} - x^\dagger$ we report on stability estimates which show that the accuracy of the regularized approximations is order optimal provided that the regularization parameter α has been chosen properly.

1. Introduction

This paper is devoted to the stable solution of nonlinear ill-posed problems

$$F(x) = y, \tag{1}$$

where $F : D(F) \to X$ is a nonlinear operator with domain $D(F) \subset X$ and X is a real Hilbert space with inner product $(\cdot, \cdot)$ and norm $\| \cdot \|$. Throughout this paper we assume that F is a *monotone* operator, that is, there holds

$$(F(x_2) - F(x_1), x_2 - x_1) \ge 0 \quad \text{for all} \quad x_1, x_2 \in D(F) \subset X. \tag{2}$$

We further assume throughout that $y^\delta \in X$ are the available noisy data with

$$\|y - y^\delta\| \le \delta \tag{3}$$

and known noise level δ, that (1) has a solution $x^\dagger$ and that F possesses a locally uniformly bounded Fréchet-derivative $F'(\cdot)$ in a ball $B_r(x^\dagger)$ of radius r around $x^\dagger \in X$.

Nonlinear ill-posed problems arise in different applications in connection with nonlinear Fredholm integral equation models, nonlinear Abel integral equation models or identification problems for ordinary and partial differential integrations. Their numerical treatment requires the application of special regularization methods. The standard regularization method is the *method of Tikhonov regularization* in which a regularized approximation x_α^δ is obtained by minimizing Tikhonov's functional

$$J_\alpha(x) = \|F(x) - y^\delta\|^2 + \alpha\|x - \overline{x}\|^2$$

with some initial guess $\overline{x} \in X$ and some properly chosen regularization parameter $\alpha > 0$. If x_α^δ is an interior point of $D(F)$, then x_α^δ satisfies the *normal operator equation*

$$F'(x)^*[F(x) - y^\delta] + \alpha(x - \overline{x}) = 0$$

of Tikhonov's functional $J_\alpha(x)$ and may be computed by the Gauss-Newton-iteration

$$x_{k+1} = x_k - (A_k^* A_k + \alpha I)^{-1}\big\{A_k^*[F(x_k) - y^\delta] + \alpha(x_k - \overline{x})\big\}$$

with some initial guess x_0, where $A_k = F'(x_k)$ and A_k^* is the adjoint of A_k.

In the case of *monotone* operators F the least squares minimization, and hence the use of the adjoint, can be avoided and one can use the simpler regularized equation

$$F(x) + \alpha(x - \overline{x}) = y^\delta. \tag{4}$$

This method is called *method of Lavrentiev regularization* [8], or *method of singular perturbation* [9]. Here the regularized approximation may be computed, e.g., by appling Newton's method which leads to the simpler iteration

$$x_{k+1} = x_k - (A_k + \alpha I)^{-1}\left\{[F(x_k) - y^\delta] + \alpha(x_k - \overline{x})\right\}.$$

Since Fréchet-differentiability implies hemicontinuity, from [1] the following result concerning existence and uniqueness of the regularized approximation of problem (4) can be obtained:

Theorem 1.1. *Let $x^\dagger \in D(F)$ be a solution of* (1) *and let $F : D(F) \to X$ be Fréchet-differentiable and monotone in a ball $B_r(x^\dagger) \subset D(F)$ with radius $r = \|\overline{x} - x^\dagger\| + \delta/\alpha$. Then the regularized problem* (4) *possesses a unique solution $x_\alpha^\delta \in B_r(x^\dagger)$.*

Lavrentiev regularization has been studied by many authors, especially for solving Volterra integral equations of the first kind, see, e.g., [2, 5, 7 - 11, 13, 16]. Here the advantage is that Lavrentiev regularization preserves the natural evolutionary structure of Volterra equations and therefore leads to quick and simple numerical procedures. One important question in the application of the regularization method (4) is the proper choice of the regularization parameter $\alpha > 0$. With too little regularization, the regularized approximations x_α^δ are highly oscillatory due to noise amplification. With too much regularization, the regularized approximations are too close to the initial guess $\overline{x}$ due to the limit relation $\lim_{\alpha\to\infty} x_\alpha^\delta = \overline{x}$. Ideally, one should select the regularization parameter α such that the total error $\|x_\alpha^\delta - x^\dagger\|$ is minimized. Since $x^\dagger$ is unknown one has to choose alternative rules which choose α from quantities that arise during calculations. Among the rules for which order optimal error bounds of Hölder type $\|x_\alpha^\delta - x^\dagger\| = O(\delta^{p/(p+1)})$ can be guaranteed, the most common in the linear case is the rule of Raus [12] which requires the knowledge of a reliable lower bound of the noise level δ. For the concept of order optimality see, e.g., [4]. The proof of order optimal error bounds in the nonlinear case requires some source condition and some nonlinearity condition. We will distinguish our studies into different cases $p = 1$ and $p \in (0, 1]$. In the case $p = 1$ our analysis requires the two assumptions

Assumption A1. There exists an element $v \in X$ such that with $A = F'(x^\dagger)$

$$\overline{x} - x^\dagger = Av.$$

Assumption A2. The Fréchet-derivative $F'(\cdot)$ is locally Lipschitz in a ball $B_r(x^\dagger) \subset D(F)$ of radius r around $x^\dagger \in X$, that is, there exists a Lipschitz constant $L \geq 0$ with

$$\|F'(x) - F'(x_0)\| \leq L\|x - x_0\| \quad \text{for all} \quad x, x_0 \in B_r(x^\dagger).$$

In the case $p \in (0, 1]$ we don't know if Lipschitz continuity of the Fréchet derivative $F'(\cdot)$ is sufficient for guaranteeing order optimal error bounds. In this case instead of A1, A2 we will exploit a weaker smoothness assumption and a stronger nonlinearity assumption:

Assumption A3. There exists an element $v \in X$ such that with $A = F'(x^\dagger)$

$$\overline{x} - x^\dagger = A^p v \quad \text{for some} \quad p \in (0, 1].$$

Assumption A4. There exists a constant $k_0 \geq 0$ such that for all x, $x_0 \in B_r(x^\dagger) \subset D(F)$ and $w \in X$ there exists some element $k(x, x_0, w) \in X$ with property

$$[F'(x) - F'(x_0)]w = F'(x_0)k(x, x_0, w) \quad \text{and} \quad \|k(x, x_0, w)\| \leq k_0\|w\|.$$

The fractional power A^p in Assumption A3 is defined by

$$A^p v := \frac{\sin p\pi}{\pi} \int_0^\infty s^{p-1}(A + sI)^{-1}Av \, \mathrm{d}s\,, \quad v \in X$$

and reduces in the selfadjoint case $A = A^* > 0$ to $A^p v = \int_0^a \lambda^p \, \mathrm{d}E_\lambda v$ where $a = \|A\|$ and $A = \int_0^a \lambda \, \mathrm{d}E_\lambda$ is the spectral decomposition of the operator A. Note that the converse results in [3] for linear ill-posed problems imply that Assumption A3 is *necessary* for the convergence rate $\|x_\alpha^\delta - x^\dagger\| = O(\delta^{p/(p+1)})$.

Let us collect two well known estimates which will be applied throughout this paper. The first estimate

$$\|F(x) - F(x_0) - F'(x_0)(x - x_0)\| \le \frac{L}{2}\|x - x_0\|^2 \tag{5}$$

follows from Assumption A2. The second estimate (see, e.g., [10]) tells us that for *monotone* operators $A \in \mathcal{L}(X)$ and $p \in [0,1]$,

$$\|(A + \alpha I)^{-1} A^p\| \le \alpha^{p-1}. \tag{6}$$

The paper is organized as follows. In Sec. 2 we derive order optimal error bounds for method (4) in case of *a priori* parameter choice which are along the lines of the paper [13] and improve well known error bounds in [1, 2, 9]. In Sec. 3 we generalize the rule of Raus for choosing α to the nonlinear case and show that this *a posteriori* rule provides order optimal error bounds $\|x_\alpha^\delta - x^\dagger\| = O(\delta^{p/(p+1)})$ for the maximal range $p \in (0,1]$.

2. A Priori Parameter Choice

2.1. Preliminary Properties

Throughout this and the next section we will consistently use the notations

$$A = F'(x^\dagger)\,, \quad A_\alpha^\delta = F'(x_\alpha^\delta)\,, \quad R_\alpha = \alpha(A + \alpha I)^{-1}\,, \quad R_\alpha^\delta = \alpha(A_\alpha^\delta + \alpha I)^{-1}.$$

Note that for monotone operators F the resulting operators A, A_α^δ, R_α and R_α^δ, respectively, are also monotone. The aim of this section is to derive order optimal error bounds for the method of Lavrentiev regularization (4) in the case of *a priori* parameter choice. Let us start with some preliminary results which can be obtained by exploiting property (2).

Proposition 2.1. *Assume the monotonicity property* (2). *Let x_α be the unique solution of the singularly perturbed operator equation*

$$F(x) + \alpha(x - \overline{x}) = y \tag{7}$$

and x_α^δ the unique solution of the singularly perturbed operator equation (4). *Then,*

$$\|x_\alpha^\delta - x_\alpha\| \le \frac{\delta}{\alpha} \quad \text{and} \quad \|x_\alpha - x^\dagger\| \le \|\overline{x} - x^\dagger\|\,. \tag{8}$$

Proof. Subtracting the equations (4) and (7) yields $F(x_\alpha^\delta) - F(x_\alpha) + \alpha(x_\alpha^\delta - x_\alpha) = y^\delta - y$ and scalar multiplication by $x_\alpha^\delta - x_\alpha$ gives

$$(F(x_\alpha^\delta) - F(x_\alpha), x_\alpha^\delta - x_\alpha) + \alpha\|x_\alpha^\delta - x_\alpha\|^2 = (y^\delta - y, x_\alpha^\delta - x_\alpha).$$

Due to assumption (2) the first summand on the left hand side is nonnegative. By neglecting this summand we obtain $\alpha\|x_\alpha^\delta - x_\alpha\|^2 \le \delta\|x_\alpha^\delta - x_\alpha\|$ and hence the first estimate of (8). Furthermore, (7) can be written in the form $F(x_\alpha) - F(x^\dagger) + \alpha(x_\alpha - x^\dagger) = \alpha(\overline{x} - x^\dagger)$. Scalar multiplication by $x_\alpha - x^\dagger$ yields

$$(F(x_\alpha) - F(x^\dagger), x_\alpha - x^\dagger) + \alpha\|x_\alpha - x^\dagger\|^2 = \alpha(\overline{x} - x^\dagger, x_\alpha - x^\dagger).$$

Due to the monotonicity property (2) the first summand on the left hand side is nonnegative. By neglecting this summand we obtain $\alpha\|x_\alpha - x^\dagger\|^2 \le \alpha\|\overline{x} - x^\dagger\|\,\|x_\alpha - x^\dagger\|$ and hence the second estimate of (8). ■

The second estimate of (8) shows that $x_\alpha \in B_r(x^\dagger)$ with $r = \|\overline{x} - x^\dagger\|$. In order to derive error bounds for $\|x_\alpha - x^\dagger\|$ in terms of α some source condition and some nonlinearity conditions are needed. We divide our studies into the cases $p = 1$ and $p \in (0, 1]$.

2.2. Error Bounds in the Case $p = 1$

In this subsection we exploit Assumptions A1 and A2 of Sec. 1 and obtain

Theorem 2.2. *Assume* (2), A1 *and* A2 *with radius* $r = \alpha\|v\|$. *Let* x_α^δ *be the unique solution of the singularly perturbed operator equation* (4). *Then, for all* $\alpha > 0$,

$$\|x_\alpha^\delta - x^\dagger\| \le \frac{\delta}{\alpha} + \Big(\|v\| + \frac{L}{2}\|v\|^2\Big)\alpha. \tag{9}$$

If α *is chosen a priori by* $\alpha \sim \sqrt{\delta}$, *then* $\|x_\alpha^\delta - x^\dagger\| = O(\sqrt{\delta})$.

Proof. Let x_α be the solution of (7). We introduce the notations $z_\alpha = x_\alpha - x^\dagger$ and $B = AR_\alpha = R_\alpha A$ with $R_\alpha = \alpha(A + \alpha I)^{-1}$. Then we have from (7), A1 and $\alpha A - \alpha B = AB$

$$\begin{aligned} F(x_\alpha) - F(x^\dagger + Bv) + \alpha(z_\alpha - Bv) &= F(x^\dagger) - F(x^\dagger + Bv) + \alpha Av - \alpha Bv \\ &= F(x^\dagger) + ABv - F(x^\dagger + Bv). \end{aligned}$$

Scalar multiplication by $z_\alpha - Bv$ yields

$$\begin{aligned} &(F(x_\alpha) - F(x^\dagger + Bv), z_\alpha - Bv) + \alpha\|z_\alpha - Bv\|^2 = \\ &(F(x^\dagger) + ABv - F(x^\dagger + Bv), z_\alpha - Bv). \end{aligned}$$

Due to the monotonicity property (2) the first summand on the left hand side is nonnegative. Consequently,

$$\alpha\|z_\alpha - Bv\|^2 \le \|F(x^\dagger) + ABv - F(x^\dagger + Bv)\|\|z_\alpha - Bv\|.$$

From Assumption A2 one gets (5). We apply this property with $x := x^\dagger + Bv$ and $x_0 := x^\dagger$ and obtain due to $\|B\| \le \alpha$ that

$$\|z_\alpha - Bv\| \le \frac{L}{2\alpha}\|Bv\|^2 \le \frac{L}{2}\alpha\|v\|^2.$$

Applying the triangle inequality $\|z_\alpha\| \le \|z_\alpha - Bv\| + \|Bv\|$ we obtain

$$\|x_\alpha - x^\dagger\| \le \Big(\|v\| + \frac{L}{2}\|v\|^2\Big)\alpha.$$

From this estimate and the first estimate of (8) we obtain (9). The convergence rate result $\|x_\alpha^\delta - x^\dagger\| = O(\sqrt{\delta})$ follows from (9) and the *a priori* parameter choice $\alpha \sim \sqrt{\delta}$.

Error bounds for $\|x_\alpha - x^\dagger\|$ have been given before in the literature under the Assumptions A1 and A2 and have the form

$$\|x_\alpha - x^\dagger\| \le \frac{2\|v\|}{2 - L\|v\|}\,\alpha$$

(see, e.g. [1,2]). This error bound requires the smallness condition $L\|v\| < 2$. Such a smallness condition is *not* necessary for our estimate.

2.3. Error Bounds in the Case $p \in (0,1]$

Now let us study the case $p \in (0,1]$ in Assumption A3. In this case we have not been able to prove order optimal convergence rate results with the nonlinearity Assumption A2. Instead we have been successful with Assumption A4 instead of A2.

Theorem 2.3. *Assume* (2), A3 *and* A4 *with radius* $r = \|\overline{x} - x^\dagger\|$. *Let* x_α^δ *be the unique solution of the singularly perturbed operator equation* (4). *Then, for all* $\alpha > 0$,

$$\|x_\alpha^\delta - x^\dagger\| \le \frac{\delta}{\alpha} + (1 + k_0)\|v\|\alpha^p. \tag{10}$$

If α *is chosen a priori by* $\alpha \sim \delta^{1/(p+1)}$, *then* $\|x_\alpha^\delta - x^\dagger\| = O(\delta^{p/(p+1)})$.

Proof. Let x_α be the solution of (7) and let us introduce the operator

$$M_\alpha = \int_0^1 F'(x^\dagger + t(x_\alpha - x^\dagger))\,\mathrm{d}t. \tag{11}$$

Obviously, $F(x_\alpha) - F(x^\dagger) = M_\alpha(x_\alpha - x^\dagger)$. Hence, equation (7) can be written in the form

$$M_\alpha(x_\alpha - x^\dagger) + \alpha(x_\alpha - x^\dagger) = \alpha(\overline{x} - x^\dagger). \tag{12}$$

Due to A1 we have $\overline{x} - x^\dagger = A^p v$. Furthermore, from (2) it follows that the inverse operators $(M_\alpha + \alpha I)^{-1}$ and $(A + \alpha I)^{-1}$, respectively, exist. Consequently,

$$\begin{aligned} x_\alpha - x^\dagger &= \alpha(M_\alpha + \alpha I)^{-1}(\overline{x} - x^\dagger) \\ &= \alpha(A + \alpha I)^{-1}A^p v + \alpha[(M_\alpha + \alpha I)^{-1} - (A + \alpha I)^{-1}]A^p v \\ &= R_\alpha A^p v + (M_\alpha + \alpha I)^{-1}(A - M_\alpha)R_\alpha A^p v. \end{aligned} \tag{13}$$

Application of Assumption A4 yields

$$\|x_\alpha - x^\dagger\| \le \|R_\alpha A^p v\| + k_0\|(M_\alpha + \alpha I)^{-1}M_\alpha\|\|R_\alpha A^p v\|.$$

Now we use the valid estimate $\|R_\alpha A^p\| \le \alpha^p$ (see (6)) and obtain

$$\|x_\alpha - x^\dagger\| \le (1 + k_0)\|v\|\alpha^p.$$

From this estimate and the first estimate of (8) we obtain (10). The convergence rate result $\|x_\alpha^\delta - x^\dagger\| = O(\delta^{p/(p+1)})$ follows from (10) and the parameter choice $\alpha \sim \delta^{1/(p+1)}$. ■

Let us compare our order optimal error bounds in Theorems 2.2 and 2.3 with corresponding bounds for *Tikhonov regularization.* For Tikhonov regularization instead of A1 and A3 the source condition

$$\overline{x} - x^\dagger = [F'(x^\dagger)^* F'(x^\dagger)]^{p/2} v\,, \quad v \in X\,, \quad p > 0 \tag{14}$$

is required. Then it can be shown that in case of *a priori* parameter choice $\alpha \sim \delta^{2/(p+1)}$ order optimal error bounds $\|x_\alpha^\delta - x^\dagger\| \sim \delta^{p/(p+1)}$ hold true

(i) for the range $p \in [1, 2]$ provided A2 and the smallness condition $L\|v\| < 1$ are satisfied (see [3, 4, 14]),

(ii) for the range $p \in (0, 1]$ provided some additional conditions concerning the nonlinear operator F similar to condition A4 are satisfied (see [6, 14]).

While for Tikhonov regularization with *a priori* parameter choice order optimal error bounds can be guaranteed for the larger range $p \in (0, 2]$, Lavrentiev regularization is simpler and does not require smallness conditions like $L\|v\| < 1$.

3. A Posteriori Parameter Choice

3.1. Preliminary Properties

A priori parameter choice is seldom suitable in practice since the choice of a good regularization parameter α requires the knowledge of the norm $\|v\|$ and the smoothness parameter p of Assumption A3. This knowledge is not necessary for *a posteriori* parameter choice. For the method of Tikhonov regularization one well known *a posteriori* rule is Morozov's discrepancy principle (see [3]) in which the regularization parameter α is chosen as the solution of the nonlinear equation $\|F(x_\alpha^\delta) - y^\delta\| = C\delta$ with some $C > 1$. However, Morozov's discrepancy principle is divergent for Lavrentiev regularization (see [16] for the linear case). An *a posteriori* rule for which order optimal convergence rates can be guaranteed has been proposed by Raus (see [12]) in the linear case with $A = A^* > 0$. This rule can be extended to the nonlinear case with monotone operators F and reads as follows:

Rule R1. Choose the regularization parameter α as the solution of the nonlinear equation

$$d(\alpha) := \|R_\alpha^\delta [F(x_\alpha^\delta) - y^\delta]\| = C\delta \quad \text{with} \quad C > 1. \tag{15}$$

In our subsequent considerations we will see that rule R1 provides order optimal error bounds for the maximal range $p \in (0, 1]$. We distinguish our study into the special case $p \in (0, 1]$ which requires Assumptions A3, A4, and $p = 1$ where Assumptions A1, A2 are necessary. Let us start our study with the justification of rule R1 for which the Assumptions A1 - A4 are not necessary.

Proposition 3.1. *Let the monotonicity property* (2) *be satisfied. If the initial guess* $\overline{x} \in X$ *satisfies* $\|F(\overline{x}) - y^\delta\| > C\delta$*, then there exists a solution* $\alpha = \alpha(\delta)$ *of equation* (15) *with*

$$\alpha \geq \alpha_1 := \frac{C-1}{\|\overline{x} - x^\dagger\|}\,\delta. \tag{16}$$

The proof of this proposition can be found in [3]. In our forthcoming considerations we always assume that rule R1 is well defined and do not state the conditions explicitly.

3.2. Error Bounds in the Case $p \in (0,1]$

In order to prove order optimal error bounds for $\|x_\alpha^\delta - x^\dagger\|$ with α chosen from rule R1 three preparatory propositions are required.

Proposition 3.2. *Let* (2) *and Assumption* A4 *with* $k_0 < 1$ *and radius* $r = \|\overline{x} - x^\dagger\|$ *be satisfied. Then, for all* $0 < \alpha_0 \leq \alpha$,

$$\|x_\alpha - x^\dagger\| \leq \frac{\|R_\alpha[F(x_\alpha) - y]\|}{(1-k_0)\alpha_0} + \|x_{\alpha_0} - x^\dagger\|. \tag{17}$$

Proof. Due to $F(x_\alpha) - F(x_{\alpha_0}) = M(x_\alpha - x_{\alpha_0})$ with $M = \int_0^1 F'(x_{\alpha_0} + t(x_\alpha - x_{\alpha_0}))\,\mathrm{d}t$ and (7) we obtain the identity

$$x_\alpha - x_{\alpha_0} = \frac{\alpha - \alpha_0}{\alpha\alpha_0} R_{\alpha_0}[F(x_\alpha) - y] + (A + \alpha_0 I)^{-1}(A - M)(x_\alpha - x_{\alpha_0}). \tag{18}$$

Since $\|R_{\alpha_0} R_\alpha^{-1}\| \leq 1$ for $\alpha_0 \leq \alpha$, we conclude that

$$\left\| \frac{\alpha - \alpha_0}{\alpha\alpha_0} R_{\alpha_0}[F(x_\alpha) - y] \right\| \leq \left| \frac{\alpha - \alpha_0}{\alpha\alpha_0} \right| \|R_\alpha[F(x_\alpha) - y]\|.$$

Consequently, from (18) we obtain due to $\alpha_0 \leq \alpha$ and A4 the estimate

$$\|x_\alpha - x_{\alpha_0}\| \leq \frac{1}{\alpha_0} \|R_\alpha[F(x_\alpha) - y]\| + k_0 \|x_\alpha - x_{\alpha_0}\|.$$

Now (17) follows from the triangle inequality $\|x_\alpha - x^\dagger\| \leq \|x_\alpha - x_{\alpha_0}\| + \|x_{\alpha_0} - x^\dagger\|$. ∎

Proposition 3.3. *Let* (2) *and Assumption* A4 *with radius* $r = \|\overline{x} - x^\dagger\| + \delta/\alpha$ *be satisfied and let* α *be chosen by rule* R1. *Then,*

$$\|R_\alpha[F(x_\alpha) - y]\| \leq (C+1)(1+k_0)\delta\,. \tag{19}$$

Proof. We use the notations $a = \|R_\alpha[F(x_\alpha) - F(x^\dagger)]\|$ and $b = \|R_\alpha^\delta[F(x_\alpha) - F(x^\dagger)]\|$, exploit the triangle inequality, apply Assumption A4 and obtain

$$\begin{aligned} a &\le \|R_\alpha^\delta[F(x_\alpha^\delta) - F(x^\dagger)]\| + \|(R_\alpha - R_\alpha^\delta)[F(x_\alpha) - F(x^\dagger)]\| \\ &= b + \|(A+\alpha I)^{-1}(A_\alpha^\delta - A)R_\alpha^\delta[F(x_\alpha) - F(x^\dagger)]\| \\ &\le (1+k_0)b. \end{aligned} \tag{20}$$

In order to estimate b in terms of δ we use the equations (4) and (7), the triangle inequality, the inequality $\|R_\alpha^\delta\| \le 1$, the first estimate of (8) as well as rule R1 and obtain

$$\begin{aligned} b &= \|R_\alpha^\delta[\alpha(\overline{x} - x_\alpha^\delta) + \alpha(x_\alpha^\delta - x_\alpha)]\| \\ &\le \|R_\alpha^\delta[F(x_\alpha^\delta) - y^\delta]\| + \alpha\|x_\alpha^\delta - x_\alpha\| \\ &\le (C+1)\delta. \end{aligned} \tag{21}$$

Now the desired estimate (19) follows from (20) and (21). ∎

In the next proposition we obtain a bound for $\alpha = \alpha(\delta)$ obtained by rule R1.

Proposition 3.4. *Let* (2), A3 *and* A4 *with radius* $r = \|\overline{x} - x^\dagger\| + \delta/\alpha$ *be satisfied and let* α *be chosen by rule* R1. *Then*

$$\alpha \ge \Big[\frac{C-1}{(1+k_0)^3\|v\|}\Big]^{\frac{1}{1+p}} \delta^{\frac{1}{1+p}}. \tag{22}$$

Proof. We use rule R1, the equations (4) and (7), the triangle inequality, the inequality $\|R_\alpha^\delta\| \le 1$ as well as the first estimate of (8) and obtain

$$\begin{aligned} C\delta &= \|R_\alpha^\delta[\alpha(\overline{x} - x_\alpha) + \alpha(x_\alpha - x_\alpha^\delta)]\| \\ &\le \|R_\alpha^\delta[\alpha(\overline{x} - x_\alpha)]\| + \alpha\|x_\alpha - x_\alpha^\delta\| \\ &\le \|R_\alpha^\delta[F(x_\alpha) - y]\| + \delta. \end{aligned} \tag{23}$$

In order to estimate the first summand on the right hand side of (23) we proceed along the lines of the proof of (23) and obtain $b \le (1+k_0)a$. Consequently,

$$(C-1)\delta \le (1+k_0)\|R_\alpha[F(x_\alpha) - F(x^\dagger)]\|.$$

In order to estimate $\|R_\alpha[F(x_\alpha) - F(x^\dagger)]\|$ we use the identity $F(x_\alpha) - F(x^\dagger) = A(x_\alpha - x^\dagger) + (M_\alpha - A)(x_\alpha - x^\dagger)$ with M_α defined by (11) and obtain due to $\|R_\alpha A\| \le \alpha$, A4 and estimate (10) with $\delta = 0$ that

$$\begin{aligned} (C-1)\delta &\le (1+k_0)\Big(\|R_\alpha A(x_\alpha - x^\dagger)\| + \|R_\alpha(M_\alpha - A)(x_\alpha - x^\dagger)\|\Big) \\ &\le (1+k_0)^2\alpha\|x_\alpha - x^\dagger\| \\ &\le (1+k_0)^3\|v\|\alpha^{p+1}. \end{aligned}$$

From this inequality we obtain (22). ∎

Now we are ready to provide order optimal error bounds for $\|x_\alpha^\delta - x^\dagger\|$ provided α is chosen from rule R1.

Theorem 3.5. *Assume* (2), A3 *and* A4 *with* $k_0 < 1$ *and radius* $r = \|\overline{x} - x^\dagger\| + \delta/\alpha$. *Let* α *be chosen by rule* R1. *Then,*

$$\|x_\alpha^\delta - x^\dagger\| \le c_p \, \|v\|^{\frac{1}{p+1}} \, \delta^{\frac{p}{p+1}} \tag{24}$$

with a constant c_p *independent of* δ *and* $\|v\|$.

Proof. Consider a fixed regularization parameter $\alpha = \alpha_0$ of the form $\alpha_0 = c_0 \left(\frac{\delta}{\|v\|}\right)^{1/(p+1)}$ with some $c_0 > 0$ and distinguish two cases. In the *first* case we assume that the solution α of rule R1 satisfies $\alpha \le \alpha_0$. In this first case we obtain from (10), (22) and the choice of α_0 that

$$\begin{aligned}\|x_\alpha^\delta - x^\dagger\| &\le \|x_\alpha - x^\dagger\| + \|x_\alpha^\delta - x_\alpha\| \\ &\le (1+k_0)\|v\|\alpha_0^p + \left[\frac{(1+k_0)^3\|v\|}{C-1}\right]^{\frac{1}{1+p}} \delta^{\frac{p}{1+p}} \\ &= \left((1+k_0)c_0^p + \left[\frac{(1+k_0)^3}{C-1}\right]^{\frac{1}{1+p}}\right) \|v\|^{\frac{1}{1+p}} \, \delta^{\frac{p}{1+p}}. \end{aligned} \tag{25}$$

In the *second* case we assume that the solution α of rule R1 satisfies $\alpha \ge \alpha_0$. In this second case we obtain from Proposition 3.2, the first estimate of (8), Theorem 2.3 with $\alpha = \alpha_0$, Proposition 3.3, Proposition 3.4 and the choice of α_0 that

$$\begin{aligned}\|x_\alpha^\delta - x^\dagger\| &\le \|x_{\alpha_0} - x^\dagger\| + \frac{\|R_\alpha[F(x_\alpha) - y]\|}{(1-k_0)\alpha_0} + \frac{\delta}{\alpha} \\ &\le (1+k_0)\|v\|\alpha_0^p + \frac{(C+1)(1+k_0)\delta}{(1-k_0)\alpha_0} + \left[\frac{(1+k_0)^3\|v\|}{C-1}\right]^{\frac{1}{1+p}} \delta^{\frac{p}{1+p}} \\ &= \left((1+k_0)c_0^p + \frac{(C+1)(1+k_0)}{(1-k_0)c_0} + \left[\frac{(1+k_0)^3}{C-1}\right]^{\frac{1}{1+p}}\right)\|v\|^{\frac{1}{1+p}} \, \delta^{\frac{p}{1+p}} . \end{aligned} \tag{26}$$

Now (24) follows from (25) and (26). ■

3.3. Error Bounds in the Case $p = 1$

In this subsection we show that under Assumptions A1 and A2 our rule R1 provides order optimal error bounds $\|x_\alpha^\delta - x^\dagger\| = O(\sqrt{\delta})$ if $L\|v\|$ is sufficiently small such that

$$c_0 := 2L\|v\|\Big(1 + \frac{L\|v\|}{2} + \frac{L^2\|v\|^2}{4}\Big)^2 < 1 \tag{27}$$

and if the constant C in rule R1 is sufficiently large such that

$$C > 1 + L\|v\|\Big(1 + \frac{L\|v\|}{2} + \frac{L^2\|v\|^2}{2} + \frac{L^3\|v\|^3}{8}\Big). \tag{28}$$

In analogy to Subs. 3.2 we need a preparatory proposition which provides a lower bound for the regularization parameter α of rule R1.

Proposition 3.6. *Let* (2), A1 *and* A2 *with radius* $r = \|\overline{x} - x^\dagger\| + \delta/\alpha$ *hold and let* α *be chosen by rule* R1 *with* C *satisfying* (28). *Then there exists a constant* k *independent of* δ *such that*

$$\alpha \geq k\sqrt{\delta}. \tag{29}$$

Proof. Exploiting (23) we obtain

$$C\delta \leq \|R^\delta_\alpha[F(x_\alpha) - y]\| + \delta \leq \|R^\delta_\alpha R^{-1}_\alpha\| \|R_\alpha[F(x_\alpha) - y]\| + \delta. \tag{30}$$

From (7) we have $(A + \alpha I)(\overline{x} - x_\alpha) = A(\overline{x} - x^\dagger) + [F(x_\alpha) - F(x^\dagger) - A(x_\alpha - x^\dagger)]$. Since due to (7) there holds $\overline{x} - x_\alpha = \frac{1}{\alpha}[F(x_\alpha) - F(x^\dagger)]$ we conclude that

$$R_\alpha[F(x_\alpha) - F(x^\dagger)] = R^2_\alpha A^2 v + R^2_\alpha[F(x_\alpha) - F(x^\dagger) - A(x_\alpha - x^\dagger)]\,. \tag{31}$$

From Assumption A2 we gets (5). We use this property with $x := x_\alpha$ and $x_0 := x^\dagger$ and obtain from $\|R_\alpha A\| \leq \alpha$, (9) with $\delta = 0$ and (31) that

$$\|R_\alpha[F(x_\alpha) - y]\| \leq \Big(1 + \frac{L\|v\|}{2} + \frac{L^2\|v\|^2}{2} + \frac{L^3\|v\|^3}{8}\Big)\|v\|\alpha^2. \tag{32}$$

In order to estimate $\|R^\delta_\alpha R^{-1}_\alpha\|$ we apply the triangle inequality, Assumption A2, the first inequality of (8) as well as (9) and obtain

$$\begin{aligned}\|R^\delta_\alpha R^{-1}_\alpha\| &= \|(A^\delta_\alpha + \alpha I)^{-1}(A - A^\delta_\alpha) + I\| \\ &\leq \frac{L}{\alpha}\Big(\|x^\delta_\alpha - x_\alpha\| + \|x_\alpha - x^\dagger\|\Big) + 1 \\ &\leq L\Big(\frac{\delta}{\alpha^2} + \|v\| + \frac{L}{2}\|v\|^2\Big) + 1. \end{aligned}\tag{33}$$

We substitute (32), (33) into (30) and obtain due to (28) the estimate (29). ■

Theorem 3.7. *Let* (2), A1 *and* A2 *with radius* $r = \|\overline{x} - x^\dagger\| + \delta/\alpha$ *be satisfied and let* α *be chosen by rule* R1. *Suppose the smallness condition* (27) *and that the constant* C *of rule R1 satisfies* (28). *Then there exists a constant* c_p *independent of* δ *such that*

$$\|x^\delta_\alpha - x^\dagger\| \leq c_p\sqrt{\delta}\,. \tag{34}$$

Proof. From the second part of (13) we have due to (31), (9) and the generalized moment inequality $\|Bv\| \leq 2\|B^2 v\|^{1/2}\|v\|^{1/2}$ that holds true for monotone B (see [10]) that

$$\begin{aligned}\|x_\alpha - x^\dagger\| &\leq \|R_\alpha A v\| + \|A - M_\alpha\|\|(A + \alpha I)^{-1}Av\| \\ &\leq \|R_\alpha A v\| + \frac{L}{2}\Big(\|v\| + \frac{L}{2}\|v\|^2\Big)\alpha\|(A + \alpha I)^{-1}Av\| \\ &= \Big(1 + \frac{L\|v\|}{2} + \frac{L^2\|v\|^2}{4}\Big)\|R_\alpha A v\| \\ &\leq 2\Big(1 + \frac{L\|v\|}{2} + \frac{L^2\|v\|^2}{4}\Big)\|R^2_\alpha A^2 v\|^{1/2}\|v\|^{1/2}. \end{aligned}\tag{35}$$

From (31) and $\|R_\alpha^2\| \le 1$ we obtain

$$\|R_\alpha^2 A^2 v\| \le \|R_\alpha[F(x_\alpha) - F(x^\dagger)]\| + \frac{L}{2}\|x_\alpha - x^\dagger\|^2. \tag{36}$$

Substituting (36) into (35), squaring both sides and rearranging terms yields

$$(1-c_0)\|x_\alpha - x^\dagger\|^2 \le 4\Big(1 + \frac{L\|v\|}{2} + \frac{L^2\|v\|^2}{4}\Big)^2 \|v\|\|R_\alpha[F(x_\alpha) - F(x^\dagger)]\| \tag{37}$$

with the constant c_0 given in (27). For estimating $\|R_\alpha[F(x_\alpha) - F(x^\dagger)]\|$ in terms of δ we use equations (4) and (7), the estimates $\|R_\alpha\| \le 1$, (8) and the valid estimate

$$\|R_\alpha(R_\alpha^\delta)^{-1}\| \le L\Big(\frac{\delta}{\alpha^2} + \|v\| + \frac{L}{2}\|v\|^2\Big) + 1,$$

which can be obtained in analogy to the proof of (33), apply rule R1 and obtain

$$\begin{aligned}\|R_\alpha[F(x_\alpha) - F(x^\dagger)]\| &\le \|R_\alpha[\alpha(\overline{x} - x_\alpha^\delta) + \alpha(x_\alpha^\delta - x_\alpha)]\| \\ &\le \|R_\alpha[F(x_\alpha^\delta) - y^\delta]\| + \alpha\|x_\alpha^\delta - x_\alpha\| \\ &\le \|R_\alpha(R_\alpha^\delta)^{-1}\|\|R_\alpha^\delta[F(x_\alpha^\delta) - y^\delta]\| + \delta \\ &\le \Big(L\frac{\delta}{\alpha^2} + 1 + L\|v\| + \frac{L^2\|v\|^2}{2}\Big)C\delta + \delta.\end{aligned} \tag{38}$$

From (29) we know that δ/α^2 is bounded by some constant. We substitute this constant into (38), substitute the resulting estimate into (37) and obtain $\|x_\alpha - x^\dagger\| \le c\sqrt{\delta}$ with some constant c independent of δ. This estimate, the first estimate of (8) and (29) provide (34). ■

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