Vietnam Journal of Mathematics **32:** *SI* (2004) 161–165

Vietnam Journal
of
MATHEMATICS

Hausdorff Dimension of Julia Sets

M. Zinsmeister

Université d'Orléans, BP 6759, 45067 Orléans Cedex 2, France

Abstract. The aim of the paper is to present one such model for which a "toy"-thermodynamic formalism may be considered. We will discuss recent development of this subject and open questions that remain.

1. Introduction

Chaotic or turbulent behavior is the rule in almost all physical phenomena and we are very far from a complete understanding. That is why physicists and mathematicians have considered simpler and presumably more tractable models to start with. Surprisingly it seems that the simplest possible models already lead to great complexity.

2. Dynamics of Complex Polynomials

A physical phenomenon is usually modelled as a continuous time dynamical system. For instance, a gas in a bottle is modelled as a huge ($N \sim 10^{23}$) number of particles moving and interacting. Formally this consists in a Hamiltonian system in $\mathbb{R}^{6N}$ which is clearly not tractable. The only possible understanding of the equilibrium of such a gas is statistical, that is through thermodynamics.

We develop here a simple thermodynamical model on a discrete dynamical system.
A discrete dynamical system is simply a system (X, T) where X is a topological space and $T : X \to X$ is a continuous mapping: the study of such a system consists in the study of (long-term) behavior of the orbits $(T^n(x))_{n\geq 0}$ in a statistical sense.

Here we take $X = \mathbb{C}$ and T is a polynomial of degree ≥ 2.
We wish to understand:
1) the dynamics of a given T,

2) how the dynamics is perturbed if T is changed, i.e. the dependence in T of the dynamics.

Before going to point 2 which will be our main concern, let us summarize the bases for 1):
Performing a preliminary affine conjugation, we may assume that T is monic and centered, i.e. that $T(z) = z^d + a_{d-2}z^{d-2} + \cdots\,,\ d \geq 2$.
If $|z|$ is large enough it is obvious that $|T(z)| \geq 2|z|$ which implies that $T^n(z) \to \infty$, we say that z escapes. It follows that the set of escaping points is an open set of the plane which is also a neighborhood of ∞.

Its complement is thus compact and non void since it must contain all the periodic points of T: it is called the filled-in Julia set and denoted by $K(T)$. The interior of $K(T)$ together with the set of escaping points is called the Fatou set of T. Its complement is the Julia set $J(T)$. The Fatou and the Julia set form a partition of the plane: the dynamics on the Fatou set appears to be (after some work!) not very interesting.

To be more precise T induces a dynamic on the set of components of the Fatou set and a deep theorem of Sullivan asserts that every such component is pre-periodic. We will thus achieve a complete understanding of the dynamics on the Fatou set by understanding the periodic components of T (we may assume that they are actually fixed since $K(T) = K(T^n),\ n \geq 1$).

Invariant components have been completely classified in the early XXth century by Fatou and Julia:
A fixed point of T^n obviously belong to $K(T)$: whether it belongs to its interior or to $J(T)$ depends on its multiplier $(T^n)'(x)$:

– If it has modulus < 1 we say that it is attracting. It then attracts neighbouring points and it thus belongs to the Fatou set and the component of the Fatou set containing it is called the immediate basin of the fixed point.

– If it has modulus > 1 then it is called reppelling: it belongs then to $J(T)$ and it can be shown that the set of repelling periodic points is dense in $J(T)$.

– The case of modulus one is the indifferent case: it splits into three subcases depending on θ such that $(T^n)'(x) = e^{2i\pi\theta}$:

- $\theta \in \mathbb{Q}$, then $x \in J(T)$ but x also belongs to the boundary of a finite number of comonents the Fatou set whose elements are attracted to x. This case is called parabolic.

- $\theta \notin \mathbb{Q}$ but the mapping T^n is loccally conjugated to the rotation being its linear part, then x is in a component of the Fatou set called a Siegel disc where T^n is conjugated to its linear part.

- $\theta \notin Q$ and T is not locally conjugated (this is called the Cremer case), in this case the fixed point belongs to the Julia set and does not create any Fatou component.

Theorem 2.1. (Fatou, Julia, see [4]) *Every bounded periodic component of the Fatou set is either a component of the immediate basin of an attracting or parabolic basin or else a Siegel disk.*

It follows that the dynamics on the Fatou set is well understood: in simple words, the trajectory of a Fatou point is "predictable". For such a point there exists $k_0 \geq k_1 \geq 1$ such that $T^{k_0+nk_1}(z)$ either converges to an attracting or parabolic periodic point or turns around an irrationaly indifferent periodic point along an analytic curve.

The situation on the Julia set is drastically different. In order to convince the reader let us consider the simplest polynomial $T(z) = z^2$. The Julia set is the unit circle and the dynamics on it consists in doubling the argument. This is a very chaotic one which can be described, using dyadic expansion, as the left shift on the space of infinite sequences of 0's and 1's, i.e. $X = \{0, 1\}^N$.

Like in nature, we build up a thermodynamic formalism to understand this turbulent behavior. Each element of X is considered as a configuration of a "lattice"-gas and an equilibrium state will be a probability measure which is shift-invariant. There are infinitely many equilibrium states. Which one would nature choose?

To figure this out we first notice that our model is driven by a potential $\varphi : X \to \mathbb{R}$ (which happens to be constant in the case $T(z) = z^2$) which is Holder continuous, X being equipped with the usual ultrametric.

For any Holder function Ψ on X we can define the **pressure**

$$P(\Psi) = \lim_{n\to\infty} \frac{1}{n} \sum_{a_1,a_2,\dots,a_n \in X} \exp\{\max |S_n\Psi(x)|,\ x \in a_1..a_n\},$$

where $a_1..a_n = \{x = (x_j);\ x_1 = a_1, \dots, x_n = a_n\}$ and $S_n\Psi$ is the Birkhoff sum

$$S_n\Psi(x) = \Psi(x) + \Psi(Tx) + \cdots + \Psi(T^{n-1}x).$$

It can be proved that

$$P(\Psi) = \max\{h_\mu + \int_X \Psi d\mu,\ \mu \in \mathcal{E}\},$$

$\mathcal{E}$ being the set of equilibrium states. Here $\int \Psi d\mu$ represents the potential energy and h_μ is the entropy defined as

$$h_\mu = \lim_{n\to\infty} -\frac{1}{n}\Big(\sum_{x_1,\dots,x_n} \mu(x_1..x_n) \ln \mu(x_1..x_n)\Big)$$

and the maximum is attained at a unique point called Gibbs state, and the equilibrium nature would choose is the Gibbs state of $-\varphi/T$ where T is temperature. This is known as Helmoltz principle: at fixed temperature nature chooses equilibrium that minimizes (in terms of classical thermodynamics) the free energy $U - TS$ where U is energy, T temperature and S is entropy (see [7] for more about this).

3. Quadratic Polynomials with One Attracting Fixed Point

What is the connection between this formalism and the geometry of Julia sets? To understand this we consider the case of the family

$$P_\lambda(z) = \lambda z + z^2.$$

We call $J_\lambda = J(P_\lambda)$. It can be shown that $\lambda \mapsto J_\lambda$ defines a holomorphic motion on $\mathbb{D}$ in the sense that for every $\lambda \in \mathbb{D}$ there exists a homeomorphism $\Phi_\lambda J_0 (= \partial\mathbb{D}) \to J_\lambda$ such that (i) $\Phi_0 = id$, (ii) $\forall z \in \mathbb{D}$, $\lambda \mapsto \Phi_\lambda(z)$ is holomorphic in $\mathbb{D}$, and (iii) $\forall z \in J_0$, $\forall \lambda \in \mathbb{D}$, $P_\lambda(\Phi_\lambda(z)) = \Phi_\lambda(P_0(z))$.

This fact allows us to view all the Julia sets $J_\lambda, \lambda \in \mathbb{D}$ on the same model X, only the potential varying with λ, namely $\varphi_\lambda = 2|\Phi_\lambda|$.

We now come to the geometry: the Julia set J_λ is fractal if $\lambda \neq 0$. This means that if one considers a lattice $\epsilon\mathbb{Z}^2$ the number of squares of this lattice that meet J_λ is of the order of ϵ^{-d}, $d \in]1,2[$ while it would be ϵ^{-1} for a C^1-curve or ϵ^{-2} for a set with nonempty interior.

The number d is called the dimension of J_λ and it is a measure of the geometric complexity of the set.

The link between this dimension and thermodynamic formalism is Bowen's formula saying that the dimension $d = d(\lambda)$ is the only zero of the function $t \mapsto P(-t\varphi_\lambda)$. In other words, if the inverse of the dimension is treated as a temperature, it is the one leading to zero pressure. Using thermodynamic formalism and operator theory Ruelle has shown that Bowen's formula implies that the function $\lambda \mapsto d(\lambda)$ is real-analytic in λ [3].

4. Boundary Behavior

The preceeding computations are only valid inside the disk. What happens when we approach the boundary?

It is remarkable that the answer is again given by a thermodynamic interpretation.

We can still define φ and the pressure. The main difference between the case $|\lambda| = 1$ and the case $\lambda \in \mathbb{D}$ is that the function "pressure versus temperature" $t \to P(-t\varphi)$, instead of decreasing from $+\infty$ to $-\infty$ decreases from $+\infty$ to $-\infty$ at some value d_0 and then remains equal to 0.

In thermodynamic terms, we have a phase transition. If $\lambda_0 = e^{2i\pi\theta}$ with $\theta \in \mathbb{Q}$ it can then be shown that d_0 is still the dimension of the corresponding Julia set. Moreover the function d is continuous along the radius landing at λ_0 [9].

It is an open question to determine the values $\lambda \in \mathbb{D}$ for which this continuity property holds.

5. Discontinuous Phenomena

The continuity described in the last paragraph remains valid for radial convergence from outside towards rational points, and the proof is essentially the same.

The behavior is radically different if one considers tangential convergence. In this case the function $\lambda \mapsto J_\lambda$ is highly non continuous at any rational; this is due to parabolic implosion, a phenomenon discovered by Douady [1].

The result is extremely precise in the case $\lambda_0 = 1$: if $\lambda_k = 1 + i\epsilon_k$ and $\epsilon_k \to 0$

in such a way that

$$-\frac{2\pi}{\epsilon_k} = N_k + \sigma + o(1)$$

with $N_k \in \mathbb{Z}$ then $J(\lambda_k)$ converges to a set $J(1, \sigma)$ containing strictly J_1. The study of the Hausdorff dimensions have been done in [2]. With the above hypothesis we have convergence of the corresponding Hausdorff dimensions and moreover

$$HD(J_1) < HD(J(1, \sigma)) < 2.$$

In particular the function d is discontinuous at 1 and the same is true at every rational point. Nevertheless it is not known if the function d restricted to $\{x = 1\} \setminus \{1\}$ has a limit as $\lambda \to 1$. This is equivalent to decide whether the function $\sigma \to HD(J(1, \sigma))$ is constant or not. In [6] it is shown that at least the function is real-analytic in σ.

Also it is proved in [8] that for $\lambda = e^{2i\pi p/q}$, $p \wedge q = 1$ then

$$\liminf_{u \to \lambda \text{ tangentially}} HD(J(\mu)) > \frac{2q}{q+1}.$$

Together with a Baire category argument, this gives a new proof of a theorem of Shishikura [5] asserting that $HD(J_\lambda) = 2$ on a dense $\mathcal{G}_\delta$ of the circle. As a biproduct there is a dense $\mathcal{G}_\delta$ subset of the circle for which the Hausdorff dimension is radially continuous. It is an open question to decide whether this set can be taken of positive Lebesgue measure.

References

1. A. Douady, Does a Julia set depend continuously on the polynomial?, *Proceedings of Symposia in Applied Mathematics* **49** (1994) 91–135.
2. A. Douady, P. Sentenac, and M. Zinsmeister, Implosion parabolique et dimension de Hausdorff, *CRAS Paris* **325** (1997) 765–772.
3. D. Ruelle, Repellers for real analytic maps, *EDTS* **2** (1982) 99–107.
4. L. Carleson and T. Gamelin, *Complex Dynamics*, Springer-Verlag, 1993.
5. M. Shishikura, The Hausdorff dimension of the boundary of the Mandelbrot sets and Julia sets, *Ann of Math.* **147** (1998) 225–267.
6. M. Urbanski and M. Zinsmeister, *Geometry of Julia Lavaurs Sets*, Indagationes Math., 2001.
7. M. Zinsmeister, *Formalisme Thermodynamique et Systemes Dynamiques Holomorphes*, Panorama et Synthèses, 1996.
8. M. Zinsmeister, Fleur de Leau-Fatou et dimension de Hausdorff, *CRAS Paris* **326** (1998) 1227–1232.