Vietnam Journal of Mathematics **33**:1 (2005) 91–95

Vietnam Journal
of
MATHEMATICS

The Bounds on Components of the Solution for Consistent Linear Systems*

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Received February 27, 2004

Abstract. For a consistent linear system $Ax = b$, where A is a diagonally dominant Z-matrix, we present the bound on components of solutions for this linear system, which generalizes the corresponding result obtained by Milaszewicz et al. [3].

1. Introduction and Definitions

In [2, 3] the authors consider the following consistent linear system

$$Ax = b, \tag{1}$$

where A is an $n \times n$ M-matrix, b is an n dimension vector in $rang(A)$. The study of the solution of the linear system (1) is very important in Leontief model of input-output analysis and in finite Markov chain (see [1, 2]). In this article we will discuss a special M -matrix linear system, when the matrix A in linear system (1) is a diagonally dominant $\overline{L}$-matrix; this matrix class often appears in input-output model and finite Markov chain (e.g., see [1]).

In order to give our main result we first introduce some definitions and notations.

Let G be a directed graph. Two vertices i and j are called strongly connected if there are paths from i to j and from j to i. A vertex is regarded as trivially strongly connected to itself. It is easy to see that strong connectivity defines an equivalence relation on vertices of G and yields a partition

$$V_1 \cup ... \cup V_k$$

*This work was supported by the Natural Science Foundation of Guangdong Province (31496), Natural Science Foundation of Universities of Guangdong Province (0119) and Excellent Talent Foundation of Guangdong Province (Q02084).

of the vertices of G. The directed subgraph G_{V_i} with the vertex set V_i of G is called a strongly connected component of G, $i = 1, ..., k$. Let $G = G(A)$ be an associated directed graph of A. A nonempty subset K of $G(A)$ is said to be a nucleus if it is a strongly connected component of $G(A)$ (see [3]). For a nucleus K, N_K denotes the set of indices involved in K.

A matrix or a vector B is nonnegative (positive) if each entry of B is nonnegative (positive, respectively). We denote them by $B \geq 0$ and $B > 0$. An $n \times n$ matrix $A = (a_{ij})$ is called a Z-matrix if for any $i \neq j$, $a_{ij} \leq 0$, a $\overline{L}$-matrix if A is a Z-matrix with $a_{ii} \geq 0$, $i = 1, ..., n$ and an M-matrix if $A = sI - B$, $B \geq 0$ and $s \geq \rho(B)$, where $\rho(B)$ denotes the spectral radius of B. Notice that A is a singular M-matrix if and only if $s = \rho(B)$. An $n \times n$ matrix $A = (a_{ij})$ is said to be diagonally dominant if $2|a_{ii}| \geq \sum_{j=1}^{n} |a_{ij}|,\ i = 1, ..., n$.

Let $N = \{1, ..., n\}$, $A \in \mathbb{R}^{n \times n}$ and α be a subset of N. We denote by $A[\alpha]$ the principal submatrix of A whose rows and columns are indexed by α. Let $x \in \mathbb{R}^n$. By $x[\alpha]$ we mean that the subvector of x whose subscripts are indexed by α.

Milaszewicz and Moledo [3] studied the above linear system and presented the following result, on which we make a slight modification.

Theorem 1.1. *Let A be a nonsingular, diagonally dominant Z-matrix. Then the solution of linear system* (1) *has the following properties:*

(i) *If $N_K \cap N_>(b) \neq \emptyset$ for each nucleus K of A, then*

$$x_i \leq D, \ \ \forall i \in N,$$

where $D = \max\{0,\ x_j : b_j > 0\}$ and $N_>(b) = \{i \in N : b_i > 0\}$.

(ii) *If $N_K \cap N_<(b) \neq \emptyset$ for each nucleus K of A, then*

$$d \leq x_i, \ \ \forall i \in N.$$

where $d = \min\{0, x_j : b_j < 0\}$ and $N_<(b) = \{i \in N : b_i < 0\}$.

Remark. Theorem 1.1 is a generalization of Theorem 7 in [2].

In this note we will extend Theorem 1.1; see Theorem 2.4.

2. The Bounds

For the rest of this note we set $N_>$, $N_<$, D and d as in Theorem 1.1. For consistent linear system (1), by $A_\geq$ and $A_\leq$ we denote the principal submatrices of A whose rows and columns are indexed by the subsets $\{i \in N : b_i \geq 0\}$ and $\{i \in N : b_i \leq 0\}$, respectively.

Now we give some lemmas which will lead to the main theorem in this note.

Lemma 2.1. *Let A be a diagonally dominant $\overline{L}$-matrix. Then A is an M-matrix.*

Proof. Since A is a diagonally dominant Z-matrix, $Ae \geq 0$, where $e = (1, 1, \ldots, 1)^t$. Let $A = sI - B$, where $s \in \mathbb{R}$ and B is nonnegative. It follows from Perron-

Frobenius Theorem on nonnegative matrices (e.g., see [1]) that there is a nonnegative nonzero vector y such that $y^tB = \rho(B)y^t$. Thus $0 \le y^tAe = (s-\rho(B^t))y^te$. Since $y^te > 0$, we have $s \ge \rho(B)$. Hence A is an M-matrix. ∎

Lemma 2.2. *Let $A \in \mathbb{R}^{n\times n}$ be an M-matrix, $b \in \mathbb{R}^n$ and $b(N_K) \ne 0$ for each nucleus K of A.*

(i) *If $A_{\ge}$ is a nonsingular M-matrix, then whenever $x(N_<(b)) > 0$ we have $x > 0$.*

(ii) *If $A_{\le}$ is a nonsingular M-matrix, then whenever $x(N_>(b)) < 0$ we have $x < 0$.*

Proof.
(i) Follows from Theorem 3.5 of [4].
(ii) By (1) we have

$$A(-x) = -b. \tag{2}$$

By (i) it is easy to see that (ii) holds. ∎

Lemma 2.3. *Let A be a diagonally dominant $\overline{L}$-matrix. If there exist a vector x and a positive vector b such that $Ax = b$, then A is a nonsingular M-matrix.*

Proof. By Lemma 2.1, A is an M-matrix. Assume that A is singular. Then so is A^t. Let $A^t = sI - B$, $s \in R$ and B is nonnegative. Then $s = \rho(B)$. It follows from Perron-Frobenius Theorem of nonnegative matrices that there is a nonnegative nonzero vector y such that $By = \rho(B)y$. Thus

$$y^tA = y^t(sI - B^t) = (s - \rho(B))y^t = 0,$$

which implies that $y^tb = y^tAx = 0$. Since $y \ge 0$, $y \ne 0$ and $b > 0$, we have $y^tb > 0$, which contradicts the assumption. Hence A is a nonsingular M-matrix. ∎

The following theorem is our main result in this note.

Theorem 2.4. *Let A be a diagonally dominant $\overline{L}$-matrix, $b(N_K) \ne 0$ for each nucleus K. Then the solution of the linear system* (1) *has the following properties:*

(i) *If $A_{\le}$ is a nonsingular M-matrix* (*or empty matrix*) *then*

$$x_i \le D, \quad \forall i \in N.$$

(ii) *If $A_{\ge}$ is a nonsingular M-matrix (or empty matrix), then*

$$x_i \ge d, \quad \forall i \in N.$$

Proof. It is enough to show that (i) holds. The proof of (ii) is similar. We consider the following three cases.

Case 1. If $N_>(b) = N$, then $b > 0$. It follows from Lemma 2.3 that A is a nonsingular M-matrix. Hence the result follows immediately from Theorem 3.1 of [3].

Case 2. If $N_>(b) = \emptyset$, then $A_{\le} = A$ is a nonsingular M-matrix. By Theorem 6.2.3 of [1] we have $A^{-1} \ge 0$. Hence $x = A^{-1}b \le 0$, which leads to our result.

Case 3. If $\emptyset \subset N_>(b) \subset N$, then we consider the following two subcases.

Subcase 3.1. If $x(N_>(b)) < 0$, then it follows $x < 0$ from Lemma 2.2 (ii), which implies that the theorem holds.

Subcase 3.2. Now we assume that there exists $j \in N_>(b)$ such that $x_j > 0$. It is enough to show that $x_j \le \max\{x_i : b_i > 0\}$.

Since $\emptyset \subset N_>(b) \subset N$, the sets $\alpha = N_>(b)$ and $\beta = \{i \in N : b_i \le 0\}$ form a partition of the set N. Hence there is a permutation matrix P such that $Pb = \begin{pmatrix} b^{(1)} \\ b^{(2)} \end{pmatrix}$, where $b^{(1)} = b[\alpha]$ and $b^{(2)} = b[\beta]$. Hence $b^{(1)} > 0$ and $b^{(2)} \le 0$. Let

$$PAP^t = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \tag{3}$$

where $A_{11} = A[\alpha]$ and $A_{22} = A[\beta] = A_{\le}$. By (1) we have $(PAP^t)Px = Pb$. Let $Px = \begin{pmatrix} x^{(1)} \\ x^{(2)} \end{pmatrix}$ be conformably with the block form (3). Then $x^{(1)} = x[\alpha]$ and $x^{(2)} = x[\beta]$. Hence $A_{21}x^{(1)} + A_{22}x^{(2)} = b^{(2)}$. Since $b^{(2)} \le 0$, we have $A_{22}x^{(2)} \le A_{21}x^{(1)}$. By the assumption that $A_{\le}$ is a nonsingular M-matrix we have $A_{22}^{-1} = A_{\le}^{-1} \ge 0$, from which we have

$$x^{(2)} \le -A_{22}^{-1}A_{21}x^{(1)}. \tag{4}$$

Since A is diagonally dominant Z-matrix, $Ae \ge 0$. Let $e = \begin{pmatrix} e^{(1)} \\ e^{(2)} \end{pmatrix}$ be conformably with the block form (3). Then $A_{21}e^{(1)} + A_{22}e^{(2)} \ge 0$, i.e., $-A_{22}^{-1}A_{21}e^{(1)} \le e^{(2)}$. Let $x_m = \max\{x_i : b_i > 0\}$. Then $x_m > 0$ and $x^{(1)} \le x_m e^{(1)}$. Notice that $-A_{22}^{-1}A_{21} \ge 0$, then by (4) we have $x^{(2)} \le -A_{22}^{-1}A_{21}x^{(1)} \le -x_m A_{22}^{-1}A_{21}e^{(1)} \le x_m e^{(2)}$, from which one can deduce that the theorem holds. ∎

Corollary 2.5. *Let A be a diagonally dominant $\overline{L}$-matrix and $b(N_K) \ne 0$ for each nucleus K. If $A_{\ge}$ and $A_{\le}$ are nonsingular, then the solution of the linear system* (1) *satisfies*

$$d \le x_i \le D, \quad \forall i \in N.$$

Proof. The result follows from Lemma 2.1, Lemma 2.2 and Theorem 2.4. ∎

Corollary 2.6. *Let A be a nonsingular, diagonally dominant $\overline{L}$-matrix, and $b(N_K) \ne 0$ for each nucleus K. Then the solution of the linear system* (1) *satisfies*

$$d \le x_i \le D, \quad \forall i \in N.$$

Proof. By Lemma 2.1, A is a nonsingular M-matrix. Since each principal submatrix of a nonsingular M-matrix is a nonsingular M-matrix, the result follows from Corollary 2.5. ∎

Corollary 2.7. *Let A be an irreducible diagonally dominant $\overline{L}$-matrix, and $b \ne 0$. Then the solution of linear system* (1) *satisfies*

$$d \leq x_i \leq D, \quad \forall i \in N.$$

Proof. The result follows immediately from Corollary 2.6. ■

Remark. If $N_K \cap N_>(b) \neq \emptyset$ or $N_K \cap N_<(b) \neq \emptyset$ for each nucleus K of A, then $b(N_K) \neq 0$ for each nucleus K of A on one hand. On the other hand, in Theorem 2.4 and Corollary 2.5 we need not to assume that A is nonsingular. Hence from the fact that each principal submatrix of a nonsingular M-matrix is also a nonsingular M-matrix, we know that Theorem 2.4 and Corollary 2.5 extend Theorem 1.1.

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