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New Characterizations and Generalizations of *PP* Rings

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Abstract. This paper consists of two parts. In the first part, it is proven that a ring R is right PP if and only if every right R-module has a monic $\mathcal{PI}$-cover, where $\mathcal{PI}$ denotes the class of all P-injective right R-modules. In the second part, for a non-empty subset X of a ring R, we introduce the notion of X-PP rings which unifies PP rings, PS rings and nonsingular rings. Special attention is paid to J-PP rings, where J is the Jacobson radical of R. It is shown that right J-PP rings lie strictly between right PP rings and right PS rings. Some new characterizations of (von Neumann) regular rings and semisimple Artinian rings are also given.

1. Introduction

A ring R is called right PP if every principal right ideal is projective, or equivalently the right annihilator of any element of R is a summand of R_R. PP rings and their generalizations have been studied in many papers such as [4, 9, 10, 12, 13, 21].

In Sec. 2 of this paper, some new characterizations of PP rings are given. We prove that a ring R is right PP if and only if every right R-module has a monic $\mathcal{PI}$-cover if and only if $\mathcal{PI}$ is closed under cokernels of monomorphisms and $E(M)/M$ is P-injective for every cyclically covered right R-module M, where $\mathcal{PI}$ denotes the class of all P-injective right R-modules.

In Sec. 3, we first introduce the notion of X-PP rings which unifies PP

rings, PS rings and nonsingular rings, where X is a non-empty subset of a ring R. Special attention is paid to the case $X = J$, the Jacobson radical of R. It is shown that right J-PP rings lie strictly between right PP rings and right PS rings. Some results which are known for PP rings will be proved to hold for J-PP rings. Then some new characterizations of (von Neumann) regular rings and semisimple Artinian rings are also given. For example, it is proven that R is regular if and only if R is right J-PP and right weakly continuous if and only if every right R-module has a $\mathcal{PI}$-envelope with the unique mapping property if and only if $\mathcal{PI}$ is closed under cokernels of monomorphisms and every cyclically covered right R-module is P-injective; R is semisimple Artinian if and only if R is a right J-PP and right (or left) Kasch ring if and only if every right R-module has an injective envelope with the unique mapping property if and only if every cyclic right R-module is both cyclically covered and P-injective. Finally, we get that R is right PS if and only if every quotient module of any mininjective right R-module is mininjective. Moreover, for an Abelian ring R, it is obtained that R is a right PS ring if and only if every divisible right R-module is mininjective, and we conclude this paper by giving an example to show that there is a non-Abelian right PS ring in which not every divisible right R-module is mininjective.

Throughout, R is an associative ring with identity and all modules are unitary. We use M_R to indicate a right R-module. As usual, $E(M_R)$ stands for the injective envelope of M_R, and $pd(M_R)$ denotes the projective dimension of M_R. We write $J = J(R)$, $Z_r = Z(R_R)$ and $S_r = Soc(R_R)$ for the Jacobson radical, the right singular ideal and the right socle of R, respectively. For a subset X of R, the left (right) annihilator of X in R is denoted by $l(X)$ $(r(X))$. If $X = \{a\}$, we usually abbreviate it to $l(a)$ $(r(a))$. We use $K \leqslant_e N$, $K \leqslant^{\max} N$ and $K \leqslant^{\oplus} N$ to indicate that K is an essential submodule, maximal submodule and summand of N, respectively. $\mathrm{Hom}(M, N)$ $(\mathrm{Ext}^n(M, N))$ means $\mathrm{Hom}_R(M, N)$ $(\mathrm{Ext}^n_R(M, N))$ for an integer $n \geq 1$. General background material can be found in [1, 6, 18, 20].

2. New Characterizations of *PP* Rings

We start with some definitions.

A pair $(\mathcal{F}, \mathcal{C})$ of classes of right R-modules is called a *cotorsion theory* [6] if $\mathcal{F}^{\perp} = \mathcal{C}$ and ${}^{\perp}\mathcal{C} = \mathcal{F}$, where $\mathcal{F}^{\perp} = \{C : \mathrm{Ext}^1(F, C) = 0 \text{ for all } F \in \mathcal{F}\}$, and ${}^{\perp}\mathcal{C} = \{F : \mathrm{Ext}^1(F, C) = 0 \text{ for all } C \in \mathcal{C}\}$.

Let $\mathcal{C}$ be a class of right R-modules and M a right R-module. A homomorphism $\phi : M \to F$ with $F \in \mathcal{C}$ is called a *$\mathcal{C}$-preenvelope* of M [6] if for any homomorphism $f : M \to F'$ with $F' \in \mathcal{C}$, there is a homomorphism $g : F \to F'$ such that $g\phi = f$. Moreover, if the only such g are automorphisms of F when $F' = F$ and $f = \phi$, the $\mathcal{C}$-preenvelope ϕ is called a *$\mathcal{C}$-envelope* of M. Following [6, Definition 7.1.6], a monomorphism $\alpha : M \to C$ with $C \in \mathcal{C}$ is said to be a *special $\mathcal{C}$-preenvelope* of M if $\mathrm{coker}(\alpha) \in {}^{\perp}\mathcal{C}$. Dually we have the definitions of a *(special) $\mathcal{C}$-precover* and a *$\mathcal{C}$-cover*. Special $\mathcal{C}$-preenvelopes (resp., special $\mathcal{C}$-precovers) are obviously $\mathcal{C}$-preenvelopes (resp., $\mathcal{C}$-precovers).

Let M be a right R-module. M is called *cyclically presented* [20, p.342] if it is isomorphic to a factor module of R by a cyclic right ideal. M is *P-injective* [14] if $\mathrm{Ext}^1(N, M) = 0$ for any cyclically presented right R-module N. M is called *cyclically covered* if M is a summand in a right R-module N such that N is a union of a continuous chain, $(N_\alpha : \alpha < \lambda)$, for a cardinal λ, $N_0 = 0$, and $N_{\alpha+1}/N_\alpha$ is a cyclically presented right R-module for all $\alpha < \lambda$ (see [19, Definition 3.3]).

Denote by $\mathcal{CC}$ ($\mathcal{PI}$) the class of all cyclically covered (P-injective) right R-modules. Then ($\mathcal{CC}$, $\mathcal{PI}$) is a complete cotorsion theory by [19, Theorem 3.4] (note that P-injective modules are exactly divisible modules in [19]). In particular, every right R-module has a special $\mathcal{PI}$-preenvelope and a special $\mathcal{CC}$-precover.

To prove the main theorem, we need the following lemma.

Lemma 2.1. *Let $\mathcal{PI}$ be closed under cokernels of monomorphisms. If $M \in \mathcal{CC}$, then $Ext^n(M, N) = 0$ for any $N \in \mathcal{PI}$ and any integer $n \geq 1$.*

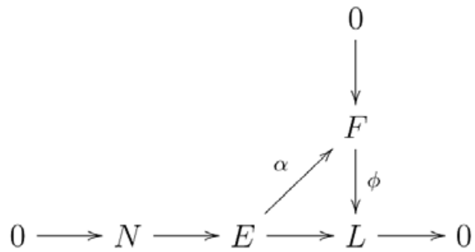
Proof. For any P-injective right R-module N, there is an exact sequence $0 \to N \to E \to L \to 0$, where E is injective. Then $\mathrm{Ext}^1(M, L) \to \mathrm{Ext}^2(M, N) \to 0$ is exact. Note that L is P-injective by hypothesis, so $\mathrm{Ext}^1(M, L) = 0$. Thus $\mathrm{Ext}^2(M, N) = 0$, and hence the result holds by induction. ∎

We are now in a position to prove

Theorem 2.2. *The following are equivalent for a ring R:*
(1) *R is a right PP ring;*
(2) *Every quotient module of any (P-)injective right R-module is P-injective;*
(3) *Every (quotient module of any injective) right R-module M has a monic $\mathcal{PI}$-cover $\phi : F \to M$;*
(4) *$\mathcal{PI}$ is closed under cokernels of monomorphisms, and every cyclically covered right R-module M has a monic $\mathcal{PI}$-cover $\phi : F \to M$;*
(5) *$\mathcal{PI}$ is closed under cokernels of monomorphisms, and $pd(M) \leqslant 1$ for every cyclically covered (cyclically presented) right R-module M;*
(6) *$\mathcal{PI}$ is closed under cokernels of monomorphisms, and $E(M)/M$ is P-injective for every cyclically covered right R-module M.*

Proof.
(1) $\Leftrightarrow$ (2) holds by [21, Theorem 2].
(2) $\Rightarrow$ (3). Let M be any right R-module. Write $F = \sum\{N \leqslant M : N \in \mathcal{PI}\}$ and $G = \oplus\{N \leqslant M : N \in \mathcal{PI}\}$. Then there exists an exact sequence $0 \to K \to G \to F \to 0$. Note that $G \in \mathcal{PI}$, so $F \in \mathcal{PI}$ by (2). Next we prove that the inclusion $i : F \to M$ is a $\mathcal{PI}$-cover of M. Let $\psi : F' \to M$ with $F' \in \mathcal{PI}$ be an arbitrary right R-homomorphism. Note that $\psi(F') \leqslant F$ by (2). Define $\zeta : F' \to F$ via $\zeta(x) = \psi(x)$ for $x \in F'$. Then $i\zeta = \psi$, and so $i : F \to M$ is a $\mathcal{PI}$-precover of M. In addition, it is clear that the identity map I_F of F is the only homomorphism $g : F \to F$ such that $ig = i$, and hence (3) follows.
(3) $\Rightarrow$ (2). Let M be any P-injective right R-module and N any submodule of M. We shall show that M/N is P-injective. Indeed, there exists an exact

sequence $0 \to N \to E \to L \to 0$ with E injective. Since L has a monic $\mathcal{PI}$-cover $\phi : F \to L$ by (3), there is $\alpha : E \to F$ such that the following exact diagram is commutative:



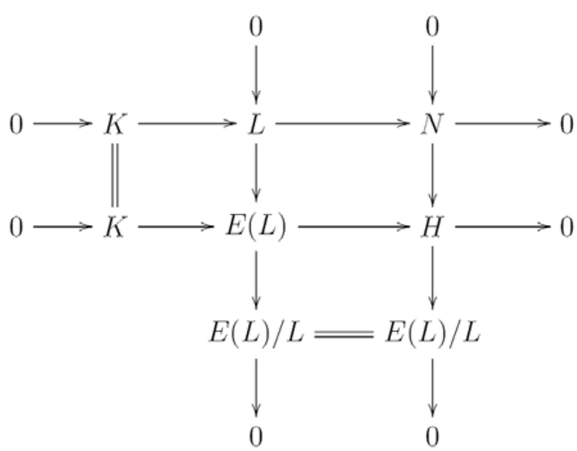
Thus ϕ is epic, and hence it is an isomorphism. Therefore L is P-injective. For any cyclically presented right R-module K, we have

$$0 = \mathrm{Ext}^1(K, L) \to \mathrm{Ext}^2(K, N) \to \mathrm{Ext}^2(K, E) = 0.$$

Therefore $\mathrm{Ext}^2(K, N) = 0$. On the other hand, the short exact sequence $0 \to N \to M \to M/N \to 0$ induces the exactness of the sequence

$$0 = \mathrm{Ext}^1(K, M) \to \mathrm{Ext}^1(K, M/N) \to \mathrm{Ext}^2(K, N) = 0.$$

Therefore $\mathrm{Ext}^1(K, M/N) = 0$, as desired.
(3) $\Rightarrow$ (4) and (2) $\Rightarrow$ (6) are clear.
(4) $\Rightarrow$ (2). Let M be any P-injective right R-module and N any submodule of M. We have to prove that M/N is P-injective. Note that N has a special $\mathcal{PI}$-preenvelope, i.e., there exists an exact sequence $0 \to N \to E \to L \to 0$ with $E \in \mathcal{PI}$ and $L \in \mathcal{CC}$. The rest of the proof is similar to that of (3) $\Rightarrow$ (2) by noting that $\mathrm{Ext}^2(K, E) = 0$ for any cyclically presented right R-module K by Lemma 2.1.
(6) $\Rightarrow$ (2). Let M be any P-injective right R-module and N any submodule of M. Note that N has a special $\mathcal{CC}$-precover, i.e., there exists an exact sequence $0 \to K \to L \to N \to 0$ with $K \in \mathcal{PI}$ and $L \in \mathcal{CC}$. We have the following pushout diagram



Since K and $E(L)$ are P-injective, so is H by (6). Note that $E(L)/L$ is P-injective by (6). Thus (6) $\Rightarrow$ (2) follows from the proof of (3) $\Rightarrow$ (2) and Lemma 2.1.

(2) $\Rightarrow$ (5). Let M be a cyclically covered right R-module. Then M admits a projective resolution

$$\cdots \to P_n \to P_{n-1} \to \cdots P_1 \to P_0 \to M \to 0.$$

Let N be any right R-module. There is an exact sequence

$$0 \to N \to E \to L \to 0,$$

where E and L are P-injective. Therefore we form the following double complex

$$\begin{array}{ccccccc}
 & 0 & & 0 & & 0 & \\
 & \uparrow & & \uparrow & & \uparrow & \\
0 \to & \mathrm{Hom}(M,L) & \to \mathrm{Hom}(P_0,L) \to & & \cdots & \to \mathrm{Hom}(P_n,L) \to & \cdots \\
 & \uparrow & & \uparrow & & \uparrow & \\
0 \to & \mathrm{Hom}(M,E) & \to \mathrm{Hom}(P_0,E) \to & & \cdots & \to \mathrm{Hom}(P_n,E) \to & \cdots \\
 & \uparrow & & \uparrow & & \uparrow & \\
 & 0 & \to \mathrm{Hom}(P_0,N) \to & & \cdots & \to \mathrm{Hom}(P_n,N) \to & \cdots \\
 & & & \uparrow & & \uparrow & \\
 & & & 0 & & 0 &
\end{array}$$

Note that, by Lemma 2.1, all rows are exact except for the bottom row since M is cyclically covered, E and L are P-injective, also note that all columns are exact except for the left column since all P_i are projective.

Using a spectral sequence argument, we know that the following two complexes

$$0 \to \mathrm{Hom}(P_0,N) \to \mathrm{Hom}(P_1,N) \to \cdots \to \mathrm{Hom}(P_n,N) \to \cdots$$

and

$$0 \to \mathrm{Hom}(M,E) \to \mathrm{Hom}(M,L) \to 0$$

have isomorphic homology groups. Thus $\mathrm{Ext}^j(M,N) = 0$ for all $j \geq 2$, and hence $pd(M) \leqslant 1$.

(5) $\Rightarrow$ (1). For any principal right ideal I of R, consider the exact sequence $0 \to I \to R \to R/I \to 0$. Since $pd(R/I) \leqslant 1$ by (5), I is projective. So R is a right PP ring. This completes the proof. ■

If R is an integral domain, then R is a Dedekind ring if and only if every cyclic R-module is a summand of a direct sum of cyclically presented modules [20, 40.5]. Here we generalize the result to the following

Proposition 2.3. *Let R be a ring such that every cyclic right R-module is cyclically covered. Then the following are equivalent:*

(1) *R is a right PP ring;*

(2) *R is a right hereditary ring.*

Proof.
(2) $\Rightarrow$ (1) is obvious.
(1) $\Rightarrow$ (2). Let N be a P-injective right R-module and I a right ideal of R. Since $(\mathcal{CC}, \mathcal{PI})$ is a cotorsion theory, $\mathrm{Ext}^1(R/I, N) = 0$ by hypothesis. So N is injective. Note that R is right hereditary if and only if every quotient module of any injective right R-module is injective, and so (2) follows from (1) and Theorem 2.2 (2). ■

3. Generalizations of *PP* Rings

Recall that R is called right PS [13] if each simple right ideal is projective. Clearly, R is right PS if and only if S_r is projective as a right R-module. R is right nonsingular if $Z_r = 0$. It is well known that right PP rings $\Rightarrow$ right nonsingular rings $\Rightarrow$ right PS rings, but no two of these concepts are equivalent (see [11, 13]).

In this section, we introduce the notion of X-PP rings which unifies PP rings, PS rings and nonsingular rings, where X is a non-empty subset of R.

Definition 3.1. *Let X be a non-empty subset of a ring R. R is called a right X-PP ring if aR is projective for any $a \in X$.*

Proposition 3.2. *A ring R is right Z_r-PP if and only if R is right nonsingular.*

Proof. Suppose R is a right Z_r-PP ring. Let $x \in Z_r$, then $r(x) \leqslant_e R_R$. By hypothesis, xR is projective. So the exact sequence $0 \to r(x) \to R_R \to xR \to 0$ is split, thus $r(x)$ is a summand of R_R. It follows that $r(x) = R$, and so $x = 0$. Thus R is a right nonsingular ring. The other direction is obvious. ■

Obviously, R is right PP if and only if R is a right R-PP ring, and R is right PS if and only if R is a right X-PP ring, where $X = \{a \in R : aR \text{ is simple}\}$. Hence the concept of X-PP rings subsumes PP rings, PS rings and nonsingular rings.

It is clear that right PP-rings are right J-PP, but the converse is false as shown by the following example.

Example 1. Let $R = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} \end{pmatrix}$. Then $J = e_{12}R$, where $e_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Note that $\mathbb{Z}/2\mathbb{Z}$ is not a projective $\mathbb{Z}$-module. Hence R is not a right PP ring by [21, Theorem 6]. Let $0 \neq x \in J$. Then it is easy to verify that $r(x) = e_{11}R$ is a summand of R_R, where $e_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. So R is a right J-PP ring.

It is known that every right PP ring is right PS. This result can be generalized to the following

Proposition 3.3. *Let R be a right J-PP ring. If $a \in R$ such that aR (or Ra) is a simple right (or left) R-module, then aR is projective. In particular, a right*

J-PP ring is right PS.

Proof. If aR is simple and $(aR)^2 \neq 0$, then $aR = eR$ for an idempotent $e \in R$ by [20, 2.7], and so aR is projective. If Ra is simple and $(Ra)^2 \neq 0$, then $Ra = Rf$ for an idempotent $f \in R$. So aR is also projective. If $(aR)^2 = 0$ or $(Ra)^2 = 0$, then $a \in J$. By hypothesis, aR is projective. ■

The next example gives a right PS ring which is not right J-PP. So right J-PP rings lie strictly between right PP rings and right PS rings.

Example 2. Let $R = \left\{ \begin{pmatrix} m & n \\ 0 & m \end{pmatrix} : m, n \in \mathbb{Z} \right\}$. Then R is a ring with the addition and the multiplication as those in ordinary matrices. Note that $J = \begin{pmatrix} 0 & \mathbb{Z} \\ 0 & 0 \end{pmatrix}$ and $S_r = 0$ by [22, Example 3.5], so R is a right PS ring. Let $x = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$. Then $x \in J$. But xR is not projective since $r(x) = J$ can not be generated by an idempotent, hence R is not a right J-PP ring.

It is known that right PP-rings are right nonsingular. However, right J-PP rings need not be right nonsingular. Indeed, there exists a right primitive ring R (hence $J = 0$) with $Z_r \neq 0$ (see [3, p. 28 - 30]). The next example gives a right nonsingular ring which is left semihereditary (hence, left J-PP) but not right J-PP.

Example 3. (Chase's Example) Let K be a regular ring with an ideal I such that, as a submodule of K_K, I is not a summand. Let $R = K/I$, which is also a regular ring. Viewing R as an (R, K)-bimodule, we can form the triangular matrix ring $T = \begin{pmatrix} R & R \\ 0 & K \end{pmatrix}$. Then T is left semihereditary but not right J-PP by the argument in [11, Example 2.34]. Moreover, since $Z(R_R) = 0, Z(K_K) = 0$, it follows that $Z(T_T) = 0$ by [8, Corollary 4.3].

Recall that a right R-module M_R is *mininjective* [15] if every homomorphism from any simple right ideal into M extends to R. M_R is *divisible* [18, 20] if $Mr = M$ for any $r \in X$ where $X = \{a \in R : r(a) = l(a) = 0\}$. M_R is said to satisfy the *$C2$-condition* if every submodule N of M that is isomorphic to a summand of M is itself a summand of M. A ring R is said to be right *P-injective* (*mininjective*) if R_R is P-injective (mininjective). R is called a right $C2$ ring if R_R satisfies the $C2$-condition.

Definition 3.4. *Let R be a ring and M a right R-module. For a non-empty subset X of R, M is said to be X-P-injective if every homomorphism $aR \to M$ extends to R for any $a \in X$. R is said to be right X-P-injective if R_R is X-P-injective. R is called a right X-$C2$ ring if R_R satisfies the $C2$-condition only for $N = aR$, $a \in X$.*

Clearly, M_R is P-injective if and only if M_R is R-P-injective, M_R is mininjective if and only if M_R is X-P-injective, where $X = \{a \in R : aR \text{ is simple}\}$, M_R is divisible if and only if M_R is X-P-injective, where $X = \{a \in R : r(a) = l(a) = 0\}$. We also note that right J-P-injective rings here are precisely right JP-injective rings in [22].

Recall that an element a in R is said to be (von Neumann) *regular* if $a = aba$ for some $b \in R$. A subset $X \subseteq R$ is said to be *regular* if every element in X is regular.

Proposition 3.5. *The following are equivalent for a non-empty subset X of R:*
(1) *Every right R-module is X-P-injective;*
(2) *aR is X-P-injective for any $a \in X$;*
(3) *R is a right X-P-injective and right X-PP ring;*
(4) *R is a right X-$C2$ and right X-PP ring;*
(5) *X is regular.*

Proof.
(1) $\Rightarrow$ (2) is clear.
(2) $\Rightarrow$ (5). Let $a \in X$. Then aR is X-P-injective. It follows that the inclusion $\iota : aR \to R$ is split. Therefore $aR \leqslant^{\oplus} R_R$, and hence a is regular.
(5) $\Rightarrow$ (1) and (3). Since X is regular, aR is a summand of R_R for any $a \in X$. Hence (1) and (3) hold.
(3) $\Rightarrow$ (4). Using [22, Lemma 1.1] and the proof of [17, Lemma 2.5 (3)], it is easy to see that a right X-P-injective ring is right X-$C2$.
(4) $\Rightarrow$ (5). Let $a \in X$. Since R is a right X-PP ring, aR is projective. So aR is isomorphic to a summand of R_R. Since R is a right X-$C2$ ring, it follows that aR is a summand of R_R. Thus a is a regular element, and so X is regular. ∎

Letting $X = \{a \in R : aR \text{ is simple}\}$ in Proposition 3.8, we get some characterizations of right *universally mininjective* rings studied by Nicholson and Yousif (see [15, Lemma 5.1]).

Recall that R is called a left SF ring if every simple left R-module is flat.

Lemmma 3.6. *If R is a left SF ring, then R is a right $C2$ ring.*

Proof. Let $I = Ra_1 + Ra_2 + \cdots + Ra_n$ be a finitely generated proper left ideal. Then there exists a maximal left ideal M containing I. It follows that R/M is a flat left R-module. By [18, Theorem 3.57], there exists $u \in M$ such that $a_i u = a_i$ $(i = 1, 2, \cdots, n)$. Thus $I(1 - u) = 0$ and hence $r(I) \neq 0$. Now suppose $aR \cong K$ where $K \leqslant^{\oplus} R_R$, then aR is projective. Hence $aR \leqslant^{\oplus} R_R$ by [2, Theorem 5.4]. So R is a right $C2$ ring. ∎

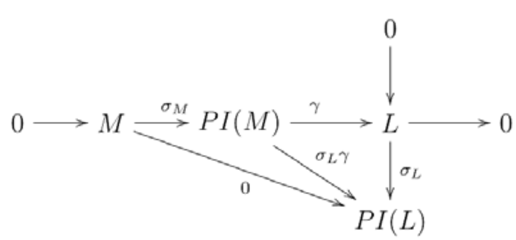
In what follows, $\sigma_M : M \to PI(M)$ ($\epsilon_M : P(M) \to M$) denotes the $\mathcal{PI}$-envelope (projective cover) of a right R-module M (if they exist). Recall that a $\mathcal{PI}$-envelope $\sigma_M : M \to PI(M)$ has the *unique mapping property* [5] if for any homomorphism $f : M \to N$, where N is P-injective, there exists a unique homomorphism $g : PI(M) \to N$ such that $g\sigma_M = f$. The concept of an injective envelope (projective cover) with the unique mapping property can be defined similarly.

Recall that a ring R is said to be *semiregular* in case R/J is regular and idempotents can be lifted modulo J. R is a right *weakly continuous* ring if R is semiregular and $J = Z_r$. By [16, p. 2435], a right PP right weakly continuous ring is regular. This conclusion remains true if we replace right PP by right J-PP as shown in the following

Theorem 3.7. *The following are equivalent for a ring R:*

(1) *R is regular;*
(2) *Every (cyclic) right R-module is P-injective;*
(3) *R is a right PP right $C2$ (or P-injective) ring;*
(4) *R is a right PP left SF ring;*
(5) *R is a right J-PP, right J-$C2$ and semiregular ring;*
(6) *R is a right J-PP right weakly continuous ring;*
(7) *Every right R-module has a $\mathcal{PI}$-envelope with the unique mapping property;*
(8) *$\mathcal{PI}$ is closed under cokernels of monomorphisms, and every cyclically covered right R-module has a $\mathcal{PI}$-envelope with the unique mapping property;*
(9) *$\mathcal{PI}$ is closed under cokernels of monomorphisms, and every cyclically covered right R-module is P-injective.*

Proof. The equivalence of (1) through (3) and (5) $\Rightarrow$ (1) follow from Proposition 3.5, (1) $\Leftrightarrow$ (4) holds by Lemma 3.6 and Proposition 3.5, (6) $\Rightarrow$ (5) follows from [16, Theorem 2.4], and (1) $\Rightarrow$ (6) through (9) is obvious.
(7) $\Rightarrow$ (2). Let M be any right R-module. There is the following exact commutative diagram



Note that $\sigma_L\gamma\sigma_M = 0 = 0\sigma_M$, so $\sigma_L\gamma = 0$ by (7). Therefore $L = \mathrm{im}(\gamma) \subseteq \ker(\sigma_L) = 0$, and hence M is P-injective.
(9) $\Rightarrow$ (2). Let M be any right R-module. Note that M has a special $\mathcal{CC}$-precover, i.e., there exists an exact sequence $0 \to K \to L \to M \to 0$ with $K \in \mathcal{PI}$ and $L \in \mathcal{CC}$. Thus $L \in \mathcal{PI}$, and $M \in \mathcal{PI}$ by (9).
(8) $\Rightarrow$ (9). Let M be a cyclically covered right R-module. By (8), there is an exact sequence

$$0 \longrightarrow M \xrightarrow{\sigma_M} PI(M) \xrightarrow{\gamma} L \longrightarrow 0,$$

where L is cyclically covered by Wakamatsu's Lemma [6, Proposition 7.2.4]. Thus M is P-injective by the proof of (7) $\Rightarrow$ (2). ■

The following two examples show that the condition that R is right J-PP (or right J-$C2$) in Theorem 3.7 is not superfluous.

Example 4. Let V be a two-dimensional vector space over a field F and $R = \left\{ \begin{pmatrix} m & n \\ 0 & m \end{pmatrix} : m \in F, n \in V \right\}$. Then R is a commutative, local, Artinian $C2$ ring, but R is not a P-injective ring by [16, p. 2438]. Hence R is a semiregular J-$C2$ ring, but it is not regular.

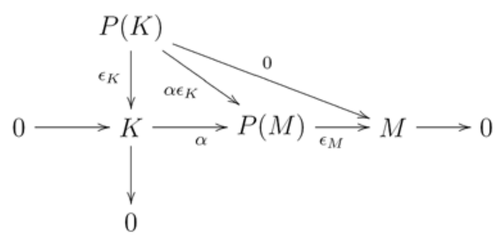
Example 5. Let F be a field and $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$. Then R is a left and right Artinian ring with $J = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$ by [16, p. 2435]. Clearly, R is a semiregular ring which is not regular. However R is a right J-PP ring. In fact, let $0 \neq x \in J$. Then it is easy to verify that $r(x) = \begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} R$ is a summand of R_R, and so xR is projective, as required.

A ring R is said to be right *Kasch* if every simple right R-module embeds in R_R, equivalently $\mathrm{Hom}(M, R) \neq 0$ for any simple right R-module M. It is known that R is semisimple Artinian if and only if R is a right PP and right (or left) Kasch ring (see [16, p. 2435]). Here we get the following

Theorem 3.8. *The following are equivalent for a ring R:*
(1) *R is a semisimple Artinian ring;*
(2) *R is a right J-PP right Kasch ring;*
(3) *R is a right J-PP left Kasch ring;*
(4) *R is a right PS right Kasch ring;*
(5) *Every right R-module has an injective envelope with the unique mapping property;*
(6) *Every right R-module has a projective cover with the unique mapping property;*
(7) *Every cyclic right R-module is both cyclically covered and P-injective.*

Proof.
(1) $\Rightarrow$ (2) through (7) is obvious.
(2) $\Rightarrow$ (4) follows from Proposition 3.3.
(4) $\Rightarrow$ (1). It suffices to show that every simple right R-module is projective. Let M be a simple right R-module. By [13, Theorem 2.4], M is either projective or $\mathrm{Hom}(M, R) = 0$ since R is right PS. Now $\mathrm{Hom}(M, R) \neq 0$ by the right Kasch hypothesis. So M is projective.
(3) $\Rightarrow$ (1). It is enough to show that every simple left ideal is projective. Let Ra be a simple left ideal. By Proposition 3.3, aR is projective. Let $r(a) = (1-e)R$, $e^2 = e \in R$. Then $a = ae$, so $Ra \subseteq Re$, and we claim that $Ra = Re$. If not, let $Ra \subseteq M \leqslant^{\max} Re$. By the left Kasch hypothesis, let $\sigma : Re/M \to {}_RR$ be monic and write $c = \sigma(e + M)$. Then $ec = c$ and $c \in r(a) = (1-e)R$ (for $ae = a \in M$) and hence $c = ec = 0$. Since σ is monic, $e \in M$, a contradiction. So $Ra = Re$ is

projective, as required.
(6) $\Rightarrow$ (1). Let M be any right R-module. There is the following exact commutative diagram



Note that $\epsilon_M \alpha \epsilon_K = 0 = \epsilon_M 0$, so $\alpha\epsilon_K = 0$ by (6). Therefore $K = \mathrm{im}(\epsilon_K) \subseteq \ker(\alpha) = 0$, and so M is projective, as required.

The proof of (5) $\Rightarrow$ (1) is similar to that of (7) $\Rightarrow$ (2) in Theorem 3.7.
(7) $\Rightarrow$ (1). By the proof of Proposition 2.3, every P-injective right R-module is injective. Thus every cyclic right R-module is injective by (7), and hence (1) follows from [11, Corollary 6.47]. ■

Note that semiprime rings are always right PS. So we have

Corollary 3.9. [7, Proposition 5.1]. *A semiprime right Kasch ring is semisimple Artinian.*

By a slight modification of the proof of [21, Theorem 2], we obtain the following

Proposition 3.10. *Let X be a non-empty subset of a ring R. The following are equivalent:*
(1) *R is a right X-PP ring;*
(2) *Every quotient module of any (X-P-)injective right R-module is X-P-injective;*
(3) *Every sum of two (X-P)-injective submodules of any right R-module is X-P-injective.*

Let $X = \{a \in R : aR \text{ is simple}\}$ (resp., J) in Proposition 3.10, we obtain the next corollary.

Corollary 3.11. *The following are equivalent for a ring R:*
(1) *R is a right PS (resp., J-PP) ring;*
(2) *Every quotient module of any mininjective (resp., J-P-injective) right R-module is mininjective (resp., J-P-injective);*
(3) *Every sum of two injective submodules of any right R-module is mininjective (resp., J-P-injective).*

We note that P-injective modules are always divisible, but the converse is not true in general. For example, let $R = \mathbb{Z}/4\mathbb{Z}$, and note that R has exactly three ideals: $0, 2R, R$. It is clear that $2R$ is a divisible R-module, but it is not P-injective.

Recall that R is called an *Abelian* (or *normal*) ring if every idempotent of R is central. If R is an Abelian ring, then R is a right PP ring if and only if every divisible right R-module is P-injective ([9, Theorem 8]). Here we have

Theorem 3.12. *Let X be a right ideal of an Abelian ring R. Then the following are equivalent:*
(1) *Every divisible right R-module is X-P-injective;*
(2) *R is a right X-PP ring.*

Proof.
(1) $\Rightarrow$ (2). Let M be an injective right R-module, then it is divisible. Thus every quotient module of M is divisible, and so it is X-P-injective by (1). Hence R is a right X-PP ring by Proposition 3.10.
(2) $\Rightarrow$ (1). Assume M is a divisible right R-module. Let $a \in X$ and $f\colon aR \to M$ be a right R-homomorphism. Since R is a right X-PP ring, $r(a) = eR$ where $e^2 = e \in R$. We claim that $a + e$ is a non-zero-divisor. In fact, let $x \in r(a + e)$, then $(a + e)x = 0$. It follows that $ex = 0$ and $ax = 0$ since R is an Abelian ring, thus $x \in r(a)$, and so $x = ex = 0$. Therefore $r(a + e) = 0$. Next, let $y \in l(a + e)$. Then $y(a + e) = 0$, so $ye = 0$ and $ya = 0$. Thus $ay \in r(ay)$. Since X is a right ideal, $ay \in X$. By hypothesis, there exists $f^2 = f \in R$ such that $r(ay) = fR$. So $ay = fay = ayf = 0$. Thus $y \in r(a)$ and so $y = ey = ye = 0$. Hence $l(a+e) = 0$.

Since M is divisible, there exists $m \in M$ such that $m(a + e) = f(a)$. Note that $f(a) = f(a(1 - e)) = f(a)(1 - e)$, so $f(a) = m(a + e)(1 - e) = ma$, and hence $f\colon aR \to M$ extends to R. This completes the proof. ∎

Corollary 3.13. *If R is an Abelian ring, then R is a right PS (resp., right J-PP) ring if and only if every divisible right R-module is mininjective (resp., J-P-injective).*

The ring R in the next example is a non-Abelian right PS ring, but not every divisible right R-module is mininjective. So the condition that R is Abelian in Corollary 3.13 cannot be removed.

Example 6. Let $R = \begin{pmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{pmatrix} = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : a, b, c \in \mathbb{Z}_2 \right\}$. It is clear that $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \neq \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ with $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ idempotent. Hence R is not an Abelian ring. Since invertible elements $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ are the only two non-zero-divisors of R, it follows that R_R is a divisible R-module. Now let $x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. It is easy to see that xR is a simple right ideal, $r(x) = \begin{pmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} R$ and $Rx = \begin{pmatrix} 0 & \mathbb{Z}_2 \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{pmatrix} = l(r(x))$. So R_R is not mininjective by [15, Lemma 1.1]. However, R is a right PP ring and hence it is right PS. In fact, it is easily checked that every element of R is either nilpotent or idempotent or invertible. Note that $x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ is the only non-zero nilpotent element and $r(x) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} R$ is a summand of R_R, and so xR is projective, as required.

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References

1. F. W. Anderson and K. R. Fuller, *Rings and Categories of Modules*, Springer-Verlag, New York, 1974.
2. H. Bass, Finitistic dimension and a homological generalization of semiprimary rings, *Trans. Amer. Math. Soc.* **95** (1960) 466–488.
3. A. W. Chatters and C. R. Hajarnavis, *Rings with Chain Conditions*, Pitman, Boston-London-Melbourne, 1980.
4. A. W. Chatters and W. M. Xue, On right duo p.p. rings. *Glasgow Math. J.* **32** (1990) 221–225.
5. N. Q. Ding, On envelopes with the unique mapping property, *Comm. Algebra* **24** (1996) 1459–1470.
6. E. E. Enochs and O. M. G. Jenda, *Relative Homological Algebra*, Walter de Gruyter, Berlin-New York, 2000.
7. C. Faith and D. V. Huynh, When self-injective rings are QF: a report on a problem, *J. Algebra Appl.* **1** (2002) 75–105.
8. K. R. Goodearl, *Ring Theory: Nonsingular Rings and Modules*, Monographs Textbooks Pure Appl. Math. 33. Marcel Dekker, Inc.: New York and Basel, 1976.
9. Y. Hirano, M. Hongan, and M. Ôhori, On right $P.P.$ rings, *Math. J. Okayama Univ.* **24** (1982) 99–109.
10. C. Huh, H. K. Kim, and Y. Lee, $P.P.$ rings and generalized $P.P.$ rings, *J. Pure Appl. Algebra* **167** (2002) 37–52.
11. T. Y. Lam, *Lectures on Modules and Rings*, Springer-Verlag, New York - Heidelberg - Berlin, 1999.
12. Y. Lee and C. Huh, Counterexamples on $P.P.$-rings. *Kyungpook Math. J.* **38** (1998) 421–427.
13. W. K. Nicholson and J. K. Watters, Rings with projective socle, *Proc. Amer. Math. Soc.* **102** (1988) 443–450.
14. W. K. Nicholson and M. F. Yousif, Principally injective rings, *J. Algebra* **174** (1995) 77–93.
15. W. K. Nicholson and M. F. Yousif, Mininjective rings, *J. Algebra* **187** (1997) 548–578.

16. W. K. Nicholson and M. F. Yousif, Weakly continuous and $C2$-rings, *Comm. Algebra* **29** (2001) 2429–2446.
17. W. K. Nicholson and M. F. Yousif, On Quasi-Frobenius Rings, In: International Symposium on Ring Theory, South Korea, June 28-July 3, 1999; G. F. Birkenmeier, J. K. Park, and Y. S. Park (Eds.), Birkhaüser, Boston, 2001, pp. 245-277.
18. J. J. Rotman, *An Introduction to Homological Algebra*, Academic Press, New York, 1979.
19. J. Trlifaj, Covers, Envelopes, and Cotorsion Theories; Lecture notes for the workshop, "Homological Methods in Module Theory", Cortona, September 10-16, 2000.
20. R. Wisbauer, *Foundations of Module and Ring Theory*, Gordon and Breach, 1991.
21. W. M. Xue, On PP rings, *Kobe J. Math.* **7** (1990) 77–80.
22. M. F. Yousif, Y. Zhou, Rings for which certain elements have the principal extension property, *Algebra Colloq.* **10** (2003) 501–512.