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Existence Theorems for Some Generalized Quasivariational Inclusion Problems

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Abstract. In this paper we give sufficient conditions for the existence of solutions of Problem (P_1) (resp. Problem (P_2)) of finding a point $(z_0, x_0) \in B(z_0, x_0) \times A(x_0)$ such that $F(z_0, x_0, x) \subset C(z_0, x_0, x_0)$ (resp. $F(z_0, x_0, x_0) \subset C(z_0, x_0, x)$) for all $x \in A(x_0)$, where A, B, C, F are set-valued maps between locally convex Hausdorff spaces. Some known existence theorems are included as special cases of the main results of the paper.

1. Introduction

Let X, Y and Z be locally convex Hausdorff topological vector spaces. Let $K \subset X$ and $E \subset Z$ be nonempty subsets. Let $A : K \longrightarrow 2^K$, $B : E \times K \longrightarrow 2^E$, $C : E \times K \times K \longrightarrow 2^Y$ and $F : E \times K \times K \longrightarrow 2^Y$ be set-valued maps with nonempty values. In this paper, we consider the existence of solutions of the following generalized quasivariational inclusion problems:

Problem (P_1): Find $(z_0, x_0) \in E \times K$ such that $x_0 \in A(x_0), z_0 \in B(z_0, x_0)$ and, for all $x \in A(x_0)$,

$$F(z_0, x_0, x) \subset C(z_0, x_0, x_0).$$

Problem (P_2): Find $(z_0, x_0) \in E \times K$ such that $x_0 \in A(x_0), z_0 \in B(z_0, x_0)$ and, for all $x \in A(x_0)$,

$$F(z_0, x_0, x_0) \subset C(z_0, x_0, x).$$

Observe that in the above models the set $C(z,\xi,x)$ is not necessarily a convex cone. This is useful for deriving many known results in quasivariational inequalities and quasivariational inclusions. We now mention some papers containing results which can be obtained from the existence theorems of the present paper. The generalized quasivariational inequality problem considered in [2, 7] corresponds to Problem (P_1) where F is single-valued and $C(z,\xi,x) \equiv \mathbb{R}_+$ (the nonnegative half-line). The paper [6] deals with Problem (P_1) where F is single-valued and $C(z,\xi,x)$ equals the sum of $F(z,\xi,x)$ and the complement of the nonempty interior of a closed convex cone. In [11] Problem (P_1) and (P_2) are considered under the assumption that $C(z,\xi,x)$ is the sum of $F(z,\xi,x)$ and a closed convex cone. Our main results formulated in Sec. 3 of this paper will include as special cases Theorem 3.1 and Corollary 3.1 of [2], Theorem 3 of [7], Theorem 2.1 of [6] and Theorems 3.1 and 3.2 of [11]. It is worth noticing that Theorems 3.1 and 3.2 of [11] are obtained under the assumptions stronger than the corresponding assumptions used in the present paper. This remark can be seen in Sec. 4. Our approach is based on a fixed point theorem of [10] which together with some necessary notions can be found in Sec. 2.

2. Preliminaries

Let X be a topological space. Each subset of X can be seen as a topological space whose topology is induced by the given topology of X. For $x \in X$, let us denote by $U(x), U_1(x),\ U_2(x), ...$ open neighborhoods of x. The empty set is denoted by $\emptyset$. A nonempty subset $Q \subset X$ is a convex cone if it is convex and if $\lambda Q \subset Q$ for all $\lambda \geq 0$.

For a set-valued map $F : X \longrightarrow 2^Y$ between two topological spaces X and Y we denote by im F and gr F the image and graph of F :

$$\text{im } F = \bigcup_{x \in X} F(x),$$
$$\text{gr}\, F = \{(x,y) \in X \times Y : y \in F(x)\}.$$

The map F is upper semicontinuous (usc) if for any $x \in X$ and any open set $N \supset F(x)$ there exists $U(x)$ such that $N \supset F(x')$ for all $x' \in U(x)$. The map F is lower semicontinuous (lsc) if for any $x \in X$ and any open set N with $F(x) \cap N \neq \emptyset$ there exists $U(x)$ such that $F(x') \cap N \neq \emptyset$ for all $x' \in U(x)$. The map F is continuous if it is both usc and lsc. The map F is closed if its graph is a closed set of $X \times Y$. The map F is compact if im F is contained in a compact set of Y. The map F is acyclic if it is usc and if, for any $x \in X, F(x)$ is nonempty, compact and acyclic. Here a topological space is called acyclic if all of its reduced Čech homology groups over rationals vanish. Observe that contractible spaces are acyclic; and hence, convex sets and star-shaped sets are acyclic.

The following known theorems will be used later.

Theorem 2.1.[10] *Let K be a nonempty subset of a locally convex Hausdorff* *topological vector space X. If $F : K \longrightarrow 2^K$ is a compact acyclic map, then F has a fixed point, i.e., there exists $x_0 \in K$ such that $x_0 \in F(x_0)$.*

Theorem 2.2. [3] *Let K be a nonempty subset of a Hausdorff topological vector space X and $t : K \longrightarrow 2^X$ a KKM-map. If for each $x \in K$, $t(x)$ is closed and, for at least one $x' \in K, t(x')$ is compact, then $\cap_{x\in K} t(x) \neq \emptyset$.*

Recall that a set-valued map $t : K \longrightarrow 2^X$ is a KKM-map if for each finite subset $\{x_1, x_2, ..., x_n\} \subset K$, we have co $\{x_1, x_2, ..., x_n\} \subset \cup_{i=1}^n F(x_i)$, where co denotes the convex hull.

3. Existence Theorems

This section is devoted to the main results of this paper: sufficient conditions for the existence of solutions of Problems (P_1) and (P_2). We begin by the following lemma.

Lemma 3.1. *Let X, Y and Z be topological spaces, and $K \subset X$ and $E \subset Z$ be nonempty subsets. Let $A : K \longrightarrow 2^K$ be a lsc map. Let $F : E \times K \times K \longrightarrow 2^Y$ be a lsc map and $C : E \times K \times K \longrightarrow 2^Y$ be a map with closed graph. Then the following set-valued maps*

$$(z, \xi) \in E \times K \mapsto s_1(z, \xi) = \{x \in K : F(z, \xi, \xi') \subset C(z, \xi, x), \forall \xi' \in A(\xi)\}$$

and

$$(z, \xi) \in E \times K \mapsto s_2(z, \xi) = \{x \in K : F(z, \xi, x) \subset C(z, \xi, \xi'), \forall \xi' \in A(\xi)\}$$

have closed graphs.

Proof. To prove that the graph of s_1 is closed it suffices to show that the complement of this graph in the topological space $E \times K \times K$ is open. In other words, we must prove that if $(\bar{z}, \bar{\xi}, \bar{x}) \notin \text{gr } s_1$ then there exist neighborhoods $U(\bar{z}), U(\bar{\xi})$ and $U(\bar{x})$ such that

$$(z, \xi, x) \notin \text{gr } s_1 \tag{3.1}$$

for all $(z, \xi, x) \in U(\bar{z}) \times U(\bar{\xi}) \times U(\bar{x})$. Indeed, since $(\bar{z}, \bar{\xi}, \bar{x}) \notin \text{gr } s_1$ there exists $\xi' \in A(\bar{\xi})$ such that

$$F(\bar{z}, \bar{\xi}, \xi') \not\subset C(\bar{z}, \bar{\xi}, \bar{x}).$$

This means that for some $\bar{y} \in F(\bar{z}, \bar{\xi}, \xi')$ we have $\bar{y} \notin C(\bar{z}, \bar{\xi}, \bar{x})$, or, equivalently, $(\bar{z}, \bar{\xi}, \bar{x}, \bar{y}) \notin \text{gr}\, C$. From this and from the closedness of gr C it follows that there exist neighborhoods $U_1(\bar{z}), U_1(\bar{\xi})$, $U(\bar{x})$ and $U(\bar{y})$ such that, for any $(z, \xi, x, y) \in U_1(\bar{z}) \times U_1(\bar{\xi}) \times U(\bar{x}) \times U(\bar{y})$,

$$(z, \xi, x, y) \notin \text{gr}\, C,$$

i.e.,

$$y \notin C(z, \xi, x). \tag{3.2}$$

Observe that $F(\bar{z}, \bar{\xi}, \xi') \cap U(\bar{y}) \neq \emptyset$ since $\bar{y} \in F(\bar{z}, \bar{\xi}, \xi') \cap U(\bar{y})$. Hence by the lower semicontinuity of F there exist neighborhoods $U(\bar{z}) \subset U_1(\bar{z})$, $U_2(\bar{\xi}) \subset U_1(\bar{\xi})$ and $U(\xi')$ such that

$$F(z, \xi, \eta) \cap U(\bar{y}) \neq \emptyset \tag{3.3}$$

for all $z \in U(\bar{z}), \xi \in U_2(\bar{\xi}), \eta \in U(\xi')$. Similarly, since $U(\xi')$ is an open set having a common point ξ' with $A(\bar{\xi})$ and since A is a lsc map there exists a neighborhood $U(\bar{\xi}) \subset U_2(\bar{\xi})$ such that

$$A(\xi) \cap U(\xi') \neq \emptyset \tag{3.4}$$

for all $\xi \in U(\bar{\xi})$. We now prove that (3.1) holds for all $(z, \xi, x) \in U(\bar{z}) \times U(\bar{\xi}) \times U(\bar{x})$. Indeed, since $\xi \in U(\bar{\xi})$ there exists $\widehat{\xi} \in A(\xi) \cap U(\xi')$ (see (3.4)). Since $(z, \xi, \widehat{\xi}) \in U(\bar{z}) \times U(\bar{\xi}) \times U(\xi')$ there exists $y \in U(\bar{y})$ with $y \in F(z, \xi, \widehat{\xi})$ (see (3.3)). Since $(z, \xi, x, y) \in U(\bar{z}) \times U(\bar{\xi}) \times U(\bar{x}) \times U(\bar{y})$ we get (3.2). Thus, for all $(z, \xi, x) \in U(\bar{z}) \times U(\bar{\xi}) \times U(\bar{x})$ there exists $\widehat{\xi} \in A(\xi)$ and $y \in F(z, \xi, \widehat{\xi})$ such that $y \notin C(z, \xi, x)$. This proves that $(z, \xi, x) \notin \text{gr } s_1$, as required. The proof of the closedness of the graph of s_1 is thus complete. We omit the similar proof of the closedness of the graph of s_2. ∎

From now on we assume that X, Y and Z are locally convex Hausdorff topological vector spaces, $K \subset X$ and $E \subset Z$ are nonempty convex subsets, and $A : K \longrightarrow 2^K$, $B : E \times K \longrightarrow 2^E$, $C : E \times K \times K \longrightarrow 2^Y$ and $F : E \times K \times K \longrightarrow 2^Y$ are set-valued maps with nonempty values. To give existence theorems for Problems (P_1) and (P_2) let us introduce the following set-valued maps $T_1,\ T_2 : E \times K \longrightarrow 2^K$ and $\tau_1,\ \tau_2 : E \times K \longrightarrow 2^{E \times K}$ by setting

$$T_1(z, \xi) = \{x \in A(\xi) : F(z, \xi, \xi') \subset C(z, \xi, x), \forall \xi' \in A(\xi)\}, \tag{3.5}$$

$$T_2(z, \xi) = \{x \in A(\xi) : F(z, \xi, x) \subset C(z, \xi, \xi'), \forall \xi' \in A(\xi)\}, \tag{3.6}$$

$$\tau_1(z, \xi) = B(z, \xi) \times T_1(z, \xi), \tag{3.7}$$

$$\tau_2(z, \xi) = B(z, \xi) \times T_2(z, \xi), \tag{3.8}$$

for all $(z, \xi) \in E \times K$. Obviously, $(z_0, x_0) \in E \times K$ is a solution of Problem (P_1) (resp. Problem (P_2)) if and only if it is a fixed point of map τ_1 (resp. τ_2). So, solving Problem (P_1) (resp. Problem (P_2)) is equivalent to finding a fixed point of map τ_1 (resp. τ_2).

Theorem 3.1. *Let $A : K \longrightarrow 2^K$ be a compact continuous map with closed values and $B : E \times K \longrightarrow 2^E$ be a compact acyclic map. Assume that $F : E \times K \times K \longrightarrow 2^Y$ is a lsc map and $C : E \times K \times K \longrightarrow 2^Y$ is a map with closed graph such that, for all $(z, \xi) \in E \times K$, the set $T_1(z, \xi)$ (resp. $T_2(z, \xi)$) is nonempty and acyclic. Then there exists a solution of Problem (P_1) (resp. Problem (P_2)).*

Proof. Let τ_1 be defined by (3.7). As we have discussed above, to prove the existence of solutions of Problem (P_1) it is enough to show that the map τ_1 has a fixed point. Such a fixed point exists by Theorem 2.1. Indeed, we first claim that T_1 is usc. Notice that, for each $(z,\xi) \in E \times K$, the set $T_1(z,\xi)$ can be rewritten as

$$T_1(z,\xi) = s_1(z,\xi) \cap A(\xi),$$

where the map $s_1 : E \times K \longrightarrow 2^K$, defined in Lemma 3.1, is closed. Hence, since the set-valued map A is usc and compact-valued it follows from this and Proposition 2 of [1, p.71] that T_1 is usc.

Observe now that τ_1 is usc with nonempty compact values since it is the product of usc maps B and T_1 with nonempty compact values (see [1, Proposition 7, p.73]). Observe also that for each $(z,\xi) \in E \times K$, the set $\tau_1(z,\xi)$ is acyclic since it is the product of two acyclic sets (see the Künneth formula in [9]). Thus, the map τ_1 is acyclic. In addition, τ_1 is a compact map since $\text{im}\ \tau_1 \subset \text{im}\ B \times \text{im}\ A$, and since A and B are compact maps. Therefore, all assumptions of Theorem 2.1 are satisfied for the set-valued map τ_1. Thus, τ_1 has a fixed point, i.e., Problem (P_1) has a solution.

To prove the existence of solutions of Problem (P_2) we use the same argument, with τ_2 instead of τ_1. ∎

From Theorem 3.1 we can obtain existence results for the following problems:

Problem (P_1') : Find $(z_0,x_0) \in E \times K$ such that $(z_0,x_0) \in B(z_0,x_0) \times A(x_0)$ and, for all $x \in A(x_0)$,

$$F(z_0,x_0,x) \cap C(z_0,x_0,x_0) = \emptyset.$$

Problem (P_2') : Find $(z_0,x_0) \in E \times K$ such that $(z_0,x_0) \in B(z_0,x_0) \times A(x_0)$ and, for all $x \in A(x_0)$,

$$F(z_0,x_0,x_0) \cap C(z_0,x_0,x) = \emptyset.$$

Before formulating these existence results let us introduce the following sets

$$T_1'(z,\xi) = \{x \in A(\xi) : F(z,\xi,\xi') \cap C(z,\xi,x) = \emptyset, \forall \xi' \in A(\xi)\}, \tag{3.9}$$
$$T_2'(z,\xi) = \{x \in A(\xi) : F(z,\xi,x) \cap C(z,\xi,\xi') = \emptyset, \forall \xi' \in A(\xi)\}. \tag{3.10}$$

Corollary 3.1. *Let A and B be as in Theorem 3.1. Assume that $F : E \times K \times K \longrightarrow 2^Y$ is a lsc map and $C : E \times K \times K \longrightarrow 2^Y$ is a map with open graph such that, for all $(z,\xi) \in E \times K$, the set $T_1'(z,\xi)$ (resp. $T_2'(z,\xi)$) is nonempty and acyclic. Then there exists a solution of Problem (P_1') (resp. Problem (P_2')).*

Proof. A point (z_0,x_0) is a solution of Problem (P_1') (resp. Problem (P_2')) if and only if it is a solution of Problem (P_1) (resp. Problem (P_2)) with C' instead of C where the map $C' : E \times K \times K \longrightarrow 2^Y$, defined by

$$C'(z,\xi,x) = Y \setminus C(z,\xi,x)$$

for all $(z,\xi,x) \in E\times K\times K$, has a closed graph. To complete our proof it suffices to apply Theorem 3.1 with C' instead of C. ∎

From Corollary 3.1 we derive the following corollary which generalizes a result given in Theorem 2.1 of [6].

Corollary 3.2. *Let A and B be as in Theorem 3.1. Let $f : E\times K\times K \longrightarrow Y$ be a single-valued continuous map and $c : E\times K \longrightarrow 2^Y$ be a set-valued map such that, for all $(z,\xi)\in E\times K$, $c(z,\xi)\neq Y$ and $c(z,\xi)$ is a closed convex cone with nonempty interior. Assume additionally that*
(i) *The map*

$$(z,\xi)\in E\times K \mapsto \text{int } c(z,\xi)$$

has an open graph.
(ii) *For all $(z,\xi)\in E\times K$, the set*

$$\{x\in A(\xi) : [f(z,\xi,A(\xi)) - f(z,\xi,x)] \cap \text{ int } c(z,\xi) = \emptyset\} \tag{3.11}$$

is acyclic.
Then there exists a solution to the following problem: Find $(z_0,x_0)\in E\times K$ such that $(z_0,x_0)\in B(z_0,x_0)\times A(x_0)$ and, for all $x\in A(x_0)$,

$$f(z_0,x_0,x) - f(z_0,x_0,x_0) \notin \text{int } c(z_0,x_0).$$

Proof. Obviously, the set (3.11) is exactly the set $T_1'(z,\xi)$ where $C : E\times K\times K \longrightarrow 2^Y$, defined by $C(z,\xi,x) = f(z,\xi,x) + \text{int } c(z,\xi)$, has an open graph. On the other hand, the set (3.11) is nonempty since $f(z,\xi,A(\xi))$ is a compact set (see [5, 8]). Therefore, by Corollary 3.1 there exists a solution of Problem (P_1'), i.e., a solution of the problem formulated in Corollary 3.2. ∎

4. Special Cases

In this section we consider some special cases of Theorem 3.1 which generalize the main results of [11].

Let α be a relation on 2^Y in the sense that α is a subset of the Cartesian product $2^Y\times 2^Y$. For two sets $M\in 2^Y$ and $N\in 2^Y$, let us write $\alpha(M,N)$ (resp. $\overline{\alpha}(M,N)$) instead of $(M,N)\in\alpha$ (resp. $(M,N)\notin\alpha$).

Lemma 4.1. *Let α be an arbitrary relation on 2^Y. Let $a\subset X$ be a nonempty compact convex subset and $f : a \longrightarrow 2^Y$ and $c : a \longrightarrow 2^Y$ be set-valued maps with nonempty values such that*
(i) *For all $\eta\in a$, the set*

$$t(\eta) = \{x\in a : \alpha(f(\eta), c(x))\}$$

is closed in a.
(ii) *For all $x\in a$, the set*

$$s(x) = \{\eta\in a : \overline{\alpha}(f(\eta), c(x))\}$$

is convex.
(iii) *For all* $x \in a$, $\alpha(f(x), c(x))$.
Then the set

$$\{x \in a : \alpha(f(\eta), c(x)),\ \forall \eta \in a\}$$

is nonempty.

Proof. This is an easy consequence of Theorem 2.2 applied to the map $t : a \longrightarrow 2^a$ defined in Lemma 4.1. ■

Remark 1. When a is not compact Lemma 4.1 remains true under the following coercivity condition: there exist a nonempty compact set $a_1 \subset a$ and a compact convex set $b \subset a$ such that, for every $x \in a \setminus a_1$, there exists $\eta \in b$ with $\overline{\alpha}(f(\eta), c(x))$.

Before going further let us introduce some notions of quasiconvexity of set-valued maps. Let $a \subset X$ be a convex subset and $D \subset Y$ be a convex cone. A map $f : a \longrightarrow 2^Y$ is said to be properly D-quasiconvex on a if for all $\eta_i \in a$, $y_i \in f(\eta_i)$ $(i = 1, 2)$ and $\mu \in (0, 1)$ there exists $y \in f(\mu\eta_1 + (1-\mu)\eta_2)$ such that

$$\text{either } y_1 \in y + D \text{ or } y_2 \in y + D. \tag{4.1}$$

Obviously, f is properly D-quasiconvex on a if it is upper D-quasiconvex on a in the sense of [11]: for all $\eta_i \in a$ $(i = 1, 2)$ and $\mu \in (0, 1)$

$$\begin{array}{ll} \text{either} & f(\eta_1) \subset f(\mu\eta_1 + (1-\mu)\eta_2) + D \\ \text{or} & f(\eta_2) \subset f(\mu\eta_1 + (1-\mu)\eta_2) + D. \end{array}$$

When f is single-valued both notions of proper D-quasiconvexity and upper D-quasiconvexity reduce to the notion of proper D-quasiconvexity of [4].

We recall also the notion of lower D-quasiconvexity of f on a [11]: for all $\eta_i \in a$ $(i = 1, 2)$ and $\mu \in (0, 1)$

$$\begin{array}{ll} \text{either} & f(\mu\eta_1 + (1-\mu)\eta_2) \subset f(\eta_1) - D \\ \text{or} & f(\mu\eta_1 + (1-\mu)\eta_2) \subset f(\eta_2) - D. \end{array}$$

Remark 2. Since D is a convex cone it is obvious that the proper D-quasiconvexity (resp. lower $(-D)$-quasiconvexity) of f implies the proper D-quasiconvexity (resp. lower $(-D)$-quasiconvexity) of $f + D$.

Lemma 4.2. *If f is properly D-quasiconvex* (*in particular, if f is upper D-quasiconvex*) *on a then*
(i) *For all $x \in a$, the set*

$$\{\eta \in a : f(\eta) \not\subset f(x) + D\} \tag{4.2}$$

is convex.
(ii) *The set*

$$\{x \in a : f(\eta) \subset f(x) + D,\ \forall \eta \in a\} \tag{4.3}$$

is convex.

Proof. To prove the convexity of the set (4.2) we must show that $\eta = \mu\eta_1 + (1-\mu)\eta_2$ belongs to the set (4.2) if $\mu \in (0,1)$ and if η_i $(i = 1, 2)$ are elements of this set, i.e., $\eta_i \in a$ and $y_i \notin f(x) + D$ for some $y_i \in f(\eta_i)$ $(i = 1, 2)$. Indeed, let $y \in f(\mu\eta_1 + (1-\mu)\eta_2)$ be such that either $y_1 \in y + D$ or $y_2 \in y + D$ (see (4.1)). If $y \in f(x) + D$ then

$$\begin{aligned} &\text{either } y_1 \in y + D \subset f(x) + D + D \subset f(x) + D \\ &\text{or} \quad\ \ y_2 \in y + D \subset f(x) + D + D \subset f(x) + D, \end{aligned}$$

which is impossible. Therefore, $y \notin f(x) + D$ which shows that $f(\eta) \not\subset f(x) + D$, i.e., η belongs to the set (4.2), as desired.

Turning to the proof of the convexity of the set (4.3) we assume that $\mu \in (0,1)$ and x_i $(i = 1, 2)$ are elements of this set, i.e., $x_i \in a$ and $f(a) \subset f(x_i) + D$ $(i = 1, 2)$. We must prove that $x = \mu x_1 + (1-\mu)x_2$ satisfies the inclusion $f(a) \subset f(x) + D$. Indeed, let $y' \in f(a)$ and $y_i \in f(x_i)$ such that $y' \in y_i + D$ $(i = 1, 2)$. By the proper quasiconvexity property there exists $y \in f(\mu x_1 + (1-\mu)x_2)$ such that

$$\text{either } y_1 \in y + D, \text{ or } y_2 \in y + D.$$

Therefore,

$$\begin{aligned} &\text{either } y' \in y_1 + D \subset y + D + D \subset f(\mu x_1 + (1-\mu)x_2) + D \\ &\text{or} \quad\ \ y' \in y_2 + D \subset y + D + D \subset f(\mu x_1 + (1-\mu)x_2) + D. \end{aligned}$$

Since this is true for arbitrary $y' \in f(a)$ we conclude that $f(a) \subset f(\mu x_1 + (1-\mu)x_2) + D$, as desired. ■

Lemma 4.3. *If f is lower $(-D)$-quasiconvex on a then*

(i) *For all $x \in a$, the set*

$$\{\eta \in a : f(x) \not\subset f(\eta) + D\}$$

is convex.

(ii) *The set*

$$\{x \in a : f(x) \subset f(\eta) + D,\ \forall \eta \in a\}$$

is convex.

Proof. Obvious. ■

Making use of Lemmas 4.1 - 4.3 we obtain the following lemma.

Lemma 4.4. *Let $a \subset X$ be a nonempty compact convex set and $D \subset Y$ be a nonempty convex cone. Let $f : a \longrightarrow 2^Y$ be lsc and properly D-quasiconvex (resp. lower $(-D)$-quasiconvex) on a. Let $c : a \longrightarrow 2^Y$ be of the form*

$$c(x) = f(x) + D,\ \forall x \in a,$$

and let c be closed.
Then the set

$$\{x \in a : f(\eta) \subset c(x), \ \forall \eta \in a\} \tag{4.4}$$
$$(resp.\ \{x \in a : f(x) \subset c(\eta), \ \forall \eta \in a\}) \tag{4.5}$$

is nonempty.

Proof. Let us prove the nonemptiness of the set (4.4) under the proper D-quasiconvexity assumption of f. Indeed, let us set in Lemma 4.1

$$\alpha(M, N) = \{(M, N) \in 2^Y \times 2^Y : M \subset N\}.$$

Then the condition (iii) of Lemma 4.1 is automatically satisfied. The condition (i) of Lemma 4.1 is assured by Lemma 3.1. Indeed, applying this lemma to the case $F(z, \xi, x) \equiv f(\xi)$ and $C(z, \xi, x) \equiv c(x)$ we see that the map

$$\xi \in a \mapsto \{x \in a : f(\xi) \subset c(x)\}$$

has closed graph; and hence the value of each point $\xi \in a$, i.e., the set $\{x \in a : f(\xi) \subset c(x)\}$, must be closed in a. The condition (ii) of Lemma 4.1 is derived from Lemma 4.2. The nonemptiness of the set (4.4) is thus proved. The nonemptiness of the set (4.5) under the lower $(-D)$-quasiconvexity property of f can be established similarly, with Lemma 4.3 instead of Lemma 4.2. ■

On the basis of Lemma 4.4 we can derive the following main results of this section.

Theorem 4.1. *Let $A : K \longrightarrow 2^K$ be a compact continuous map with closed convex values and $B : E \times K \longrightarrow 2^E$ be a compact acyclic map. Assume that $F : E \times K \times K \longrightarrow 2^Y$ is a lsc map and $C : E \times K \times K \longrightarrow 2^Y$ is a map with closed graph. Assume additionally that C is of the form*

$$C(z, \xi, x) = F(z, \xi, x) + D(z, \xi), \ \forall (z, \xi, x) \in E \times K \times K \tag{4.6}$$

where, for all $(z, \xi) \in E \times K$, $D(z, \xi)$ is a convex cone and $F(z, \xi, \cdot)$ is properly $D(z, \xi)$-quasiconvex (resp. lower $(-D(z, \xi))$-quasiconvex) on $A(\xi)$.

Then there exists a solution of Problem (P_1) (resp. Problem (P_2)).

Proof. Let us prove the existence of a solution of Problem (P_1). We fix $(z, \xi) \in E \times K$ and we remark that in our case

$$T_1(z, \xi) = \{x \in A(\xi) : F(z, \xi, \xi') \subset F(z, \xi, x) + D(z, \xi), \ \forall \xi' \in A(\xi)\}.$$

Then by Lemma 4.4 $T_1(z, \xi)$ is nonempty. Also, it is acyclic since it is convex by Lemma 4.2. Therefore, by Theorem 3.1 there exists a solution of Problem (P_1). The proof of the existence of a solution of Problem (P_2) is similar, with Lemma 4.3 instead of Lemma 4.2. ■

Before formulating corollaries of Theorem 4.1 let us recall some notions. Let a be a convex subset of X, D be a convex cone of Y, and $f : a \longrightarrow 2^Y$ be a set-valued map. We say that f is D-upper semicontinuous (resp. D-lower semicontinuous) if $f + D$ is usc (resp. lsc). We say that f is D-continuous if it is both D-upper semicontinuous and D-lower semicontinuous. We say that f is D-closed if $f + D$ is closed.

Corollary 4.1. *Let $A : K \longrightarrow 2^K$ be a compact continuous map with closed convex values and $B : E \times K \longrightarrow 2^E$ be a compact acyclic map. Let $D \subset Y$ be a convex cone and $F : E \times K \times K \longrightarrow 2^Y$ be a set-valued map such that*

(i) *F is D-lower semicontinuous.*
(ii) *F is D-closed.*
(iii) *For all $(z, \xi) \in E \times K$, $F(z, \xi, \cdot)$ is properly D-quasiconvex (resp. lower $(-D)$-quasiconvex) on $A(\xi)$.*

Then there exists $(z_0, x_0) \in E \times K$ such that $(z_0, x_0) \in B(z_0, x_0) \times A(x_0)$ and, for all $x \in A(x_0)$,

$$F(z_0, x_0, x) \subset F(z_0, x_0, x_0) + D \tag{4.7}$$

$$(\text{resp. } F(z_0, x_0, x_0) \subset F(z_0, x_0, x) + D). \tag{4.8}$$

Proof. Assume that F is D-lower semicontinuous and, for all $(z, \xi) \in E \times K$, $F(z, \xi, \cdot)$ is properly D-quasiconvex on $A(\xi)$. Observe from D-lower semicontinuity property and Remark 2 that $F' = F + D$ is lower semicontinuous and, for all $(z, \xi) \in E \times K$, $F'(z, \xi, \cdot)$ is properly D-quasiconvex on $A(\xi)$. Applying Theorem 4.1 with F' instead of F and with $D(z, \xi) \equiv D$ we see that there exists $(z_0, x_0) \in E \times K$ such that $(z_0, x_0) \in B(z_0, x_0) \times A(x_0)$ and, for all $x \in A(x_0)$,

$$F'(z_0, x_0, x) \subset F'(z_0, x_0, x_0) + D.$$

From this inclusion we derive (4.7) since $F(z_0, x_0, x) \subset F'(z_0, x_0, x)$ and $F'(z_0, x_0, x_0) + D = F'(z_0, x_0, x_0)$. The first conclusion of Corollary 4.1 is thus proved. The second one can be proved by the same argument (under the lower $(-D)$-quasiconvexity assumption). ■

Remark 3. Since D is a convex cone it is easy to check that f is D-lower semicontinuous on a if f is lower $(-D)$-continuous on a in the sense of [11]: for any $\bar{x} \in a$ and for any neighborhood $U(0_Y)$ of the origin of Y there exists a neighborhood $U(\bar{x})$ such that

$$f(\bar{x}) \subset f(x) + U(0_Y) + D, \quad \forall x \in U(\overline{x}).$$

We recall also the notion of upper D-continuity of f on a in the sense of [11]: for any $\bar{x} \in a$ and for any neighborhood $U(0_Y)$ of the origin of Y there exists a neighborhood $U(\bar{x})$ such that

$$f(x) \subset f(\bar{x}) + U(0_Y) + D, \quad \forall x \in U(\overline{x}).$$

Corollary 4.2. *Let $A : K \longrightarrow 2^K$ be a compact continuous map with closed convex values and $B : E \times K \longrightarrow 2^E$ be a compact acyclic map. Let $D \subset Y$*

be a closed convex cone and $F : E \times K \times K \longrightarrow 2^Y$ *be a upper* D*-continuous and lower* $(-D)$*-continuous map with nonempty compact valued such that, for all* $(z,\xi) \in E \times K$, $F(z,\xi,\cdot)$ *is properly* D*-quasiconvex (resp. lower* $(-D)$*-quasiconvex) on* $A(\xi)$*.*

Then the conclusion of Corollary 4.1 is true.

Proof. Observe from Remark 3 that F is D-lower semicontinuous. It is easy to verify that $F+D$ is closed (i.e., F is D-closed) since F is an upper D-continuous map with compact values and D is a closed convex cone. Our conclusion is now derived from Corollary 4.1. ∎

Remark 4. The results given in Corollary 4.2 were established in [11, Theorems 3.1 and 3.2] under the assumptions stronger than those of Corollary 4.2. Namely, in addition to the assumptions of Corollary 4.2 it is required in [11] that
(i) The dual cone of D has a weak* compact base
(ii) F has convex values
(iii) The map $B(z,\xi)$ does not depend on z.

Remark 5. Corollary 4.2 includes as special cases Theorem 3.1 and Corollary 3.1 in [2], and Theorem 3 in [7].

We conclude our paper by the following example, where the map F does not have convex values and the map $B(z,\xi)$ depends on z.

Example 4.1. Let us consider Problem (P_1) with $X = Y = Z = \mathbb{R}$, $E = K = [0,1]$, $D(z,\xi) \equiv \mathbb{R}_+$, $A(\xi) = [1-\xi, 1]$, $B(z,\xi) = \{1 - z\xi\}$, and $F(z,\xi,x) = \{z(\xi^3 - x^3), z(\xi^3 + x^2)\}$. Then, it is easy to verify that all assumptions of Corollary 4.2 are satisfied. Hence, there exists $(z_0,x_0) = \left(\frac{1}{2}, 1\right) \in B(z_0,x_0) \times A(x_0)$ such that

$$\begin{aligned}\left\{z_0(x_0^3 - x^3), z_0(x_0^3 + x^2)\right\} &= \left\{\frac{1}{2}(1 - x^3), \frac{1}{2}(1 + x^2)\right\} \\ &\subset \{0,1\} + \mathbb{R}_+ \\ &= \{0, z_0(x_0^3 + x_0^2)\} + \mathbb{R}_+,\ \forall x \in [0,1],\end{aligned}$$

i.e.,

$$F(z_0,x_0,x) \subset F(z_0,x_0,x_0) + D,\ \forall x \in A(x_0).$$

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