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nX-Complementary Generations of the Rudvalis Group Ru

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Abstract Let G be a finite group and nX a conjugacy class of elements of order n in G. G is called $nX-$complementary generated if, for every $x \in G - \{1\}$, there is a $y \in nX$ such that $G = < x, y >$.

In [20] the question of finding all positive integers n such that a given non-abelian finite simple group G is nX-complementary generated was posed. In this paper we answer this question for the sporadic group Ru. In fact, we prove that for any element order n of the sporadic group Ru, Ru is nX-complementary generated if and only if $n \geq 3$.

1. Introduction

A group G is said to be (l, m, n)-generated if it can be generated by two elements x and y such that $o(x) = l, o(y) = m$ and $o(xy) = n$. In this case G is the quotient of the triangle group $T(l, m, n)$ and for any permutation π of S_3, the group G is also $((l)\pi, (m)\pi, (n)\pi)$-generated. Therefore we may assume that $l \leq m \leq n$. By [5], if the non-abelian simple group G is (l, m, n)-generated, then either $G \cong A_5$ or $\frac{1}{l} + \frac{1}{m} + \frac{1}{n} < 1$. Hence for a non-abelian finite simple group G and divisors l, m, n of the order of G such that $\frac{1}{l} + \frac{1}{m} + \frac{1}{n} < 1$, it is natural to ask if G is a (l, m, n)-generated group. The motivation for this question came from the calculation of the genus of finite simple groups [26]. It can be shown that the problem of finding the genus of a finite simple group can be reduced to one of generations(for details see [23]).

In a series of papers, [16 - 21] Moori and Ganief established all possible (p, q, r)-generations and nX-complementary generations, where p, q, r are distinct primes, of the sporadic groups J_1, J_2, J_3, HS, McL, Co_3, Co_2, and F_{22}. Also, the first author in [2 - 4] and [8 - 14] (joint works), did the same for the sporadic groups Co_1, Th, $O'N$, Ly, Suz and He. The motivation for this study is outlined in these papers and the reader is encouraged to consult these papers for background material as well as basic computational techniques.

Throughout this paper we use the same notation as in [1, 8, 10, 11]. In particular, $\Delta(G) = \Delta(lX, mY, nZ)$ denotes the structure constant of G for the conjugacy classes lX, mY, nZ, whose value is the cardinality of the set $\Lambda = \{(x, y) \mid xy = z\}$, where $x \in lX$, $y \in mY$ and z is a fixed element of the conjugacy class nZ. Also, $\Delta^\star(G) = \Delta^\star_G(lX, mY, nZ)$ and $\Sigma(H)$ denote the number of pairs $(x, y) \in \Lambda$ such that $G = \langle x, y \rangle$ and $\langle x, y \rangle \subseteq H$, respectively. The number of pairs $(x, y) \in \Lambda$ generating a subgroup H of G will be given by $\Sigma^\star(H)$ and the centralizer of a representative of lX will be denoted by $C_G(lX)$. A general conjugacy class of a subgroup H of G with elements of order n will be denoted by nx. Clearly, if $\Delta^\star(G) > 0$, then G is (lX, mY, nZ)-generated and (lX, mY, nZ) is called a generating triple for G. The number of conjugates of a given self-normalizing subgroup H of G containing a fixed element z is given by $\chi_{N_G(H)}(z)$, where $\chi_{N_G(H)}$ is the permutation character of G with action on the conjugates of H (cf. [24]). In most cases we will calculate this value from the fusion map from $N_G(H)$ into G stored in GAP [22].

Let G be a group and nX a conjugacy class of elements of order n in G. Following Woldar [25], the group G is said to be nX-complementary generated if, for any arbitrary non-identity element $x \in G$, there exists a $y \in nX$ such that $G = \langle x, y \rangle$. The element $y = y(x)$ for which $G = \langle x, y \rangle$ is called complementary.

It is an easy fact that, for any positive integer n, $T(2, 2, n) \cong D_{2n}$, the dihedral group of order $2n$. This shows that a non-dihedral group cannot be $2X$-complementary generated.

Now we discuss techniques that are useful in resolving generation type questions for finite groups. A useful result that we shall often use is a result from Conder, Wilson and Woldar [6], as follows:

Lemma 1.1. *If G is nX-complementary generated and $(sY)^k = nX$, for some integer k, then G is sY-complementary generated.*

Further useful results that we shall use are:

Lemma 1.2. ([19]) *Let G be a $(2X, sY, tZ)$-generated simple group then G is $(sY, sY, (tZ)^2)$-generated.*

Lemma 1.3. *Let G be a finite simple group and H a maximal subgroup of G containing a fixed element x. Then the number h of conjugates of H containing x is $\chi_H(x)$, where χ_H is the permutation character of G with action on the* *conjugates of H. In particular,*

$$h = \sum_{i=1}^{m} \frac{|C_G(x)|}{|C_H(x_i)|},$$

where $x_1, x_2, \cdots, x_m$ are representatives of the H-conjugacy classes that fuse to the G-conjugacy class of x.

We calculated h for suitable triples in Table 3. Throughout this paper our notation is standard and taken mainly from [1, 6, 8]. In this paper, we will prove the following theorem:

Theorem. *The Rudvalis group Ru is nX-complementary generated if and only if $n \geq 3$.*

2. nX-Complementary Generations for Ru

In this section we obtain all of the nX-complementary generations of the Rudvalis group Ru. We will use the maximal subgroups of Ru listed in the ATLAS extensively, especially those with order divisible by 29. We listed in Table 1, all the maximal subgroups of Ru and in Table 3, the fusion maps of these maximal subgroups into Ru (obtained from GAP) that will enable us to evaluate $\Delta^{\star}_{Ru}(pX, qY, rZ)$, for prime classes pX, qY and rZ. In this table h denotes the number of conjugates of the maximal subgroup H containing a fixed element z (see Lemma 1.4). For basic properties of the group Ru and information on its maximal subgroups the reader is referred to [7]. It is a well known fact that Ru has exactly 15 conjugacy classes of maximal subgroups, as listed in Table 1.

Table 1. The Maximal Subgroup of Ru

Group	*Order*	*Group*	*Order*
$2F_4(2)'.2$	$2^{12}.3^3.5^2.13$	$2^6 : U_3(3) : 2$	$2^{12}.3^3.7$
$(2^2 \times Sz(8)) : 3$	$2^8.3.5.7.13$	$2^{3+8} : L_3(2)$	$2^{14}.3.7$
$U_3(5).2$	$2^5.3^2.5^3.7$	$2.2^{4+6} : S_5$	$2^{14}.3.5$
$L_2(25).2^2$	$2^5.3.5^2.13$	A_8	$2^6.3^2.5.7$
$L_2(29)$	$2^2.3.5.7.29$	$5^2 : 4S_5$	$2^5.3.5^3$
$3.A_6.2^2$	$2^5.3^3.5$	$5^{1+2} : 2^5$	$2^5.5^3$
$L_2(13).2$	$2^3.3.7.13$	$A_6.2^2$	$2^5.3^2.5$
$5 : (4 \times A_5)$	$2^4.3.5^2$		

In [25], Woldar proved that every sporadic simple group is pX-complementary generated, for the greatest prime divisor p of the order of the group. Also, by another result from [25], a group G is nX-complementary generated if for every conjugacy class pY of prime order elements in G there is a conjugacy class tZ

such that G is (pY, nX, tZ)-generated. By the mentioned result of Woldar Ru is $29X$-complementary generated, for $X \in \{A, B\}$.

Lemma 2.1. *The sporadic group Ru is not $4Z$-complementary generated, for $Z \in \{A, B, C, D\}$. It is not $2X$-complementary generated, for $X \in \{A, B\}$.*

Proof. Since Ru is simple and every finite simple group is not isomorphic to some dihedral group, Ru is not $(2X, 2X, nY)$-generated, for all classes of involutions and any Ru-class nY. Thus, Ru is not $2X-$complementary generated. Set $V = \{A, B, C, D\}$ and consider the conjugacy class $29B$. If $Z \in V$ and pY is an arbitrary prime class of Ru, then by Table 1 and Table 3, there is no maximal subgroup of Ru that contains $(pY, 4Z, 29B)$-generated proper subgroups. Therefore, $\Delta^{\star}_{Ru}(pY, 4Z, 29B) = \Delta_{Ru}(pY, 4Z, 29B) > 0$, and so Ru is $(pY, 4Z, 29B)$-generated. This shows that the Rudvalis group Ru is $4Z$-complementary generated, for $Z \in V$. ∎

Table 2. The Structure Constants of the Group Ru

pX	$\Delta(2A,3A,pX)$	$\Delta(2A,3B,pX)$	$\Delta(2A,5A,pX)$	$\Delta(2A,5B,pX)$
$7A$	252	56	504	1911
$13A$	364	260	1456	2405
$29A$	203	29	551	1914
pX	$\Delta(2A,7A,pX)$	$\Delta(2A,13A,pX)$	$\Delta(2B,3A,pX)$	$\Delta(2B,3B,pX)$
$7A$	-	-	560	168
$13A$	19695	-	520	52
$29A$	21489	10904	609	232
pX	$\Delta(2B,5A,pX)$	$\Delta(2B,5B,pX)$	$\Delta(2B,7A,pX)$	$\Delta(2B,13A,pX)$
$7A$	1232	4144	-	-
$13A$	728	4680	45136	-
$29A$	1537	5191	44486	25955
pX	$\Delta(3A,5A,pX)$	$\Delta(3A,5B,pX)$	$\Delta(3A,7A,pX)$	$\Delta(3A,13A,pX)$
$7A$	67704	225036	-	-
$13A$	71656	226460	2414620	-
$29A$	67512	227679	2411669	1298997
pX	$\Delta(5A,7A,pX)$	$\Delta(5A,13A,pX)$	$\Delta(5B,7A,pX)$	$\Delta(5B,13A,pX)$
$13A$	5050552	-	17170439	-
$29A$	5212025	2850613	17375176	9502923
pX	$\Delta(7A,13A,pX)$	-	-	-
$29A$	100214894	-	-	-

Theorem 2.2. *The Rudvalis group Ru is pX-complementary generated, if p is an odd prime divisor of $|Ru|$.*

Proof. By Woldar's result, mentioned above, the group Ru is $29X$-complementary generated for $X \in \{A, B\}$. So, it is enough to investigate the prime divisors of $|Ru|$ distinct from 2 and 29. Set $\mathcal{A} = \{2A, 5A, 13A\}$ and consider the conjugacy class $29A$. Our main proof will consider a number of cases:

Case 1. Ru is 3A-complementary generated. If $pY \in \mathcal{A}$ then by Table 1 and Table 3, there is no maximal subgroup of Ru that contains $(pY, 3A, 29A)$-generated proper subgroups. Therefore, $\Delta^{\star}_{Ru}(pY, 3A, 29A) = \Delta_{Ru}(pY,\ 3A,\ 29A) > 0$, and so Ru is $(pY, 3A, 29A)$-generated. On the other hand, by Lemma 1.3, since Ru is $(2A, 3A, 29A)-$generated, it is $(3A, 3A, (29A)^2 = 29B)$-generated. We now assume that pY is a prime class of Ru, different from $2A, 3A, 5A$ and $13A$.

Table 3. The Partial Fusion Map of $L_2(29)$ into Ru

$L_2(29)$-class	2a	3a	5a	5b	7a	7b	7c	29a	29b
$\to Ru$	2B	3A	5B	5B	7A	7A	7A	29A	29B
h							6	1	1

By Table 3, $L_2(29)$ is the only class of maximal subgroups containing elements of order 29. Consider the triple $(2B, 3A, 29A)$. Thus $\Delta_{Ru}(2B, 3A, 29A) = 609$ and $\Sigma(L_2(29)) = 29$. From Table 3, we calculate further that $\Delta^{\star}(Ru) \geq 609 - 1(29) > 0$ and the generation of Ru by this triple follows. We now consider the triple $(5B, 3A, 29A)$. By Table 2, $\Delta_{Ru}(5B, 3A, 29A) = 227679$ and $\Sigma(L_2(29)) = 116$. From Table 3, we calculate further that $\Delta^{\star}(Ru) \geq 227679 - 1(116) > 0$ and the generation of Ru by this triple follows. For the conjugacy class 7A, using a similar argument as in above, we can see that $(7A, 3A, 29A)$ is a generating triple for Ru. On the other hand, there is no maximal subgroup containing the conjugacy classes $3A, 13A$ and $29A$. This shows that $\Delta^{\star}_{Ru}(13A, 3A, 29A) = \Delta_{Ru}(13A, 3A, 29A) > 0$.

For the cases $29A$ and $29B$, we apply the Woldar's result and the fact that the relation R introduced above is symmetric.

Case 2. Ru is pX-complementary generated, for $pX \in \{5A, 13A\}$. Using Table 3, we can see that there is no maximal subgroup containing the conjugacy classes $5A$ and $29A$ or $13A$ and $29A$. This shows that $\Delta^{\star}(Ru) = \Delta(Ru) > 0$. Thus, the Rudvalis group Ru is $5A$- and $13A$-complementary generated.

Case 3. Ru is 5B-complementary generated. If $pY \in \mathcal{A}$ then by Table 1 and Table 3, there is no maximal subgroup of Ru that contains $(pY, 5B, 29A)$-generated proper subgroups. Therefore, $\Delta^{\star}_{Ru}(pY, 5B, 29A) = \Delta_{Ru}(pY,\ 5B,\ 29A) > 0$, and so Ru is $(pY, 5B, 29A)$-generated. On the other hand, by Lemma 1.3, since Ru is $(2A, 5B, 29A)$-generated, it is $(5B, 5B, (29A)^2 = 29A)$-generated. Suppose that pY is an arbitrary prime class of Ru, different from $2A, 5A, 5B$ and $13A$. Amongst the maximal subgroups of Ru with order divisible by 29, the only maximal subgroups with non-empty intersection with any conjugacy class in this triple are isomorphic to $L_2(29)$. By tedious calculations, similar to those in Case 1, we can see that $\Delta^{\star}_{Ru}(pY, 5B, 29A) > 0$ and so Ru is $5B$-complementary generated.

Case 4. Ru is 7A-complementary generated. If $pY \in \mathcal{A}$ then by Table 1 and Table 3, there is no maximal subgroup of Ru that contains $(pY, 7A, 29A)$-generated proper subgroups. Therefore, $\Delta^\star_{Ru}(pY, 7A, 29A) = \Delta_{Ru}(pY,\ 7A,\ 29A) > 0$, and so Ru is $(pY, 7A, 29A)$-generated. On the other hand, by Lemma 1.2, since Ru is $(2A, 7A, 29A)$-generated, it is $(7A, 7A, (29A)^2 = 29A)$-generated. Suppose that pY is an arbitrary prime class of Ru, different from $2A, 5A, 7A$ and $13A$. Amongst the maximal subgroups of Ru with order divisible by 29, the only maximal subgroups with non-empty intersection with any conjugacy class in this triple are isomorphic to $L_2(29)$. Using a similar calculations, as in Case 1, we can see that $\Delta^\star_{Ru}(pY, 7A, 29A) > 0$ and so Ru is $7A$-complementary generated. This completes the proof. ∎

We are now ready to prove the main result of this paper:

Theorem. *The Rudvalis group Ru is nX-complementary generated if and only if $n \geq 3$.*

Proof. If nX is a conjugacy class of Ru with $n \notin \{1, 2\}$, then n is divisible by at least one of $3, 4, 5, 7, 13, 29$. So the result follows from Lemma 2.1, Theorem 2.2 and elementary considerations. ∎

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References

1. M. Aschbacher, Sporadic groups, Cambridge University Press, U. K., 1997.
2. A. R. Ashrafi, Generating pairs for the Held group He, *J. Appl. Math. & Computing* **10** (2002) 167–174.
3. A. R. Ashrafi anf G. A. Moghani, nX-Complementary generations of the Suzuki group Suz, *Bul. Acad. Stiinte Repub. Mold. Mat.* **40** (2002) 61–70.
4. A. R. Ashrafi, (p, q, r)-generations and nX-complementary generations of the Thompson group Th, *SUT J. Math.* **39**(2003) 41–54.
5. M. D. E. Conder, Some results on quotients of triangle groups, *Bull. Austral. Math. Soc.* **30** (1984) 73–90.
6. M. D. E. Conder, R. A. Wilson, and A. J. Woldar, The symmetric genus of sporadic groups, *Proc. Amer. Math. Soc.* **116** (1992) 653–663.
7. J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker, and R. A. Wilson, *Atlas of Finite Groups,* Oxford Univ. Press (Clarendon), Oxford, 1985.
8. M. R. Darafsheh and A. R. Ashrafi, $(2, p, q)$-Generation of the Conway group Co_1, *Kumamoto J. Math.* **13** (2000) 1–20.

9. M. R. Darafsheh and A. R. Ashrafi, Generating pairs for the sporadic group Ru, *J. Appl. Math & Computing* **12** (2003) 143–154.
10. M. R. Darafsheh, A. R. Ashrafi, and G. A. Moghani, (p, q, r)-Generations of the Conway group Co_1, for odd p, *Kumamoto J. Math.* **14** (2001) 1-20.
11. M. R. Darafsheh, A. R. Ashrafi, and G. A. Moghani, (p, q, r)-Generations and nX-Complementary generations of the sporadic group Ly, *Kumamoto J. Math.* **16** (2003) 13–25.
12. M. R. Darafsheh, A. R. Ashrafi, and G. A. Moghani, (p, q, r)-Generations of the sporadic group $O'N$, *Groups St. Andrews 2001 in Oxford,* Vol. 1, 101–109;*LMS lecture note series* Vol. **304**, Cambridge University Press, Cambridge, 2003.
13. M. R. Darafsheh, A. R. Ashrafi, and G. A. Moghani, (p, q, r)-Generations of the sporadic group $O'N$, *Southeast Asian Bull. Math.* **28** (2004) 1011–1019.
14. M. R. Darafsheh, G. A. Moghani, and A. R. Ashrafi, nX-Complementary generations of the Conway group Co_1, *Acta Math. Vietnam.* **29** (2004) 57–75.
15. M. R. Darafsheh and G. A. Moghani, nX-complementary Generations of the Sporadic Group He, *Italian J. Pure and Appl. Math.* **16** (2004) 145–154.
16. S. Ganief and J. Moori, (p, q, r)-Generations of the smallest Conway group Co_3, *J. Algebra* **188** (1997) 516–530.
17. S. Ganief and J. Moori, (p, q, r)-Generations and nX-complementary generations of the sporadic groups HS and McL, *J. Algebra* **188** (1997) 531–546.
18. S. Ganief and J. Moori, Generating pairs for the Conway groups Co_2 and Co_3, *J. Group Theory* **1** (1998) 237–256.
19. S. Ganief and J. Moori, 2-Generations of the Forth Janko Group J_4, *J. Algebra* **212** (1999) 305-322.
20. J. Moori, (p, q, r)-Generations for the Janko groups J_1 and J_2, *Nova J. Algebra and Geometry* **2** (1993) 277–285.
21. J. Moori, $(2, 3, p)$-Generations for the Fischer group F_{22}, *Comm. Algebra* **2** (1994) 4597–4610.
22. M. Schonert et al., GAP, Groups, Algorithms and Programming, Lehrstuhl D fur Mathematik, RWTH, Aachen, 1992.
23. A. J. Woldar, Representing M_{11}, M_{12}, M_{22} and M_{23} on surfaces of least genus, *Comm. Algebra* **18** (1990) 15–86.
24. A. J. Woldar, Sporadic simple groups which are Hurwitz, *J. Algebra* **144** (1991) 443–450.
25. A. J. Woldar, 3/2-Generation of the Sporadic simple groups, *Comm. Algebra* **22** (1994) 675–685.
26. A. J. Woldar, On Hurwitz generation and genus actions of sporadic groups, *Illinois Math. J.* **33** (1989) 416–437.