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Using Boundary-Operator Method for Approximate Solution of a Boundary Value Problem (BVP) for Triharmonic Equation*

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Abstract In this paper we propose and study an iterative method for solving a BVP for a triharmonic type equation. It is based on using a boundary-domain operator defined on pairs of boundary and domain functions in combination with parametric extrapolation technique. This method iteratively reduces the BVP for sixth order equation to a sequence of BVPs for Poisson equation.

1. Introduction

In earlier papers we developed the boundary operator method for constructing and investigating the convergence of a domain decomposition method for a BVP for second order elliptic equation with discontinuous coefficients [1], and an iterative method for the Dirichlet problem for the biharmonic type equation $\Delta^2 u - a\Delta u + bu = f$ when $a^2 - 4b \geq 0$ [2]. In the case if the latter condition is not satisfied, for example for the equation $\Delta^2 u + bu = f$ describing the bend of a plate on elastic foundation, the boundary operator method does not work. Therefore, for treating this case in [3] we have introduced boundary-domain operator defined on pairs of domain functions and boundary functions. With the help of this operator the BVP for biharmonic type equation is reduced to a

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sequence of BVPs for Poisson equation.

In this paper the boundary-domain operator method is used for constructing and studying an iterative method for the following BVP for triharmonic equation

$$\begin{aligned} &\Delta^3 u - au = f(x), \quad x \in \Omega, \\ &u|_\Gamma = 0, \quad \Delta u|_\Gamma = 0, \\ &\frac{\partial u}{\partial \nu}\Big|_\Gamma = 0, \end{aligned} \tag{1}$$

where Δ is the Laplace operator, Ω is a bounded domain in $R^n (n \geq 2)$, Γ is the sufficiently smooth boundary of Ω, ν is the outward normal to Γ and a is a positive number. The solvability and smoothness of the solution of problem (1) follows from the general theory of elliptic problems (see [5]), namely, if $f \in H^s(\Omega)$ then there exists a unique solution $u \in H^{s+6}(\Omega)$. Here, as usual, $H^s(\Omega)$ is Sobolev space.

2. Reduction of BVP to Boundary-Domain Operator Equation

We set

$$\Delta u = v, \Delta v = w$$

and

$$\varphi = au \tag{2}$$

and denote by w_0 the trace of w on Γ , i.e. $w_0 = w|_\Gamma$. Then from (1) we come to the sequence of problems

$$\begin{aligned} &\Delta w = f + \varphi, \quad x \in \Omega, \quad w|_\Gamma = w_0, \\ &\Delta v = w, \quad x \in \Omega, \quad v|_\Gamma = 0, \\ &\Delta u = v, \quad x \in \Omega, \quad u|_\Gamma = 0, \end{aligned} \tag{3}$$

where the functions φ and w_0 are temporarily undefined. The solution u from the above problems should satisfy the last condition in (1) and the relation (2), i.e.

$$au = \varphi, \quad \frac{\partial u}{\partial \nu}\Big|_\Gamma = 0. \tag{4}$$

Now, we introduce the operator B defined on pairs of boundary functions w_0 and domain functions φ

$$z = \begin{pmatrix} w_0 \\ \varphi \end{pmatrix}$$

by the formula

$$Bz = \begin{pmatrix} -a\frac{\partial u}{\partial \nu} \\ \varphi - au \end{pmatrix}, \tag{5}$$

where u is found from the sequence of problems

$$\begin{aligned} \Delta w = \varphi, \quad x \in \Omega, \quad w|_\Gamma = w_0, \\ \Delta v = w, \quad x \in \Omega, \quad v|_\Gamma = 0, \\ \Delta u = v, \quad x \in \Omega, \quad u|_\Gamma = 0. \end{aligned} \tag{6}$$

Notice that the operator B primarily defined on smooth functions is extended by continuity on the whole space $H = L_2(\Gamma) \times L_2(\Omega)$. Its properties will be investigated later.

Theorem 1.
a) *Suppose that* u *is the solution of the original Problem* (1) *and*

$$w_0 = \Delta^2 u|_\Gamma, \quad \varphi = au. \tag{7}$$

Then the pair of functions $z = (w_0, \varphi)^T$, *where* T *denotes transpose, satisfies the operator equation*

$$Bz = F, \tag{8}$$

where

$$F = \begin{pmatrix} a\frac{\partial u_2}{\partial \nu} \\ au_2 \end{pmatrix}, \tag{9}$$

u_2 *being determined from the problems*

$$\begin{aligned} \Delta w_2 = f, \quad x \in \Omega, \quad w_2|_\Gamma = 0, \\ \Delta v_2 = w_2, \quad x \in \Omega, \quad v_2|_\Gamma = 0, \\ \Delta u_2 = v_2, \quad x \in \Omega, \quad u_2|_\Gamma = 0. \end{aligned} \tag{10}$$

b) *Conversely, each pair of functions* $z = (w_0, \varphi)^T$, *which is the solution of the equation* (8) - (10) *uniquely defines a function* u *which is the solution of the Problem* (1) *such that the relation* (7) *is valid.*

Proof. The Part a) of the theorem is easy proved if after reducing the Problem (1) to the sequence of the problems (3) we represent

$$(u, v, w) = (u_1, v_1, w_1) + (u_2, v_2, w_2),$$

where u_1, v_1, w_1 satisfy the problems (6) and u_2, v_2, w_2 satisfy (10) and take into account the definition of the operator B.

For proving Part b) let u_1 be the solution of (6). Then by the definition of B we have

$$Bz = \begin{pmatrix} -a\frac{\partial u_1}{\partial \nu} \\ \varphi - au_1 \end{pmatrix}.$$

Take into account (9), from (8) we obtain

$$\frac{\partial(u_1 + u_2)}{\partial \nu} = 0, \quad \varphi - a(u_1 + u_2) = 0.$$

Now, it is easy to verify that the function $u = u_1 + u_2$ is the solution of Problem (1) and there holds the relation (7).
The theorem is proved. ∎

Now, let us study the properties of B in the space H with the scalar product

$$(z, \bar{z}) = (w_0, \bar{w}_0)_{L_2(\Gamma)} + (\varphi, \bar{\varphi})_{L_2(\Omega)}$$

for the elements $z = (w_0, \varphi)^T$ and $\bar{z} = (\bar{w}_0, \bar{\varphi})^T$.

Property 1. *B is symmetric and positive in H.*

Proof. For any functions z and $\bar{z}$ belonging to H we have

$$(Bz, \bar{z}) = \int_\Gamma -a\frac{\partial u}{\partial \nu}\bar{w}_0 d\Gamma + \int_\Omega (\varphi - au)\bar{\varphi} dx. \tag{11}$$

Taking into account the expression of Bz given by (5)-(6) and of $B\bar{z}$ by the same formula, where all the functions are marked with a bar over, we make transformations of the first intergral

$$\begin{aligned} J_1 &= \int_\Gamma -a\frac{\partial u}{\partial \nu}\bar{w}_0 d\Gamma = \int_\Gamma -a\frac{\partial u}{\partial \nu}\bar{w} d\Gamma = \int_\Gamma (au\frac{\partial \bar{w}}{\partial \nu} - a\frac{\partial u}{\partial \nu}\bar{w}) d\Gamma \\ &= a\int_\Omega (u\Delta\bar{w} - \bar{w}\Delta u) dx = a\int_\Omega (u\bar{\varphi} - v\Delta\bar{v}) dx \\ &= a\int_\Omega u\bar{\varphi} dx + a\int_\Omega \text{grad}v.\text{grad}\bar{v} dx. \end{aligned}$$

From here and (11) it follows that

$$(Bz, \bar{z}) = a\int_\Omega \text{grad}v.\text{grad}\bar{v} dx + \int_\Omega \varphi\bar{\varphi} dx = (B\bar{z}, z).$$

It means that B is symmetric in H.

Furthermore, we have

$$(Bz, z) = a\int_\Omega |\text{grad}v|^2 dx + \int_\Omega \varphi^2 dx \geq 0.$$

Therefore, $(Bz, z) = 0$ if and only if $\varphi = 0$ and $\text{grad}\, v = 0$. Since $v|_\Gamma = 0$ we have $v = 0$ in Ω. This implies $w_0 = 0$. Hence $z = 0$, and the positiveness of the operator B is proved.

Property 2. *B can be decomposed into the sum of a symmetric, positive, completely continuous operator and a projection operator, namely,*

$$B = B_0 + I_2, \tag{12}$$

where B_0 and I_2 are defined as follows

$$z = \begin{pmatrix} w_0 \\ \varphi \end{pmatrix}, \quad B_0 z = \begin{pmatrix} -a\frac{\partial u}{\partial \nu} \\ -au \end{pmatrix}, \quad I_2 z = \begin{pmatrix} 0 \\ \varphi \end{pmatrix}, \tag{13}$$

u being defined from (6).

The complete continuity of B_0 is easily followed from the embedding theorems of Sobolev spaces (see, e.g., [5]). The analogous technique was used in our earlier works [1, 3].

Property 3. *B is bounded in H.*

This fact is a direct corollary of Property 2.

Since $B = B^* > 0$ but is not completely continuous in H the use of two-layer iterative schemes to the equation (8) does not guarantee its convergence. Hence, in the next section we will disturb this equation and apply the parametric extrapolation method (see [1 - 4]) for constructing approximate solution for Problem (1).

3. Construction of Approximate Solution of BVP (1) Via a Perturbed Problem

We associate with the original problem (1) the following perturbed problem

$$\begin{aligned} \Delta^3 u_\delta - a u_\delta &= f(x), \quad x \in \Omega, \\ u_\delta|_\Gamma &= 0, \quad \Delta u_\delta|_\Gamma = 0, \\ \left(-a\frac{\partial u_\delta}{\partial \nu} + \delta \Delta^2 u_\delta\right)\Big|_\Gamma &= 0, \end{aligned} \tag{14}$$

where δ is a small parameter.

Theorem 2. *Suppose that $f \in H^{s-6}(\Omega)$, $s \geq 6$. Then for the solution of the problem* (14) *there holds the following asymptotic expansion*

$$u_\delta = u + \sum_{i=1}^{N} \delta^i y_i + \delta^{N+1} y_\delta, \quad x \in \Omega, \quad 0 \leqslant 3N \leqslant s - 5/2, \tag{15}$$

where $y_0 = u$ is the solution of (1), *y_i $(i = 1, ..., N)$ are functions independent of δ, $y_i \in H^{s-3i}(\Omega)$, $y_\delta \in H^{s-3N}(\Omega)$ and*

$$\|y_\delta\|_{H^2(\Omega)} \leqslant C_1, \tag{16}$$

C_1 being independent of δ.

Proof. Under the assumption of the theorem, by [5] there exists a unique solution $u \in H^s(\Omega)$ of the problem (14). After substituting (15) into (14) and balancing coefficients of like powers of δ we see that y_i and y_δ satisfy the following problems

$$\begin{aligned} \Delta^3 y_i - a y_i &= 0, \quad x \in \Omega, \\ y_i|_\Gamma &= 0, \quad \Delta y_i|_\Gamma = 0, \\ a\frac{\partial y_i}{\partial \nu}\Big|_\Gamma &= \Delta^2 y_{i-1}\Big|_\Gamma, \quad i = 1, ..., N, \end{aligned} \tag{17}$$

$$
\begin{aligned}
\Delta^3 y_\delta - a y_\delta = 0, \quad x \in \Omega, \\
y_\delta|_\Gamma = 0, \quad \Delta y_\delta|_\Gamma = 0, \\
\Big(-a\frac{\partial y_\delta}{\partial \nu} + \delta \Delta^2 y_\delta\Big)\Big|_\Gamma = -\Delta^2 y_N\Big|_\Gamma .
\end{aligned}
\tag{18}
$$

Once again, using [5] it is not difficult to establish successively that (17) has a unique solution $y_i \in H^{s-3i}(\Omega)$ and (18) has a unique solution $y_\delta \in H^{s-3N}(\Omega)$. Clearly, y_i $(i = 1, ..., N)$ do not depend on δ . It remains to estimate y_δ. For this purpose we reduce (18) to a boundary operator equation. We set

$$\Delta y_\delta = v_\delta,\ \Delta v_\delta = w_\delta$$

and

$$w_\delta|_\Gamma = w_{\delta 0}, \quad a y_\delta = \varphi_\delta.$$

Then we get

$$
\begin{aligned}
\Delta w_\delta = \varphi_\delta, \quad x \in \Omega, \quad w_\delta|_\Gamma = w_{\delta 0}, \\
\Delta v_\delta = w_\delta, \quad x \in \Omega, \quad v_\delta|_\Gamma = 0, \\
\Delta y_\delta = v_\delta, \quad x \in \Omega, \quad y_\delta|_\Gamma = 0.
\end{aligned}
\tag{19}
$$

It is easy to see that the pair of functions $z_\delta = (w_{\delta 0}, \varphi_\delta)^T$ satisfies the operator equation

$$B z_\delta + \delta I_1 z_\delta = h, \tag{20}$$

where

$$h = \begin{pmatrix} -\Delta^2 y_N\big|_\Gamma \\ 0 \end{pmatrix}, \quad I_1 z_\delta = \begin{pmatrix} w_{\delta 0} \\ 0 \end{pmatrix}.$$

Using Lemma 1 in Appendix we have

$$(B z_\delta, z_\delta) \leqslant (B z_0, z_0), \tag{21}$$

where z_0 is the solution of the equation $B z_0 = h$. This equation has a solution because it is the one that Problem (18) with $\delta = 0$ may be reduced to.

In Sec. 2, when investigating the properties of B we have established that

$$(B z_\delta, z_\delta) = a \int\limits_\Omega |\operatorname{grad} v_\delta|^2 dx + \int\limits_\Omega {\varphi_\delta}^2 dx. \tag{22}$$

In view of the Fridrichs inequality we have

$$\int\limits_\Omega |\operatorname{grad} v_\delta|^2 dx \geq C_2 \|v_\delta\|^2_{L_2(\Omega)}. \tag{23}$$

On the other hand, since y_δ satisfies the last problem in (19) there holds the estimate

$$\|y_\delta\|_{H^2(\Omega)} \leqslant C_3 \|v_\delta\|_{L_2(\Omega)}$$

From here and (23), (22) and (21) we obtain

$$\|y_\delta\|_{H^2(\Omega)} \leqslant C_1,$$

where $C_1 = \frac{C_3}{\sqrt{aC_2}}(Bz_0, z_0)^{1/2}$, C_2, C_3 and z_0 being independent of δ. Thus, the theorem is proved. ■

As usual (see [1 - 4]), we construct an approximate solution of the original problem (1) by the formula

$$U^E = \sum_{i=1}^{N+1} \gamma_i u_{\delta/i}, \tag{24}$$

where

$$\gamma_i = \frac{(-1)^{N+1-i} i^{N+1}}{i!(N+1-i)!},$$

$u_{\delta/i}$ is the solution of (14) with the parameter δ/i $(i = 1, ..., N+1)$. Then, it is easy to obtain the following

Theorem 3. *For the approximate solution in the form* (24) *for the original Problem* (1) *there holds the estimate*

$$\|U^E - u\|_{H^2(\Omega)} \leqslant C\delta^{N+1},$$

where u is the solution of (1), *C is a constant independent of δ.*

4. Iterative Method for Solving Problem (14)

First we notice that in the same way as for the original Problem (1), Problem (14) may be reduced to the operator equation

$$B_\delta z_\delta = F, \tag{25}$$

for $z_\delta = (w_{\delta 0}, \varphi_\delta)^T$, where $w_{\delta 0} = \Delta^2 u_\delta|_\Gamma$, $\varphi_\delta = au_\delta, B_\delta = B + \delta I_1$, B and F are defined as in Theorem 1. Clearly, B_δ is bounded and

$$B_\delta = B_\delta^* \geq \delta I, \tag{26}$$

where I is the identity operator.

For solving (25) we can apply the general theory of two-layer iterative scheme for equation with symmetric, positive definite operator [6]. Namely, we consider the iterative scheme

$$\frac{z_\delta^{(k+1)} - z_\delta^{(k)}}{\tau_{\delta,k+1}} + B_\delta z_\delta^{(k)} = F, \tag{27}$$

where $\tau_{\delta,k+1}$ is the Chebyshev collection of parameters according to bounds $\gamma_\delta^{(1)} = \delta$, $\gamma_\delta^{(2)} = \delta + \|B\|$ (see [6] for detail). In the case of simple iteration

$$\tau_{\delta,k} \equiv \tau_{\delta,0} = \frac{2}{\gamma_\delta^{(1)} + \gamma_\delta^{(2)}}$$

we get

$$\|z_\delta^{(k)} - z_\delta\|_H \leqslant (\rho_\delta)^k \|z_\delta^{(0)} - z_\delta\|_H, \tag{28}$$

where

$$\rho_\delta = \frac{1-\xi_\delta}{1+\xi_\delta},\ \xi_\delta = \frac{\gamma_\delta^{(1)}}{\gamma_\delta^{(2)}}$$

and as above $H = L_2(\Gamma) \times L_2(\Omega)$.

The iterative scheme (27) may be realized by the following algorithm

(i) Given a start approximation $z_\delta^{(0)} = (w_{\delta 0}^{(0)}, \varphi_\delta^{(0)})^T$.

(ii) Knowing $z_\delta^{(k)} = (w_{\delta 0}^{(k)}, \varphi_\delta^{(k)})^T,\ (k = 0, 1, ...)$, solve successively three problems

$$\begin{aligned}
\Delta w_\delta^{(k)} &= f + \varphi_\delta^{(k)}, \quad x \in \Omega, \quad w_\delta^{(k)}|_\Gamma = w_{\delta 0}^{(k)},\\
\Delta v_\delta^{(k)} &= w_\delta^{(k)}, \quad x \in \Omega, \quad v_\delta^{(k)}|_\Gamma = 0,\\
\Delta u_\delta^{(k)} &= v_\delta^{(k)}, \quad x \in \Omega, \quad u_\delta^{(k)}|_\Gamma = 0.
\end{aligned} \tag{29}$$

iii) Compute the new approximation of $z_\delta^{(k+1)} = (w_{\delta 0}^{(k+1)}, \varphi_\delta^{(k+1)})^T$

$$\begin{aligned}
w_{\delta 0}^{(k+1)} &= w_{\delta 0}^{(k)} + \tau_{\delta, k+1}\Big(a \frac{\partial u_\delta^{(k)}}{\partial \nu}\Big|_\Gamma - \delta w_{\delta 0}^{(k)}\Big), \quad x \in \Gamma,\\
\varphi_\delta^{(k+1)} &= \varphi_\delta^{(k)} + \tau_{\delta, k+1}(a u_\delta^{(k)} - \varphi_\delta^{(k)}), \quad x \in \Omega.
\end{aligned}$$

Using estimates for the solution of elliptic problems [5] and taking into account (28) we get the estimate

$$\|u_\delta^{(k)} - u_\delta\|_{H^{5/2}(\Omega)} \leqslant C(\rho_\delta)^k \|z_\delta^{(0)} - z_\delta\|_H,$$

where C is a constant independent of δ and $z_\delta = (\Delta^2 u_\delta|_\Gamma, a u_\delta)^T$ as was mentioned in the beginning of the section.

Appendix

Lemma 1. *Suppose A is a linear, symmetric and positive operator, P is a nonnegative operator in a Hilbert space H with the scalar product $(.,.)$. Let u_δ and u_0 be the solutions of the equations*

$$Au_\delta + \delta P u_\delta = h, \quad \delta > 0, \tag{30}$$

$$Au_0 = h. \tag{31}$$

Then there holds the estimate

$$(Au_\delta, u_\delta) \leqslant (Au_0, u_0). \tag{32}$$

Proof. From (30) and (31) we have

$$Au_\delta + \delta P u_\delta = Au_0.$$

Scalarly multiplying both sides of the above equality by u_δ and taking into account the nonnegativeness of P we get

$$(Au_\delta, u_\delta) \leqslant (Au_0, u_\delta). \tag{33}$$

From the inequality

$$(A(u_\delta - u_0), u_\delta - u_0) = (Au_\delta, u_\delta) - 2(Au_0, u_\delta) + (Au_0, u_0) \geqslant 0$$

we have

$$2(Au_0, u_\delta) \leqslant (Au_\delta, u_\delta) + (Au_0, u_0).$$

Now, from the above inequality and (33) the estimate (32) follows and the lemma is proved.

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