On an Invariant-Theoretic Description of the Lambda Algebra

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Dedicated to Professor Huỳnh Mùi on the occasion of his sixtieth birthday

Abstract The purpose of this paper is to give a mod-$p$ analogue of the Lomonaco invariant-theoretic description of the lambda algebra for $p$ an odd prime. More precisely, using modular invariants of the general linear group $GL_n = GL(n, \mathbb{F}_p)$ and its Borel subgroup $B_n$, we construct a differential algebra $Q_-$ which is isomorphic to the lambda algebra $\Lambda = \Lambda_p$.

Introduction

For the last few decades, the modular invariant theory has been playing an important role in stable homotopy theory. Singer [9] gave an interpretation for the dual of the lambda algebra $\Lambda_p$, which was introduced by the six authors [1], in terms of modular invariant theory of the general linear group at the prime $p = 2$. In [8], Hung and the author gave a mod-$p$ analogue of the Singer invariant-theoretic description of the dual of the lambda algebra for $p$ an odd prime. Lomonaco [6] also gave an interpretation for the lambda algebra in terms of modular invariant theory of the Borel subgroup of the general linear group at $p = 2$.

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The purpose of this paper is to give a mod-$p$ analogue of the Lomonaco invariant-theoretic description of the lambda algebra for $p$ an odd prime. More precisely, using modular invariants of the general linear group $GL_n = GL(n, F_p)$ and its Borel subgroup $B_n$, we construct a differential algebra $Q_-$ which is isomorphic to the lambda algebra $\Lambda = \Lambda_p$. Here and in what follows, $F_p$ denotes the prime field of $p$ elements. Recall that, $\Lambda_p$ is the $E_1$-term of the Adams spectral sequence of spheres for $p$ an odd prime, whose $E_2$-term is $\text{Ext}^*_{\mathcal{A}(p)}(F_p, F_p)$ where $\mathcal{A}(p)$ denotes the mod $p$ Steenrod algebra, and $E_\infty$-term is a graded algebra associated to the $p$-primary components of the stable homotopy of spheres.

It should be noted that the idea for the invariant-theoretic description of the lambda algebra is due to Lomonaco, who realizes it for $p = 2$ in [6]. In this paper, we develop his work for $p$ any odd prime. Our main contributions are the computations at odd degrees, where the behavior of the lambda algebra is completely different from that for $p = 2$.

The paper contains 4 sections. Sec. 1 is a preliminary on the modular invariant theory and its localization. In Sec. 2 we construct the differential algebra $Q_-$ by using modular invariant theory and show that $Q_-$ can be presented by a set of generators and some relations on them. In Sec. 3 we recall some results on the lambda algebra and show that it is isomorphic to a differential subalgebra $Q_-$ of $Q$. Finally, in Sec. 4 we give an $F_p$-vector space basis for $Q_-$.

1. Preliminaries on the Invariant Theory

For an odd prime $p$, let $E_n$ be an elementary abelian $p$-group of rank $n$, and let

$$H^*(BE^n) = E(x_1, x_2, \ldots, x_n) \otimes F_p(y_1, y_2, \ldots, y_n)$$

be the mod-$p$ cohomology ring of $E^n$. It is a tensor product of an exterior algebra on generators $x_i$ of dimension 1 with a polynomial algebra on generators $y_i$ of dimension 2. Here and throughout the paper, the coefficients are taken over the prime field $F_p$ of $p$ elements.

Let $GL_n = GL(n, F_p)$ and $B_n$ be its Borel subgroup consisting of all invertible upper triangular matrices. These groups act naturally on $H^*(BE^n)$. Let $S$ be the multiplicative subset of $H^*(BE^n)$ generated by all elements of dimension 2 and let

$$\Phi_n = H^*(BE^n)_S$$

be the localization of $H^*(BE^n)$ obtained by inverting all elements of $S$. The action of $GL_n$ on $H^*(BE^n)$ extends to an action of its on $\Phi_n$. We recall here some results on the invariant rings $\Gamma_n = \Phi_{G_n}$ and $\Delta_n = \Phi_{B_n}$.

Let $L_{k,s}$ and $M_{k,s}$ denote the following graded determinants (in the sense of Mui [3])
for $0 \leq s \leq k \leq n$ and $M_{k,k} = 0$. We set $L_k = L_{k,k}, 1 \leq k \leq n, L_0 = 1$. Recall that $L_k$ is invertible in $\Phi_n$.

As is well known $L_{k,s}$ is divisible by $L_k$. Dickson invariants $Q_{k,s}$ and Mui invariants $R_{k,s}, V_k, 0 \leq s \leq k$, are defined by

$$Q_{k,s} = \frac{L_{k,s}}{L_k}, \quad R_{k,s} = M_{k,s}L_k^{p-2}, \quad V_k = L_k/L_{k-1}.$$  

Note that $\dim Q_{k,s} = 2(p^k - p^s), \quad \dim R_{k,s} = 2(p^k - p^s) - 1, \quad \dim V_k = 2p^{k-1}, \quad Q_{k,0} = L_k^{p-1}, \quad L_k = V_kV_{k-1} \ldots V_2V_1$.

From the results in Dickson [2] and Mui [3, 4.17] we observe

**Theorem 1.1.** (see Singer [9])

$$\Gamma_n = E(R_{n,0}, R_{n,1}, \ldots, R_{n,n-1}) \otimes \mathbb{F}_p(Q_{n,0}^{\pm 1}, Q_{n,1}, \ldots, Q_{n,n-1}).$$

Following Li–Singer [7], we set

$$N_k = M_{k,k-1}L_k^{p-2}, \quad W_k = V_k^{p-1}, 1 \leq k \leq n.$$  

Then we have

**Theorem 1.2.** (see Li–Singer [7])

$$\Delta_n = E(N_1, N_2, \ldots, N_n) \otimes \mathbb{F}_p(W_1^{\pm 1}, W_2^{\pm 1}, \ldots, W_n^{\pm 1}).$$

For latter use, we set

$$t_k = N_k/Q_{k-1,0}^{p-1}, \quad w_k = W_k/Q_{k-1,0}^{p-1}, 1 \leq k \leq n.$$  

Observe that $\dim t_k = 2p - 3, \quad \dim w_k = 2p - 2$. From Theorem 1.2 we obtain
Corollary 1.3.

\[ \Delta_n = E(t_1, t_2, \ldots, t_n) \otimes \mathbb{F}_p(w_1^{\pm 1}, w_2^{\pm 1}, \ldots, w_n^{\pm 1}). \]

Moreover, from Dickson [2], Mui [3], we have

**Proposition 1.4.**

(i) \( Q_{n,s} = Q_{n-1,s-1}^p + Q_{n-1,0}^{p-1}Q_{n-1,s}w_n, \)

(ii) \( R_{n,s} = Q_{n-1,0}^{p-1}(R_{n-1,s}w_n + Q_{n-1,s}^t n). \)

2. The Algebra \( Q \)

In this section, we construct the differential algebra \( Q \) by using modular invariant theory. In Sec. 4, we will show that the lambda algebra is isomorphic to a subalgebra of \( Q \).

**Definition 2.1.** Let \( \Delta_n \) be as in Sec. 1. Set

\[ \Delta = \bigoplus_{n \geq 0} \Delta_n. \]

Here, by convention, \( \Delta_0 = \mathbb{F}_p. \) This is a direct sum of vector spaces over \( \mathbb{F}_p. \)

**Remark.** For \( I = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n, i_1, i_2, \ldots, i_n) \) with \( \varepsilon_j = 0, 1, i_j \in \mathbb{Z}, \) set

\[ w^I = t_1^{\varepsilon_1}t_2^{\varepsilon_2} \cdots t_n^{\varepsilon_n}w_1^{i_1+\varepsilon_1}w_2^{i_2+\varepsilon_2} \cdots w_n^{i_n+\varepsilon_n}, \]

even in the case when some of \( \varepsilon_j \) or \( i_j \) are zero. For example, the element \( t_1 \in \Delta_2 \) will be written as \( t_1t_2^0w_1^0w_2^0, \) to be distinguished from \( t_1 \in \Delta_1, \) since \( t_1 \neq t_1t_2^0w_1^0w_2^0. \) For any \( n > 0 \) we have a monomial

\[ t_1^0t_2^0 \ldots t_n^0w_1^0w_2^0 \ldots w_n^0 \in \Delta_n \]

which is the identity of \( \Delta_n. \) All these elements are distinct in \( \Delta. \)

Now we equip \( \Delta \) with an algebra structure as follows. For any non-negative integers \( k, \ell, \) we define an isomorphism of algebras

\[ \mu_{k,\ell} : \Delta_k \otimes \Delta_{\ell} \to \Delta_{k+\ell} \]

by setting

\[ \mu_{k,\ell}(t_1^{e_1}t_2^{e_2} \cdots t_k^{e_k}w_1^{i_1+\varepsilon_1}w_2^{i_2+\varepsilon_2} \cdots w_k^{i_k+\varepsilon_k} \otimes t_1^{\sigma_1}t_2^{\sigma_2} \cdots t_{\ell}^{\sigma_{\ell}}w_1^{j_1+\varepsilon_1}w_2^{j_2+\varepsilon_2} \cdots w_{\ell}^{j_{\ell}+\varepsilon_{\ell}}) = t_1^{e_1}t_2^{e_2} \cdots t_k^{e_k}t_{k+1}^{\sigma_1}t_{k+2}^{\sigma_2} \cdots t_{k+\ell}^{\sigma_{\ell}}w_1^{i_1+\varepsilon_1}w_2^{i_2+\varepsilon_2} \cdots w_{k+\ell}^{i_k+\varepsilon_k} \]

for any \( i_1, i_2, \ldots, i_k, j_1, j_2, \ldots, j_{\ell} \in \mathbb{Z}, \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k, \sigma_1, \sigma_2, \ldots, \sigma_{\ell} = 0, 1. \)

We assemble \( \mu_{k,\ell}, \) \( k, \ell \geq 0, \) to obtain a multiplication

\[ \mu : \Delta \otimes \Delta \to \Delta. \]

This multiplication makes \( \Delta \) into an algebra.

For simplicity, we denote \( \mu(x \otimes y) = x \ast y \) for any elements \( x, y \in \Delta. \)
Definition 2.2. Let $\Gamma$ denote the two-sided ideal of $\Delta$ generated by all elements of the forms

$$
t_1^0t_2^0w_1^{-1}w_2^0Q_{2,0}^aQ_{2,1}^b,
$$

$$
t_1^0t_2^0w_1^{-1}w_2^0R_{2,0}Q_{2,0}^aQ_{2,1}^b - R_{2,1}Q_{2,0}^aQ_{2,1}^b,
$$

$$
t_1^0t_2^0w_1^{-1}w_2^0R_{2,1}Q_{2,0}^aQ_{2,1}^b - R_{2,0}Q_{2,0}^aQ_{2,1}^b,
$$

$$
t_1^0t_2^0w_1^{-1}w_2^0R_{2,1}Q_{2,0}^aQ_{2,1}^b,
$$

where $a, b \in \mathbb{Z}$, $b \geq 0$.

We define $Q = \Delta/\Gamma$ to be the quotient of $\Delta$ by the ideal $\Gamma$.

For any non-negative integer $n$, we define a homomorphism

$$
\tilde{\delta}_n : \Delta_n \rightarrow \Delta_{n+1}
$$

by setting

$$
\tilde{\delta}_n(x) = -t_1w_1^{-1} * x + (-1)^{1 \dim x} t_1w_1^{-1},
$$

for any homogeneous element $x \in \Delta_n$. By assembling $\tilde{\delta}_n, n \geq 0$, we obtain an endomorphism

$$
\tilde{\delta} : \Delta \rightarrow \Delta.
$$

Theorem 2.3. The endomorphism $\tilde{\delta} : \Delta \rightarrow \Delta$ induces an endomorphism $\delta : Q \rightarrow Q$ which is a differential.

Proof. Let $u \in \Delta_n$ be a homogeneous element and suppose $u \in \Gamma$. From the definition of $\Gamma$ we see that $u$ is a sum of elements of the form

$$
u_i * s_i * z_i,
$$

where $u_i \in \Delta_{n_i}$, $z_i \in \Delta_{n_i-2}$ and $s_i$ is one of the elements given in Definition 2.2. Then $\tilde{\delta}(u)$ is a sum of elements of the form

$$
-t_1w_1^{-1} * u_i * s_i * z_i + (-1)^{1 \dim u_i} u_i * s_i * z_i * t_1w_1^{-1}.
$$

Since $t_1w_1^{-1} * u_i \in \Delta_{n_i+1}$, $z_i * t_1w_1^{-1} \in \Delta_{n_i-1}$, we obtain $\tilde{\delta}(u) \in \Gamma$. So, $\tilde{\delta}$ induces an endomorphism

$$
\delta : Q \rightarrow Q.
$$

Now we prove that $\delta \tilde{\delta} = 0$. It suffices to check that if $x \in \Delta_n$ is a homogeneous element then $\delta \tilde{\delta}(x) \in \Gamma$. In fact, from the definition of $\tilde{\delta}$ we have

$$
\tilde{\delta}(x) = t_1t_2w_1^{-1}w_2^{-1} * x - x * t_1t_2w_1^{-1}w_2^{-1}.
$$

A direct computation using Proposition 1.4 shows that

$$R_{2,0}Q_{2,0}^{-1} = t_1^0t_2^0w_1^{-1}w_2^{-1},
$$

$$R_{2,1}Q_{2,0}^{-1} = t_1^0t_2^0w_1^{-1}w_2^{-1}.$$
From these, we have
\[ t_1^0 w_1^0 w_2^0 R_{2,0} R_{2,1} Q_{2,0}^{-2} = t_1 t_2 w_1^{-1} w_2^{-1}. \]

Hence we obtain
\[ \bar{\delta}(x) = t_1^0 w_1^0 w_2^0 R_{2,0} R_{2,1} Q_{2,0}^{-2} * x - x * t_1^0 w_1^0 w_2^0 R_{2,0} R_{2,1} Q_{2,0}^{-2} \in \Gamma. \]

The theorem is proved. \[\Box\]

Now we give a new system of generators for \( Q \).

Let \( T \) be the free associative algebra over \( \mathbb{F}_p \) generated by \( x_{i+1} \) of degree \( 2(p-1)i - 1 \) and \( y_{i+1} \) of degree \( 2(p-1)i \), for any \( i \in \mathbb{Z} \).

It is easy to see that there exists a unique derivation \( D : T \to T \) satisfying
\[ D(x_i) = x_{i-1}, \quad D(y_i) = y_{i-1}, \quad i \in \mathbb{Z}. \]

(Recall that \( D \) is called a derivation if \( D(uv) = D(u)v + uD(v) \), for any \( u, v \in T \).)

Denote by \( D^n = D \circ D \circ \ldots \circ D \) the composite of \( n \)-copies of \( D \).

For simplicity, we set
\[ x_i^\varepsilon = \begin{cases} x_i, & \varepsilon = 1 \\ y_i, & \varepsilon = 0. \end{cases} \]

By induction on \( n \) we easily obtain

**Lemma 2.4.** Under the above notation, we have
\[ D^n(x_i^\varepsilon, x_j^\varepsilon) = \sum_{k=0}^{n} \binom{n}{k} x_i^\varepsilon_{1-k} x_j^\varepsilon_{2-n+k}. \]

Here \( \binom{n}{k} \) denotes the binomial coefficient.

We define a homomorphism of algebras \( \pi : T \to Q \) by setting
\[ \pi(x_{i+1}) = t_1 w_1^{-1}, \quad \pi(y_{i+1}) = t_1^0 w_1^1, \quad i \in \mathbb{Z}. \]

That means \( \pi(x_i^\varepsilon) = t_1^\varepsilon w_1^{i-\varepsilon} \) for any \( i \in \mathbb{Z}, \varepsilon = 0, 1 \).

**Proposition 2.5.** The homomorphism \( \pi : T \to Q \) is an epimorphism. Its kernel is the two-sided ideal of \( T \) generated by all elements of the forms
\[ D^n(y_{pi} y_{i+1}), \quad D^n(x_{pi} y_{i+1}), \quad D^n(y_{pi+1} x_{i+1} - x_{pi+1} y_{i+1}), \quad D^n(x_{pi+1} x_{i+1}), \]

with \( n \geq 0, i \in \mathbb{Z} \).

**Proof.** It is easy to see that \( \pi \) is an epimorphism. Now we prove the remaining part of the proposition.
By a direct computation we obtain

\[ Q_{2,0}^a Q_{2,1}^b = \sum_{k=0}^{b} \binom{b}{k} t_1^0 t_2^0 u_1^{p(a) - b + k} w_2^{a + b - k} \]

\[ R_{2,0} Q_{2,0}^a Q_{2,1}^b = \sum_{k=0}^{b} \binom{b}{k} t_1 t_2 u_1^{p(a + b + 1) - b + k - 1} w_2^{a + b + 1 - k} + \sum_{k=0}^{b} \binom{b}{k} t_1^0 t_2^0 u_1^{p(a + b + 1) - b + k} w_2^{a + b - k} \]

\[ R_{2,1} Q_{2,0}^a Q_{2,1}^b = \sum_{k=0}^{b} \binom{b}{k} t_1^0 t_2^0 u_1^{p(a + b + 1) - b + k - 1} w_2^{a + b - k} \]

\[ R_{2,0} R_{2,1} Q_{2,0}^a Q_{2,1}^b = \sum_{k=0}^{b} \binom{b}{k} t_1 t_2 u_1^{p(a + b + 2) - b + k - 2} w_2^{a + b + 1 - k} \]

Using Lemma 2.4 and the definition of \( \pi \) we have

\[ \pi(D^n(y_{pi} y_{i+1})) = \pi\left( \sum_{k=0}^{n} \binom{n}{k} y_{pi - n + k} y_{i + 1 - k} \right) \]

\[ = \sum_{k=0}^{n} \binom{n}{k} t_1^0 t_2^0 w_1^{pi - n + k} w_2^{j - k} \]

\[ = t_1^0 t_2^0 w_1^{pi - n + k} w_2^{j - k} \sum_{k=0}^{n} \binom{n}{k} t_1^0 t_2^0 w_1^{pi - n + k} w_2^{j - k} \]

\[ = t_1^0 t_2^0 w_1^{pi - n + k} w_2^{j - k} Q_{2,0}^n Q_{2,1}^n \]

\[ = 0 \text{ in } Q. \]

By an argument analogous to the previous one, we get

\[ \pi(D^n(x_{pi} y_{i+1})) = t_1^0 t_2^0 w_1^{pi - n + k} w_2^{j - k} Q_{2,0}^n Q_{2,1}^n - R_{2,0} Q_{2,0}^{i - n - 1} Q_{2,1}^n = 0 \text{ in } Q \]

\[ \pi(D^n(y_{pi+1} x_{i+1} - x_{pi+1} y_{i+1})) = (2 R_{2,0} Q_{2,0}^{i - n - 1} Q_{2,1}^n - R_{2,0} Q_{2,0}^{i - n - 1} Q_{2,1}^n) = 0 \text{ in } Q \]

\[ \pi(D^n(x_{pi+1} x_{i+1})) = -t_1^0 t_2^0 w_1^{pi + 1} w_2^{j - k} R_{2,0} R_{2,1} Q_{2,0}^{i - n - 2} Q_{2,1}^n = 0 \text{ in } Q. \]

From these and the definition of \( \Gamma \) we obtain the proposition. \( \blacksquare \)

### 3. The Lambda Algebra and the Modular Invariant Theory

In this section, we show that the lambda algebra, which is introduced by the six authors of [1], is isomorphic to a subalgebra of \( Q \).

Let \( \Lambda \) denote the graded free associative algebra over \( \mathbb{F}_p \) with generators \( \lambda_{i-1} \) of dimension \(-2(p - 1)i + 1\) and \( \mu_{i-1} \) of dimension \(-2(p - i), i \geq 0\), subject to
the relations:

\[
\sum_{k=0}^{n} \binom{n}{k} \lambda_{k+p_1-1} \lambda_{i+n-k-1} = 0 \tag{1}
\]
\[
\sum_{k=0}^{n} \binom{n}{k} (\mu_{k+p_1-1} \lambda_{i+n-k-1} - \lambda_{k+p_1-1} \mu_{i+n-k-1}) = 0 \tag{2}
\]
\[
\sum_{k=0}^{n} \binom{n}{k} \lambda_{k+p_1} \mu_{n-k-1} = 0 \tag{3}
\]
\[
\sum_{k=0}^{n} \binom{n}{k} \mu_{k+p_1} \mu_{i+n-k-1} = 0 \tag{4}
\]

for \( i,n \geq 0 \). By \( \Lambda \) we mean the subalgebra of \( \bar{\Lambda} \) generated by \( \lambda_{i-1}, \lambda_i, \mu_{i-1}, \mu_i \geq 0 \).

We note that this definition is the same as that given in [1], but we are writing the product in the order opposite to that used in [1].

For simplicity, we denote \( \lambda_i^{\varepsilon} = \begin{cases} \lambda_i, & \varepsilon = 1 \\ \mu_i, & \varepsilon = 0 \end{cases} \) for any \( i \geq -1 \). We set

\[
\lambda(\varepsilon_1, \varepsilon_2, i, n) = \sum_{k=0}^{n} \binom{n}{k} (\lambda_{k+p_1-1}^{\varepsilon_1} \lambda_{i+n-k-1}^{\varepsilon_2} - \varepsilon_2 (1 - \varepsilon_1) \lambda_{k+p_1-1}^{\varepsilon_2} \lambda_{i+n-k-1}^{\varepsilon_1}),
\]

for any \( \varepsilon_1, \varepsilon_2, i, n \) with \( \varepsilon_1, \varepsilon_2 = 0, 1 \) and \( i,n \geq 0 \). Then the defining relations (1) - (4) become

\[
\lambda(\varepsilon_1, \varepsilon_2, i, n) = 0. \tag{5}
\]

Then we can consider \( \Lambda \) as the free graded associative algebra over \( \mathbb{F}_p \) with generators \( \lambda_i^{\varepsilon}, i \geq -1 \), subject to the relation (5) with \( i \geq -\varepsilon_1 \).

**Definition 3.1.** A sequence \( I = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n, i_1, i_2, \ldots, i_n) \), \( \varepsilon_j = 0, 1, i_j \geq 0 \), is said to be admissible if

\[
pi_j \geq i_{j+1} + \varepsilon_j, \ 1 \leq j < n, \text{ and } i_n \geq \varepsilon_n.
\]

In this case, the associated monomial \( \lambda_I = \lambda_{i_1}^{\varepsilon_1} \lambda_{i_2}^{\varepsilon_2} \ldots \lambda_{i_n}^{\varepsilon_n} \) is also said to be admissible.

**Theorem 3.2.** (Bousfield et al. [1]) The admissible monomials form an additive basis for \( \Lambda \).

**Definition 3.3.** The homomorphism \( \bar{d} : \bar{\Lambda} \rightarrow \bar{\Lambda} \) is defined by

\[
\bar{d}(x) = -\lambda_{-1} x + (-1)^{\dim x} x \lambda_{-1},
\]
for any homogeneous element $x \in \Lambda$.

In $\bar{\Lambda}$, we have $\lambda_{-1} \lambda_{-1} = 0$, hence $\bar{d}d = 0$. So $\bar{d}$ is a differential on $\bar{\Lambda}$. From the defining relations (1)-(4) we obtain

\begin{align*}
\bar{d}(\lambda_0) &= 0, \\
\bar{d}(\mu_{-1}) &= 0, \\
\bar{d}(\lambda_{n-1}) &= \sum_{k=1}^{n-1} \binom{n}{k} \lambda_{k-1} \lambda_{n-k-1}, \\
\bar{d}(\mu_{n-1}) &= \lambda_{n-1} \mu_{-1} + \sum_{k=1}^{n-1} \binom{n}{k} \left( \lambda_{k-1} \mu_{n-k-1} - \mu_{k-1} \lambda_{n-k-1} \right) - \mu_{-1} \lambda_{n-1},
\end{align*}

for any $n \geq 2$. From these, we obtain $\bar{d}(\lambda_{n-1}^\varepsilon) \in \Lambda, n \geq \varepsilon$, so $\bar{d}$ passes to a differential $d$ on $\Lambda$.

Now we describe the algebra $\Lambda$ in terms of modular invariants.

**Definition 3.4.** We define $Q_-$ to be the subalgebra of $Q$ generated by all elements $x_{i+1}^\varepsilon$ with $i \leq -\varepsilon$.

For any $\varepsilon_1, \varepsilon_2 = 0, 1$, $n \geq 0$, $i \in \mathbb{Z}$, we set

$$x(\varepsilon_1, \varepsilon_2, i, n) = D^n(x_{p_1+\varepsilon_2}^{\varepsilon_2} x_{i+1}^{\varepsilon_1} - \varepsilon_2 (1 - \varepsilon_1) x_{p_1+\varepsilon_2}^{\varepsilon_1} x_{i+1}^{\varepsilon_1}).$$

Then the defining relations of $Q$ become

$$x(\varepsilon_1, \varepsilon_2, i, n) = 0.$$  \hfill (6)

So we can consider $Q_-$ as the free graded associative algebra over $\mathbb{F}_p$ with generators $x_{i+1}^\varepsilon$, $i \leq -\varepsilon$, subject to the relation (6) with $i \leq -\varepsilon_1$.

**Theorem 3.5.** As a graded differential algebra, $\Lambda$ is isomorphic to $Q_-$. 

**Proof.** We define a homomorphism of algebras

$$\Phi : \Lambda \to Q_-$$

by setting

$$\Phi(\lambda_{i-1}^\varepsilon) = x_{-i+1}^\varepsilon,$$

for any $i \geq -\varepsilon$. From the definition of $Q_-$ we easily obtain

$$\Phi(\lambda(\varepsilon_1, \varepsilon_2, i, n)) = x(\varepsilon_1, \varepsilon_2, -i, n)$$

for any $\varepsilon_1, \varepsilon_2 = 0, 1, i, n \geq 0, i \geq \varepsilon_1$. Hence, the homomorphism $\Phi$ is well defined.

Now we define a homomorphism of algebras

$$\Psi : Q_- \to \Lambda,$$

by setting $\Psi(x_{i+1}^\varepsilon) = \lambda_{i-1}^\varepsilon$, for any $i \leq -\varepsilon$. It is easy to check that

$$\Psi(x(\varepsilon_1, \varepsilon_2, i, n)) = \lambda(\varepsilon_1, \varepsilon_2, -i, n),$$
for any $\varepsilon_1, \varepsilon_2 = 0, 1, n \geq 0, i \leq -\varepsilon_1$. So, the homomorphism $\Psi$ is well defined. Obviously, we have

$$\Phi \circ \Psi = 1_{Q_-}, \quad \Psi \circ \Phi = 1_{\Lambda}.$$ 

Hence $\Phi$ is an isomorphism of algebras.

Finally we prove that $\Phi$ preserves the differential structure. We have

$$\Phi(\delta(\lambda_{n-1})) = \Phi\left(\sum_{k=1}^{n-1} \binom{n}{k} \lambda_{k-1} \lambda_{n-k-1}\right)$$

$$= \sum_{k=1}^{n-1} \binom{n}{k} x_{-k+1} x_{k-n+1}$$

$$= d(x_{-n+1})$$

$$= d\Phi(\lambda_{n-1}),$$

for any $n \geq 1$. Similarly, we obtain

$$\Phi(\delta(\mu_{n-1})) = d\Phi(\mu_{n-1}),$$

for any $n \geq 0$. So $\Phi$ is an isomorphism of differential algebras. The theorem is proved. 

\section*{4. An Additive Basis for $Q$}

For $J = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n, j_1, j_2, \ldots, j_n)$, with $\varepsilon_k = 0, 1, j_k \in \mathbb{Z}, k = 1, 2, \ldots, n$, we set

$$x_J = x_{j_1+1}^{\varepsilon_1} x_{j_2+1}^{\varepsilon_2} \cdots x_{j_n+1}^{\varepsilon_n}.$$ 

**Definition 4.1.** The monomial $x_J$ is said to be admissible if

$$j_k \geq pj_{k+1} + \varepsilon_{k+1}, k = 1, 2, \ldots, n.$$ 

Denote by $J_n$ the set of all sequences $J$ such that $x_J$ is admissible.

We note that if $j_k \leq -\varepsilon_k, k = 1, 2, \ldots, n$, then $x_J$ is admissible if and only if $\lambda_{-J}$ is admissible in $\Lambda$. Here

$$-J = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n, -j_1, -j_2, \ldots, -j_n).$$

From the relation $D^n(x_{pi+1}x_{i+1}) = 0$ in $Q$ we have

$$x_{pi-n+1}x_{i+1} = -\sum_{k=0}^{n-1} \binom{n}{k} x_{pi-k+1}x_{i+1-n+k}. \quad (7)$$

Applying relations of the same form to those terms of the right hand side of (7) which are not admissible, after finitely many steps we obtain an expression of the form

$$x_{pi-n+1}x_{i+1} = \sum a_{n,k} x_{pi-k+1}x_{i+1-n+k}, \quad (8)$$
where \( a_{n,k} \in \mathbb{F}_p \) and all the monomials appearing on the right hand side are admissible (see the proof of Lemma 4.2). That means
\[
a_{n,k} = 0 \text{ if } (p + 1)k \geq pn.
\]
By an argument analogous to the previous one, we get
\[
x_{pi} - n y_{i+1} = \sum b_{n,k} x_{pi-k} y_{i+1-n+k}, \tag{9}
\]
\[
x_{pi} - n+1 y_{i+1} = \sum c_{n,k} y_{pi-k+1} x_{i+1-n+k} + \sum c'_{n,k} x_{pi-k+1} y_{i+1-n+k}, \tag{10}
\]
\[
y_{pi} - n y_{i+1} = \sum d_{n,k} x_{pi-k} y_{i+1-n+k}, \tag{11}
\]
where \( b_{n,k}, c_{n,k}, c'_{n,k}, d_{n,k} \in \mathbb{F}_p, b_{n,k} = c_{n,k} = d_{n,k} = 0 \) if \( (p + 1)k \geq pn \), \( c'_{n,k} = 0 \) if \( (p + 1)k > pn \).

**Lemma 4.2.** If \( k < 0 \) then \( a_{n,k} = b_{n,k} = c_{n,k} = c'_{n,k} = d_{n,k} = 0 \).

**Proof.** For simplicity, we only prove \( a_{n,k} = 0 \). The others can be obtained by a similar argument. Let \( x_{pi} - n y_{i+1} \) be an inadmissible term in the right hand side of (7). Then \( (p + 1)\ell \geq pn, 0 \leq \ell < n \). Set \( m = (p + 1)\ell - pn = p(i - n - \ell) - pi + \ell \geq 0 \). Then we have
\[
x_{pi} - \ell x_{i+1-n+\ell} = x_{p(i-n+\ell)-m+1}(i-n+\ell+1).
\]
Applying (7) we get
\[
x_{pi} - \ell x_{i+1-n+\ell} = -\sum_{j=0}^{m-1} \binom{m}{j} x_{p(i-n+\ell)-j+1} x_{i-n+\ell+1-m-j}.
\]
We have
\[
\ell - m + j \geq \ell - m = \ell - (p + 1)\ell + pn = p(n - \ell) > 0.
\]
Therefore in (8) the coefficient \( a_{n,k} \) such that \( a_{n,k} \neq 0 \), with the lowest possible \( k \) is \( a_{n,0} \). Hence \( a_{n,k} = 0 \) if \( k < 0 \).

The main result of this section is

**Theorem 4.3.** The set
\[
X = \bigcup_{n \geq 0} \{ x_J : J \in \mathcal{J}_n \}
\]
is an \( \mathbb{F}_p \)-vector space basis for \( Q \).

**Proof.** We first prove that \( X \) spans \( Q \). Let \( x_J \) be a monomial in \( Q \). We apply the relations (8)-(11) and Lemma 4.2 to the inadmissible pairs in \( x_J \) and after
a finite number of steps we can write \( x_J \) as a linear combination of monomials of the form

\[ x_J = x_J' \cdot x_J'', \]

where \( x_J' \) is an admissible monomial involving generators \( x_i^{\varepsilon+1} \) with \( i > -\varepsilon \) and \( x_J'' \) is a monomial involving generators \( x_i^{\varepsilon+1} \) with \( i \leq -\varepsilon \). Then \( x_J'' \in Q_- \).

Using Theorem 3.5 we get

\[ \Psi(x_J'') = \sum \alpha_u \lambda_{-J_u}, \]

where \( \alpha_u \in \mathbb{F}_p \) and \( \lambda_{-J_u} \) is an admissible monomial in \( \Lambda \). From this we obtain

\[ x_J'' = \sum \alpha_u x_{J_u}, \]

where \( x_{J_u} \) is an admissible monomial in \( Q \) (see Definition 4.1). It is easy to see that the monomial \( x_J' \cdot x_{J_u} \) is admissible in \( Q \). Therefore \( X \) spans \( Q \).

We now prove that the set \( X \) is linearly independent. Suppose that

\[ m \sum_{u=1}^{m} a_u x_{J_u} = 0 \text{ in } Q, \]

with \( a_u \in \mathbb{F}_p, J_u \in J_n, u = 1, 2, \ldots, m \). Then we have

\[ m \sum_{u=1}^{m} a_u w^{J_u} \in \Gamma. \]

We order the set \( \{ w^J : J \in J_n \} \) by agreeing that

\[ w^J_1 > w^J_2 \text{ if and only if } J_1 > J_2. \]

Here the order in \( \mathbb{Z}^{2n} \) is the antilexicographical one.

Suppose that there is an index \( u \) such that \( a_u \neq 0 \). Let \( w^J_u \) be the greatest monomial of all monomials \( w^{J_u} \) such that \( a_u \neq 0 \) and assume that \( J = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n, j_1, j_2, \ldots, j_n) \). Since \( \sum_{u=1}^{m} a_u w^{J_u} \in \Gamma \), \( w^J \) is a term in the expression of elements of the form

\[ t^{\varepsilon_1}_1 \cdots t^{\varepsilon_n}_n w_{1}^{j_1+1} \cdots w_{k-1}^{j_k+1+\varepsilon_k-1} * z * t^{\varepsilon_k+2}_1 \cdots t^{\varepsilon_n}_n w_{1}^{j_k+2+\varepsilon_k+2} \cdots w_{n-k-2}^{j_{n-k-2}+\varepsilon_n}, \]

where \( 1 \leq k \leq n \) and \( z \) is one of the elements given in Definition 2.3.

If \( z = \alpha_1 \alpha_2 w_1^{-1} w_2^{a} Q_{2,0} Q_{2,1}^b \) then

\[ z = \sum_{j=0}^{b} \binom{b}{j} \alpha_1 \alpha_2 w_1^{p(a+b) - b + j - 1} w_2^{a + b - j}. \]

Since \( w^J \) is the greatest monomial, from this we get \( j_k = p(a+b) - b - 1 \), \( j_{k+1} = a + b \), \( \varepsilon_{k+1} = 0 \). Hence

\[ p(j_{k+1} + \varepsilon_{k+1}) = p(a + b) > p(a + b) - b - 1 = j_k. \]
If \( z = t_1^0 t_2^0 w_1^{-1} w_2^0 R_{2,0} Q_{2,1}^2 - R_{2,1} Q_{2,0}^2 \) then
\[
z = \sum_{j=0}^b \binom{b}{j} t_1^0 t_2^0 w_1^{p(a+b) - j - 2} w_2^{a+b+1-j}.
\]

Hence \( j_k = p(a+b+1) - b - 1 \), \( j_{k+1} = a + b + 1 \), \( \varepsilon_{k+1} = 0 \), and
\[
p j_{k+1} + \varepsilon_{k+1} = p(a+b+1) > p(a+b+1) - b - 1 = j_k.
\]

If \( z = 2t_1^0 t_2^0 w_1^2 R_{2,1} Q_{2,0}^2 - R_{2,0} Q_{2,0}^2 \) then
\[
z = \sum_{j=0}^b \binom{b}{j} t_1^0 t_2^0 w_1^{p(a+b+1) - j - 2} w_2^{a+b+1-j}.
\]

From this we obtain \( j_k = p(a+b+1) - b \), \( j_{k+1} = a + b + 1 \), \( \varepsilon_{k+1} = 1 \). Hence
\[
p j_{k+1} + \varepsilon_{k+1} = p(a+b+1) + 1 > p(a+b+1) - b = j_k.
\]

If \( z = t_1^0 t_2^0 w_1^2 R_{2,0} R_{2,1} Q_{2,0}^2 \) then
\[
z = \sum_{j=0}^b \binom{b}{j} t_1^0 t_2^0 w_1^{p(a+b+2) - j - 1} w_2^{a+b+1-k}.
\]

Hence \( j_k = p(a+b+2) - b \), \( j_{k+1} = a + b + 2 \), \( \varepsilon_{k+1} = 1 \). From this it follows that
\[
p j_{k+1} + \varepsilon_{k+1} = p(a+b+2) + 1 > p(a+b+2) - b = j_k.
\]

Therefore \( x, j \) is inadmissible. This contradicts the fact that \( x, j \) is admissible. Hence, the theorem is proved.

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**References**


