Vietnam Journal of Mathematics **33**:1 (2005) 19–32

Vietnam Journal
of
MATHEMATICS

On an Invariant-Theoretic Description of the Lambda Algebra*

Nguyen Sum

Department of Mathematics, University of Quy Nhon,
170 An Duong Vuong, Quy Nhon, Binh Dinh, Vietnam

Received May 12, 2003
Revised September 15, 2004

Dedicated to Professor Huỳnh Mùi on the occasion of his sixtieth birthday

Abstract The purpose of this paper is to give a mod-p analogue of the Lomonaco invariant-theoretic description of the lambda algebra for p an odd prime. More precisely, using modular invariants of the general linear group $GL_n = GL(n, \mathbb{F}_p)$ and its Borel subgroup B_n, we construct a differential algebra Q_- which is isomorphic to the lambda algebra $\Lambda = \Lambda_p$.

Introduction

For the last few decades, the modular invariant theory has been playing an important role in stable homotopy theory. Singer [9] gave an interpretation for the dual of the lambda algebra Λ_p, which was introduced by the six authors [1], in terms of modular invariant theory of the general linear group at the prime $p = 2$. In [8], Hung and the author gave a mod-p analogue of the Singer invariant-theoretic description of the dual of the lambda algebra for p an odd prime. Lomonaco [6] also gave an interpretation for the lambda algebra in terms of modular invariant theory of the Borel subgroup of the general linear group at $p = 2$.

*This work was supported in part by the Vietnam National Research Program Grant 140801.

The purpose of this paper is to give a mod-p analogue of the Lomonaco invariant-theoretic description of the lambda algebra for p an odd prime. More precisely, using modular invariants of the general linear group $GL_n = GL(n, \mathbb{F}_p)$ and its Borel subgroup B_n, we construct a differential algebra Q_- which is isomorphic to the lambda algebra $\Lambda = \Lambda_p$. Here and in what follows, $\mathbb{F}_p$ denotes the prime field of p elements. Recall that, Λ_p is the E_1-term of the Adams spectral sequence of spheres for p an odd prime, whose E_2-term is $\mathrm{Ext}^*_{\mathcal{A}(p)}(\mathbb{F}_p, \mathbb{F}_p)$ where $\mathcal{A}(p)$ denotes the mod p Steenrod algebra, and E_∞-term is a graded algebra associated to the p-primary components of the stable homotopy of spheres.

It should be noted that the idea for the invariant-theoretic description of the lambda algebra is due to Lomonaco, who realizes it for $p = 2$ in [6]. In this paper, we develope of his work for p any odd prime. Our main contributions are the computations at odd degrees, where the behavior of the lambda algebra is completely different from that for $p = 2$.

The paper contains 4 sections. Sec. 1 is a preliminary on the modular invariant theory and its localization. In Sec. 2 we construct the differential algebra Q by using modular invariant theory and show that Q can be presented by a set of generators and some relations on them. In Sec. 3 we recall some results on the lambda algebra and show that it is isomorphic to a differential subalgebra Q_- of Q. Finally, in Sec. 4 we give an $\mathbb{F}_p$-vector space basis for Q.

1. Preliminaries on the Invariant Theory

For an odd prime p, let E^n be an elementary abelian p-group of rank n, and let

$$H^*(BE^n) = E(x_1, x_2, \dots, x_n) \otimes \mathbb{F}_p(y_1, y_2, \dots, y_n)$$

be the mod-p cohomology ring of E^n. It is a tensor product of an exterior algebra on generators x_i of dimension 1 with a polynomial algebra on generators y_i of dimension 2. Here and throughout the paper, the coefficients are taken over the prime field $\mathbb{F}_p$ of p elements.

Let $GL_n = GL(n, \mathbb{F}_p)$ and B_n be its Borel subgroup consisting of all invertible upper triangular matrices. These groups act naturally on $H^*(BE^n)$. Let S be the multiplicative subset of $H^*(BE^n)$ generated by all elements of dimension 2 and let

$$\Phi_n = H^*(BE^n)_S$$

be the localization of $H^*(BE^n)$ obtained by inverting all elements of S. The action of GL_n on $H^*(BE^n)$ extends to an action of its on Φ_n. We recall here some results on the invariant rings $\Gamma_n = \Phi_n^{GL_n}$ and $\Delta_n = \Phi_n^{B_n}$.

Let $L_{k,s}$ and $M_{k,s}$ denote the following graded determinants (in the sense of Mui [3])

$$L_{k,s} = \begin{vmatrix} y_1 & y_2 & \dots & y_k \\ y_1^p & y_2^p & \dots & y_k^p \\ \vdots & \vdots & \dots & \vdots \\ y_1^{p^{s-1}} & y_2^{p^{s-1}} & \dots & y_k^{p^{s-1}} \\ y_1^{p^{s+1}} & y_2^{p^{s+1}} & \dots & y_k^{p^{s+1}} \\ \vdots & \vdots & \dots & \vdots \\ y_1^{p^k} & y_2^{p^k} & \dots & y_k^{p^k} \end{vmatrix},$$

$$M_{k,s} = \begin{vmatrix} x_1 & x_2 & \dots & x_k \\ y_1 & y_2 & \dots & y_k \\ y_1^p & y_2^p & \dots & y_k^p \\ \vdots & \vdots & \dots & \vdots \\ y_1^{p^{s-1}} & y_2^{p^{s-1}} & \dots & y_k^{p^{s-1}} \\ y_1^{p^{s+1}} & y_2^{p^{s+1}} & \dots & y_k^{p^{s+1}} \\ \vdots & \vdots & \dots & \vdots \\ y_1^{p^{k-1}} & y_2^{p^{k-1}} & \dots & y_k^{p^{k-1}} \end{vmatrix}.$$

for $0 \le s \le k \le n$ and $M_{k,k} = 0$. We set $L_k = L_{k,k}, 1 \le k \le n, L_0 = 1$. Recall that L_k is invertible in Φ_n.

As is well known $L_{k,s}$ is divisible by L_k. Dickson invariants $Q_{k,s}$ and Mui invariants $R_{k,s}, V_k,\ 0 \le s \le k$, are defined by

$$Q_{k,s} = L_{k,s}/L_k, \quad R_{k,s} = M_{k,s}L_k^{p-2}, \quad V_k = L_k/L_{k-1}.$$

Note that $\dim Q_{k,s} = 2(p^k - p^s),\ \dim R_{k,s} = 2(p^k - p^s) - 1,\ \dim V_k = 2p^{k-1}$, $Q_{k,0} = L_k^{p-1},\ L_k = V_k V_{k-1} \dots V_2 V_1$.

From the results in Dickson [2] and Mui [3, 4.17] we observe

Theorem 1.1. (see Singer [9])

$$\Gamma_n = E(R_{n,0}, R_{n,1}, \dots, R_{n,n-1}) \otimes \mathbb{F}_p(Q_{n,0}^{\pm 1}, Q_{n,1}, \dots, Q_{n,n-1}).$$

Following Li–Singer [7], we set

$$N_k = M_{k,k-1}L_k^{p-2}, \quad W_k = V_k^{p-1}, 1 \le k \le n.$$

Then we have

Theorem 1.2. (see Li–Singer [7])

$$\Delta_n = E(N_1, N_2, \dots, N_n) \otimes \mathbb{F}_p(W_1^{\pm 1}, W_2^{\pm 1}, \dots, W_n^{\pm 1}).$$

For latter use, we set

$$t_k = N_k/Q_{k-1,0}^{p-1}, \quad w_k = W_k/Q_{k-1,0}^{p-1}, \quad 1 \le k \le n.$$

Observe that $\dim t_k = 2p - 3,\ \dim w_k = 2p - 2$. From Theorem 1.2 we obtain

Corollary 1.3.

$$\Delta_n = E(t_1, t_2, \dots , t_n) \otimes \mathbb{F}_p(w_1^{\pm 1}, w_2^{\pm 1}, \dots , w_n^{\pm 1}).$$

Moreover, from Dickson [2], Mui [3], we have

Proposition 1.4.
(i) $Q_{n,s} = Q^p_{n-1,s-1} + Q^{p-1}_{n-1,0} Q_{n-1,s} w_n$,
(ii) $R_{n,s} = Q^{p-1}_{n-1,0}(R_{n-1,s}w_n + Q_{n-1,s}t_n)$.

2. The Algebra Q

In this section, we construct the differential algebra Q by using modular invariant theory. In Sec. 4, we will show that the lambda algeba is isomorphic to a subalgebra of Q.

Definition 2.1. *Let Δ_n be as in Sec. 1. Set*

$$\Delta = \bigoplus_{n\geq 0} \Delta_n.$$

Here, by convention, $\Delta_0 = \mathbb{F}_p$. This is a direct sum of vector spaces over $\mathbb{F}_p$.

Remark. For $I = (\varepsilon_1, \varepsilon_2, \dots , \varepsilon_n, i_1, i_2, \dots , i_n)$ with $\varepsilon_j = 0, 1, i_j \in \mathbb{Z}$, set

$$w^I = t_1^{\varepsilon_1} t_2^{\varepsilon_2} \dots t_n^{\varepsilon_n} w_1^{i_1+\varepsilon_1} w_2^{i_2+\varepsilon_2} \dots w_n^{i_n+\varepsilon_n},$$

even in the case when some of ε_j or i_j are zero. For example, the element $t_1 \in \Delta_2$ will be written as $t_1t_2^0w_1^0w_2^0$, to be distinguished from $t_1 \in \Delta_1$, since $t_1 \neq t_1t_2^0w_1^0w_2^0$. For any $n > 0$ we have a monomial

$$t_1^0t_2^0 \dots t_n^0w_1^0w_2^0 \dots w_n^0 \in \Delta_n$$

which is the identity of Δ_n. All these elements are distinct in Δ.

Now we equip Δ with an algebra structure as follows. For any non-negative integers k, ℓ, we define an isomorphism of algebras

$$\mu_{k,\ell} : \Delta_k \otimes \Delta_\ell \to \Delta_{k+\ell}$$

by setting

$$\begin{aligned}&\mu_{k,\ell}(t_1^{\varepsilon_1} t_2^{\varepsilon_2} \dots t_k^{\varepsilon_k} w_1^{i_1+\varepsilon_1} w_2^{i_2+\varepsilon_2} \dots w_k^{i_k+\varepsilon_k} \otimes t_1^{\sigma_1} t_2^{\sigma_2} \dots t_\ell^{\sigma_\ell} w_1^{j_1+\sigma_1} w_2^{j_2+\sigma_2} \dots w_\ell^{j_\ell+\sigma_\ell})\\ &= t_1^{\varepsilon_1} t_2^{\varepsilon_2} \dots t_k^{\varepsilon_k} t_{k+1}^{\sigma_1} t_{k+2}^{\sigma_2} \dots t_{k+\ell}^{\sigma_\ell} w_1^{i_1+\varepsilon_1} w_2^{i_2+\varepsilon_2} \dots w_k^{i_k+\varepsilon_k} w_{k+1}^{j_1+\sigma_1} w_{k+2}^{j_2+\sigma_2} \dots w_{k+\ell}^{j_\ell+\sigma_\ell},\end{aligned}$$

for any $i_1, i_2, \dots , i_k, j_1, j_2, \dots , j_\ell \in \mathbb{Z}, \varepsilon_1, \varepsilon_2, \dots , \varepsilon_k, \sigma_1, \sigma_2, \dots , \sigma_\ell = 0, 1$.

We assemble $\mu_{k,\ell},\ k,\ \ell \geq 0$, to obtain a multiplication

$$\mu : \Delta \otimes \Delta \ \to \Delta.$$

This multiplication makes Δ into an algebra.

For simplicity, we denote $\mu(x \otimes y) = x * y$ for any elements $x, y \in \Delta$.

Definition 2.2. *Let Γ denote the two-sides ideal of Δ generated by all elements of the forms*

$$t_1^0 t_2^0 w_1^{-1} w_2^0 Q_{2,0}^a Q_{2,1}^b,$$
$$t_1^0 t_2^0 w_1^{-1} w_2^0 R_{2,0} Q_{2,0}^a Q_{2,1}^b - R_{2,1} Q_{2,0}^a Q_{2,1}^b,$$
$$2t_1^0 t_2^0 w_1 w_2^0 R_{2,1} Q_{2,0}^a Q_{2,1}^b - R_{2,0} Q_{2,0}^a Q_{2,1}^b,$$
$$t_1^0 t_2^0 w_1 w_2^0 R_{2,0} R_{2,1} Q_{2,0}^a Q_{2,1}^b,$$

where $a, b \in \mathbb{Z}, b \geq 0$.

We define

$$Q = \Delta/\Gamma$$

to be the quotient of Δ by the ideal Γ.

For any non-negative integer n, we define a homomorphism

$$\bar{\delta}_n : \Delta_n \to \Delta_{n+1}$$

by setting

$$\bar{\delta}_n(x) = -t_1 w_1^{-1} * x + (-1)^{\dim x} x * t_1 w_1^{-1},$$

for any homogeneous element $x \in \Delta_n$. By assembling $\bar{\delta}_n, n \geq 0$, we obtain an endomorphism

$$\bar{\delta} : \Delta \to \Delta.$$

Theorem 2.3. *The endomorphism $\bar{\delta} : \Delta \to \Delta$ induces an endomorphism $\delta : Q \to Q$ which is a differential.*

Proof. Let $u \in \Delta_n$ be a homogeneous element and suppose $u \in \Gamma$. From the definition of Γ we see that u is a sum of elements of the form

$$u_i * s_i * z_i,$$

where $u_i \in \Delta_{n_i},\ z_i \in \Delta_{n-n_i-2}$ and s_i is one of the elements given in Definition 2.2. Then $\bar{\delta}(u)$ is a sum of elements of the form

$$-t_1 w_1^{-1} * u_i * s_i * z_i + (-1)^{\dim u} u_i * s_i * z_i * t_1 w_1^{-1}.$$

Since $t_1 w_1^{-1} * u_i \in \Delta_{n_i+1},\ z_i * t_1 w_1^{-1} \in \Delta_{n-n_i-1}$, we obtain $\bar{\delta}(u) \in \Gamma$. So, $\bar{\delta}$ induces an endomorphism

$$\delta : Q \to Q.$$

Now we prove that $\delta\delta = 0$. It suffices to check that if $x \in \Delta_n$ is a homogeneous element then $\bar{\delta}\bar{\delta}(x) \in \Gamma$. In fact, from the definition of $\bar{\delta}$ we have

$$\bar{\delta}\bar{\delta}(x) = t_1 t_2 w_1^{-1} w_2^{-1} * x - x * t_1 t_2 w_1^{-1} w_2^{-1}.$$

A direct computation using Proposition 1.4 shows that

$$R_{2,0} Q_{2,0}^{-1} = t_1 t_2^0 w_1^{-1} w_2^0 + t_1^0 t_2 w_1^0 w_2^{-1},$$
$$R_{2,1} Q_{2,0}^{-1} = t_1^0 t_2 w_1^{-1} w_2^{-1}.$$

From these, we have

$$t_1^0 t_2^0 w_1 w_2^0 R_{2,0} R_{2,1} Q_{2,0}^{-2} = t_1 t_2 w_1^{-1} w_2^{-1}.$$

Hence we obtain

$$\bar{\delta}\bar{\delta}(x) = t_1^0 t_2^0 w_1 w_2^0 R_{2,0} R_{2,1} Q_{2,0}^{-2} * x - x * t_1^0 t_2^0 w_1 w_2^0 R_{2,0} R_{2,1} Q_{2,0}^{-2} \in \Gamma.$$

The theorem is proved. ■

Now we give a new system of generators for Q.

Let T be the free associative algebra over $\mathbb{F}_p$ generated by x_{i+1} of degree $2(p-1)i-1$ and y_{i+1} of degree $2(p-1)i$, for any $i \in \mathbb{Z}$.

It is easy to see that there exists a unique derivation $D : T \to T$ satisfying

$$D(x_i) = x_{i-1}, \quad D(y_i) = y_{i-1}, \quad i \in \mathbb{Z}.$$

(Recall that D is called a derivation if $D(uv) = D(u)v + uD(v)$, for any $u, v \in T$.)

Denote by $D^n = D \circ D \circ \ldots \circ D$ the composite of n-copies of D.

For simplicity, we set

$$x_i^\varepsilon = \begin{cases} x_i, & \varepsilon = 1 \\ y_i, & \varepsilon = 0. \end{cases}$$

By induction on n we easily obtain

Lemma 2.4. *Under the above notation, we have*

$$D^n(x_{q_1}^{\varepsilon_1} x_{q_2}^{\varepsilon_2}) = \sum_{k=0}^{n} \binom{n}{k} x_{q_1-k}^{\varepsilon_1} x_{q_2-n+k}^{\varepsilon_2}.$$

Here $\binom{n}{k}$ denotes the binomial coefficient.

We define a homomorphism of algebras $\pi : T \to Q$ by setting

$$\pi(x_{i+1}) = t_1 w_1^{i-1}, \pi(y_{i+1}) = t_1^0 w_1^i, \quad i \in \mathbb{Z}.$$

That means $\pi(x_{i+1}^\varepsilon) = t_1^\varepsilon w_1^{i-\varepsilon}$ for any $i \in \mathbb{Z},\ \varepsilon = 0, 1$.

Proposition 2.5. *The homomorphism $\pi : T \to Q$ is an epimorphism. Its kernel is the two-sides ideal of T generated by all elements of the forms*

$$\begin{aligned} &D^n(y_{pi} y_{i+1}),\\ &D^n(x_{pi} y_{i+1}),\\ &D^n(y_{pi+1} x_{i+1} - x_{pi+1} y_{i+1}),\\ &D^n(x_{pi+1} x_{i+1}), \end{aligned}$$

with $n \geq 0,\ i \in \mathbb{Z}$.

Proof. It is easy to see that π is an epimorphim. Now we prove the remaining part of the proposition.

By a direct computation we obtain

$$
\begin{aligned}
Q_{2,0}^{a}Q_{2,1}^{b} &= \sum_{k=0}^{b}\binom{b}{k}t_1^0t_2^0w_1^{p(a+b)-b+k}w_2^{a+b-k}\\
R_{2,0}Q_{2,0}^{a}Q_{2,1}^{b} &= \sum_{k=0}^{b}\binom{b}{k}t_1t_2^0w_1^{p(a+b+1)-b+k-1}w_2^{a+b+1-k}\\
&\quad+\sum_{k=0}^{b}\binom{b}{k}t_1^0t_2w_1^{p(a+b+1)-b+k}w_2^{a+b-k}\\
R_{2,1}Q_{2,0}^{a}Q_{2,1}^{b} &= \sum_{k=0}^{b}\binom{b}{k}t_1^0t_2w_1^{p(a+b+1)-b+k-1}w_2^{a+b-k}\\
R_{2,0}R_{2,1}Q_{2,0}^{a}Q_{2,1}^{b} &= \sum_{k=0}^{b}\binom{b}{k}t_1t_2w_1^{p(a+b+2)-b+k-2}w_2^{a+b+1-k}.
\end{aligned}
$$

Using Lemma 2.4 and the definition of π we have

$$
\begin{aligned}
\pi(D^n(y_{pi}y_{i+1})) &= \pi\Big(\sum_{k=0}^{n}\binom{n}{k}y_{pi-n+k}y_{i+1-k}\Big)\\
&= \sum_{k=0}^{n}\binom{n}{k}t_1^0t_2^0w_1^{pi-n+k-1}w_2^{i-k}\\
&= t_1^0t_2^0w_1^{-1}w_2^0\sum_{k=0}^{n}\binom{n}{k}t_1^0t_2^0w_1^{pi-n+k}w_2^{i-k}\\
&= t_1^0t_2^0w_1^{-1}w_2^0Q_{2,0}^{i-n}Q_{2,1}^{n}\\
&= 0 \ \text{ in } Q.
\end{aligned}
$$

By an argument analogous to the previous one, we get

$$
\begin{aligned}
\pi(D^n(x_{pi}y_{i+1})) &= t_1^0t_2^0w_1^{-1}w_2^0R_{2,0}Q_{2,0}^{i-n-1}Q_{2,1}^{n} - R_{2,1}Q_{2,0}^{i-n-1}Q_{2,1}^{n} = 0 \ \text{ in } Q\\
\pi(D^n(y_{pi+1}x_{i+1} - x_{pi+1}y_{i+1})) &= (2t_1^0t_2^0w_1w_2^0R_{2,1} - R_{2,0})Q_{2,0}^{i-n-1}Q_{2,1}^{n} = 0 \ \text{ in } Q\\
\pi(D^n(x_{pi+1}x_{i+1})) &= -t_1^0t_2^0w_1w_2^0R_{2,0}R_{2,1}Q_{2,0}^{i-n-2}Q_{2,1}^{n} = 0 \ \text{ in } Q.
\end{aligned}
$$

From these and the definition of Γ we obtain the proposition. ∎

3. The Lambda Algebra and the Modular Invariant Theory

In this section, we show that the lambda algebra, which is introduced by the six authors of [1], is isomorphic to a subalgebra of Q.

Let $\bar{\Lambda}$ denote the graded free associative algebra over $\mathbb{F}_p$ with generators λ_{i-1} of dimension $-2(p-1)i+1$ and μ_{i-1} of dimension $-2(p-i), i \geq 0$, subject to the relations:

$$\sum_{k=0}^{n}\binom{n}{k}\lambda_{k+pi-1}\lambda_{i+n-k-1}=0 \qquad (1)$$

$$\sum_{k=0}^{n}\binom{n}{k}\big(\mu_{k+pi-1}\lambda_{i+n-k-1}-\lambda_{k+pi-1}\mu_{i+n-k-1}\big)=0 \qquad (2)$$

$$\sum_{k=0}^{n}\binom{n}{k}\lambda_{k+pi}\mu_{i+n-k-1}=0 \qquad (3)$$

$$\sum_{k=0}^{n}\binom{n}{k}\mu_{k+pi}\mu_{i+n-k-1}=0 \qquad (4)$$

for $i,n\geq 0$. By Λ we mean the subalgebra of $\bar{\Lambda}$ generated by $\lambda_{i-1}, i>0$ and $\mu_{i-1}, i\geq 0$.

We note that this definition is the same as that given in [1], but we are writing the product in the order opposite to that used in [1].

For simplicity, we denote

$$\lambda_i^{\varepsilon}=\begin{cases}\lambda_i, & \varepsilon=1\\ \mu_i, & \varepsilon=0,\end{cases}$$

for any $i\geq -1$. We set

$$\lambda(\varepsilon_1,\varepsilon_2,i,n)=\sum_{k=0}^{n}\binom{n}{k}\big(\lambda^{\varepsilon_1}_{k+pi-\varepsilon_2}\lambda^{\varepsilon_2}_{i+n-k-1}-\varepsilon_2(1-\varepsilon_1)\lambda^{\varepsilon_2}_{k+pi-\varepsilon_2}\lambda^{\varepsilon_1}_{i+n-k-1}\big),$$

for any $\varepsilon_1,\varepsilon_2,i,n$ with $\varepsilon_1,\varepsilon_2=0,1$ and $i,n\geq 0$. Then the defining relations (1) - (4) become

$$\lambda(\varepsilon_1,\varepsilon_2,i,n)=0. \qquad (5)$$

Then we can consider Λ as the free graded associative algebra over $\mathbb{F}_p$ with generators $\lambda^{\varepsilon}_{i-1},\ i\geq\varepsilon$, subject to the relation (5) with $i\geq -\varepsilon_1$.

Definition 3.1. *A sequence $I=(\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_n,i_1,i_2,\ldots,i_n),\ \varepsilon_j=0,1,\ i_j\geq 0$, is said to be admissible if*

$$pi_j\geq i_{j+1}+\varepsilon_j,\ 1\leq j<n,\ \textit{and}\ i_n\geq\varepsilon_n.$$

In this case, the associated monomial $\lambda_I=\lambda^{\varepsilon_1}_{i_1-1}\lambda^{\varepsilon_2}_{i_2-1}\ldots\lambda^{\varepsilon_n}_{i_n-1}$ is also said to be admissible

Theorem 3.2. (Bousfield et al. [1]) *The admissible monomials form an additive basis for Λ.*

Definition 3.3. *The homomorphism $\bar{d}:\bar{\Lambda}\ \to\bar{\Lambda}$ is defined by*

$$\bar{d}(x)=-\lambda_{-1}x+(-1)^{\dim x}x\lambda_{-1},$$

for any homogeneous element $x \in \bar{\Lambda}$.

In $\bar{\Lambda}$, we have $\lambda_{-1}\lambda_{-1} = 0$, hence $\bar{d}\bar{d} = 0$. So $\bar{d}$ is a differential on $\bar{\Lambda}$. From the defining relations (1)-(4) we obtain

$$\begin{aligned}
&\bar{d}(\lambda_0) = 0,\ \bar{d}(\mu_{-1}) = 0,\ \bar{d}(\mu_0) = \lambda_0\mu_{-1} - \mu_{-1}\lambda_0,\\
&\bar{d}(\lambda_{n-1}) = \sum_{k=1}^{n-1}\binom{n}{k}\lambda_{k-1}\lambda_{n-k-1},\\
&\bar{d}(\mu_{n-1}) = \lambda_{n-1}\mu_{-1} + \sum_{k=1}^{n-1}\binom{n}{k}\big(\lambda_{k-1}\mu_{n-k-1} - \mu_{k-1}\lambda_{n-k-1}\big) - \mu_{-1}\lambda_{n-1},
\end{aligned}$$

for any $n \geq 2$. From these, we obtain $\bar{d}(\lambda^{\varepsilon}_{n-1}) \in \Lambda, n \geq \varepsilon$, so $\bar{d}$ passes to a differential d on Λ.

Now we describe the algebra Λ in terms of modular invariants.

Definition 3.4. *We define Q_- to be the subalgebra of Q generated by all elements x^{ε}_{i+1} with $i \leq -\varepsilon$.*

For any $\varepsilon_1, \varepsilon_2 = 0, 1,\ n \geq 0,\ i \in \mathbb{Z}$, we set

$$x(\varepsilon_1, \varepsilon_2, i, n) = D^n\big(x^{\varepsilon_1}_{pi+\varepsilon_2}x^{\varepsilon_2}_{i+1} - \varepsilon_2(1-\varepsilon_1)x^{\varepsilon_2}_{pi+\varepsilon_2}x^{\varepsilon_1}_{i+1}\big).$$

Then the defining relations of Q become

$$x(\varepsilon_1, \varepsilon_2, i, n) = 0. \tag{6}$$

So we can consider Q_- as the free graded associative algebra over $\mathbb{F}_p$ with generators $x^{\varepsilon}_{i+1},\ i \leq -\varepsilon$, subject to the relation (6) with $i \leq -\varepsilon_1$.

Theorem 3.5. *As a graded differential algebra, Λ is isomorphic to Q_-.*

Proof. We define a homomorphism of algebras

$$\Phi : \Lambda \to Q_-$$

by setting

$$\Phi(\lambda^{\varepsilon}_{i-1}) = x^{\varepsilon}_{-i+1},$$

for any $i \geq -\varepsilon$. From the definition of Q_- we easily obtain

$$\Phi\big(\lambda(\varepsilon_1, \varepsilon_2, i, n)\big) = x(\varepsilon_1, \varepsilon_2, -i, n)$$

for any $\varepsilon_1, \varepsilon_2 = 0, 1, i, n \geq 0, i \geq \varepsilon_1$. Hence, the homomorphism Φ is well defined.

Now we define a homomorphism of algebras

$$\Psi : Q_- \to \Lambda,$$

by setting $\Psi(x^{\varepsilon}_{i+1}) = \lambda^{\varepsilon}_{-i-1}$, for any $i \leq -\varepsilon$. It is easy to check that

$$\Psi\big(x(\varepsilon_1, \varepsilon_2, i, n)\big) = \lambda(\varepsilon_1, \varepsilon_2, -i, n),$$

for any $\varepsilon_1, \varepsilon_2 = 0, 1, n \geq 0, i \leq -\varepsilon_1$. So, the homomorphism Ψ is well defined. Obviously, we have

$$\Phi \circ \Psi = 1_{Q_-}, \quad \Psi \circ \Phi = 1_\Lambda.$$

Hence Φ is an isomorphism of algebras.

Finally we prove that Φ preserves the differential structure. We have

$$\begin{aligned}\Phi(\delta(\lambda_{n-1})) &= \Phi\big(\sum_{k=1}^{n-1}\binom{n}{k}\lambda_{k-1}\lambda_{n-k-1}\big)\\ &= \sum_{k=1}^{n-1}\binom{n}{k}x_{-k+1}x_{k-n+1}\\ &= d(x_{-n+1})\\ &= d\Phi(\lambda_{n-1}),\end{aligned}$$

for any $n \geq 1$. Similarly, we obtain

$$\Phi(\delta(\mu_{n-1})) = d\Phi(\mu_{n-1}),$$

for any $n \geq 0$. So Φ is an isomorphism of differential algebras. The theorem is proved. ■

4. An Additive Basis for Q

For $J = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n, j_1, j_2, \dots, j_n)$, with $\varepsilon_k = 0, 1, j_k \in \mathbb{Z}, k = 1, 2, \dots, n$, we set

$$x_J = x_{j_1+1}^{\varepsilon_1} x_{j_2+1}^{\varepsilon_2} \dots x_{j_n+1}^{\varepsilon_n}.$$

Definition 4.1. *The monomial x_J is said to be admissible if*

$$j_k \geq pj_{k+1} + \varepsilon_{k+1}, k = 1, 2, \dots, n.$$

Denote by $\mathcal{J}_n$ the set of all sequences J such that x_J is admissible.

We note that if $j_k \leq -\varepsilon_k,\ k = 1, 2, \dots, n$, then x_J is admissible if and only if λ_{-J} is admissible in Λ. Here

$$-J = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n, -j_1, -j_2, \dots, -j_n).$$

From the relation $D^n(x_{pi+1}x_{i+1}) = 0$ in Q we have

$$x_{pi-n+1}x_{i+1} = -\sum_{k=0}^{n-1}\binom{n}{k}x_{pi-k+1}x_{i+1-n+k}. \tag{7}$$

Applying relations of the same form to those terms of the right hand side of (7) which are not admissible, after finitely many steps we obtain an expression of the form

$$x_{pi-n+1}x_{i+1} = \sum a_{n,k}x_{pi-k+1}x_{i+1-n+k}, \tag{8}$$

where $a_{n,k} \in \mathbb{F}_p$ and all the monomials appearing on the right hand side are admissible (see the proof of Lemma 4.2). That means

$$a_{n,k} = 0 \text{ if } (p+1)k \geq pn.$$

By an argument analogous to the previous one, we get

$$\begin{aligned} x_{pi-n}y_{i+1} &= \sum b_{n,k}x_{pi-k}y_{i+1-n+k}, &(9)\\ x_{pi-n+1}y_{i+1} &= \sum c_{n,k}y_{pi-k+1}x_{i+1-n+k}\\ &\quad + \sum c'_{n,k}x_{pi-k+1}y_{i+1-n+k}, &(10)\\ y_{pi-n}y_{i+1} &= \sum d_{n,k}x_{pi-k}y_{i+1-n+k}, &(11)\end{aligned}$$

where $b_{n,k}, c_{n,k}, c'_{n,k}, d_{n,k} \in \mathbb{F}_p$, $b_{n,k} = c_{n,k} = d_{n,k} = 0$ if $(p+1)k \geq pn$, $c'_{n,k} = 0$ if $(p+1)k > pn$.

Lemma 4.2. *If $k < 0$ then $a_{n,k} = b_{n,k} = c_{n,k} = c'_{n,k} = d_{n,k} = 0$.*

Proof. For simplicity, we only prove $a_{n,k} = 0$. The others can be obtained by a similar argument. Let $x_{pi-\ell+1}x_{i+1-n+\ell}$ be an inadmissible term in the right hand side of (7). Then $(p+1)\ell \geq pn, 0 \leq \ell < n$. Set $m = (p+1)\ell - pn = p(i-n+\ell) - pi + \ell \geq 0$. Then we have

$$x_{pi-\ell+1}x_{i+1-n+\ell} = x_{p(i-n+\ell)-m+1}x_{(i-n+\ell)+1}.$$

Applying (7) we get

$$\begin{aligned} x_{pi-\ell+1}x_{i+1-n+\ell} &= -\sum_{j=0}^{m-1}\binom{m}{j}x_{p(i-n+\ell)-j+1}x_{i-n+\ell+1-m+j}\\ &= -\sum_{j=0}^{m-1}\binom{m}{j}x_{pi-(\ell-m+j)+1}x_{i+1-n+(\ell-m+j)}.\end{aligned}$$

We have

$$\ell - m + j \geq \ell - m = \ell - (p+1)\ell + pn = p(n-\ell) > 0.$$

Therefore in (8) the coefficient $a_{n,k}$ such that $a_{n,k} \neq 0$, with the lowest possible k is $a_{n,0}$. Hence $a_{n,k} = 0$ if $k < 0$.

The main result of this section is

Theorem 4.3. *The set*

$$\mathcal{X} = \bigcup_{n\geq 0}\{x_J : J \in \mathcal{J}_n\}$$

is an $\mathbb{F}_p$-vector space basis for Q.

Proof. We first prove that $\mathcal{X}$ spans Q. Let x_J be a monomial in Q. We apply the relations (8)-(11) and Lemma 4.2 to the inadmissible pairs in x_J and after a finite number of steps we can write x_J as a linear combination of monomials of the form

$$x_{J'}x_{J''},$$

where $x_{J'}$ is an admissible monomial involving generators x^{ε}_{i+1} with $i > -\varepsilon$ and $x_{J''}$ is a monomial involving generators x^{ε}_{i+1} with $i \leq -\varepsilon$. Then $x_{J''} \in Q_-$. Using Theorem 3.5 we get

$$\Psi(x_{J''}) = \sum \alpha_u \lambda_{-Ju},$$

where $\alpha_u \in \mathbb{F}_p$ and λ_{-J_u} is an admissible monomial in Λ. From this we obtain

$$x_{J''} = \sum \alpha_u x_{Ju},$$

where x_{J_u} is an admissible monomial in Q (see Definition 4.1). It is easy to see that the monomial $x_{J'}x_{J_u}$is admissible in Q. Therefore $\mathcal{X}$ spans Q.

We now prove that the set $\mathcal{X}$ is linearly independent. Suppose that

$$\sum_{u=1}^{m} a_u x_{J_u} = 0 \text{ in } Q,$$

with $a_u \in \mathbb{F}_p, J_u \in \mathcal{J}_n, u = 1, 2, \dots, m$. Then we have

$$\sum_{u=1}^{m} a_u w^{J_u} \in \Gamma.$$

We order the set $\{w^J : J \in \mathcal{J}_n\}$ by agreeing that

$$w^{J_1} > w^{J_2} \text{ if and only if } J_1 > J_2.$$

Here the order in $\mathbb{Z}^{2n}$ is the antilexicographical one.

Suppose that there is an index u such that $a_u \neq 0$. Let w^J be the greatest monomial of all monomials w^{J_u} such that $a_u \neq 0$ and assume that $J = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n, j_1, j_2, \dots, j_n)$. Since $\sum_{u=1}^{n} a_u w^{J_u} \in \Gamma$, w^J is a term in the expression of elements of the form

$$t_1^{\varepsilon_1} \dots t_{k-1}^{\varepsilon_{k-1}} w_1^{j_1+\varepsilon_1} \dots w_{k-1}^{j_{k-1}+\varepsilon_{k-1}} * z * t_1^{\varepsilon_{k+2}} \dots t_{n-k-2}^{\varepsilon_n} w_1^{j_{k+2}+\varepsilon_{k+2}} \dots w_{n-k-2}^{j_n+\varepsilon_n},$$

where $1 \leq k \leq n-2$ and z is one of the elements given in Definition 2.3.

If $z = t_1^0 t_2^0 w_1^{-1} w_2^0 Q_{2,0}^a Q_{2,1}^b$ then

$$z = \sum_{j=0}^{b} \binom{b}{j} t_1^0 t_2^0 w_1^{p(a+b)-b+j-1} w_2^{a+b-j}.$$

Since w^J is the greatest monomial, from this we get $j_k = p(a+b)-b-1$, $j_{k+1} = a+b$, $\varepsilon_{k+1} = 0$. Hence

$$pj_{k+1} + \varepsilon_{k+1} = p(a+b) > p(a+b) - b - 1 = j_k.$$

If $z = t_1^0 t_2^0 w^{-1} w_2^0 R_{2,0} Q_{2,0}^a Q_{2,1}^b - R_{2,1} Q_{2,0}^a Q_{2,1}^b$ then

$$z = \sum_{j=0}^{b} \binom{b}{j} t_1 t_2^0 w_1^{p(a+b)-b+j-2} w_2^{a+b+1-j}.$$

Hence $j_k = p(a+b+1) - b - 1,\ j_{k+1} = a+b+1,\ \varepsilon_{k+1} = 0$, and

$$p_{j_{k+1}} + \varepsilon_{k+1} = p(a+b+1) > p(a+b+1) - b - 1 = j_k.$$

If $z = 2t_1^0 t_2^0 w_1 w_2^0 R_{2,1} Q_{2,0}^a Q_{2,1}^b - R_{2,0} Q_{2,0}^a Q_{2,1}^b$ then

$$z = \sum_{j=0}^{b} \binom{b}{j} t_1^0 t_2 w_1^{p(a+b+1)-b+j} w_2^{a+b-j} - \sum_{j=0}^{b} \binom{b}{j} t_1 t_2^0 w_1^{p(a+b+1)-b+j-1} w_2^{a+b+1-j}.$$

From this we obtain $j_k = p(a+b+1) - b,\ j_{k+1} = a+b+1,\ \varepsilon_{k+1} = 1$. Hence

$$pj_{k+1} + \varepsilon_{k+1} = p(a+b+1) + 1 > p(a+b+1) - b = j_k.$$

If $z = t_1^0 t_2^0 w_1 w_2^0 R_{2,0} R_{2,1} Q_{2,0}^a Q_{2,1}^b$ then

$$z = \sum_{j=0}^{b} \binom{b}{j} t_1 t_2 w_1^{p(a+b+2)-b+k-1} w_2^{a+b+1-k}.$$

Hence $j_k = p(a+b+2) - b, j_{k+1} = a+b+2, \varepsilon_{k+1} = 1$. From this it follows that

$$pj_{k+1} + \varepsilon_{k+1} = p(a+b+2) + 1 > p(a+b+2) - b = j_k.$$

Therefore x_J is inadmissible. This contradicts the fact that x_J is admissible. Hence, the theorem is proved. ∎

Acknowledgements. The author expresses his warmest thank to Professor Nguyễn H. V. Hưng for helpful suggestions which lead him to this paper.

References

1. A. K. Bousfield, E. B. Curtis, D. M. Kan, D. G. Quillen, D. L. Rector, and J. W. Schlesinger, The mod-p lower central series and the Adams spectral sequence, *Topology* **5** (1966) 331–342.
2. L. E. Dickson, A fundamental system of invariants of the general modular linear group with a solution of the form problem, *Trans. Amer. Math. Soc.* **12** (1911) 75–98.
3. Huynh Mui, Modular invariant theory and the cohomology algebras of symmetric groups, *J. Fac. Sci. Univ. Tokyo Sec. IA Math.* **22** (1975) 319–369.
4. Huynh Mui, Dickson invariants and Milnor basis of the Steenrod algebra, *Colloq. Math. Soc. Janos Bolyai, Topology and Appl. Eger, Hungary* **41** (1983) 345–355.
5. Huynh Mui, Cohomology operations derived from modular invariants, *Math. Z.* **193** (1986) 151–163.

6. L. A. Lomonaco, Invariant theory and the total squaring operation, *Ph. D. Thesis, Univ. of Warwick,* September, 1986.
7. H. H. Li and W. M. Singer, Resolutions of modules over the Steenrod algebra and the classical theory of invariants, *Math. Z.* **18** (1982) 269–286.
8. Nguyen H. V. Hung and Nguyen Sum, On Singer's invariant-theoretic description of the lambda algebra: A mod p analogue, *J. Pure and Appl. Algebra* **99** (1995) 297–329.
9. W. M. Singer, Invariant theory and the lambda algebra, *Trans. Amer. Math. Soc.* **280** (1983) 673–693.
10. N. E. Steenrod and D. B. A. Epstein, Cohomology operations, *Ann. of Math.* No. 50, Princeton University Press, 1962.