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On the Almost Sure Convergence of Weighted Sums of I.I.D. Random Variables

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Abstract. We generalize some theorems of Chow and Lai [2] to general weighted sums of i.i.d. random variables. A characterization of moment conditions like $Ee^{\alpha|X|^\beta}|X|^\gamma < \infty$ or $E|X|^\alpha(\log^+|X|)^\beta < \infty$ is also given.

1. Introduction

Let $X_1, X_2, ...$ be independent identically distributed random variables with zero means. Let $(a_{nk}),\ n, k = 1, 2, ...,$ be any array of real numbers and (m_n) be any sequence of positive integers such that $m_n \to \infty$. The problem is to find best conditions for almost sure convergence to zero of

$$S_n = \sum_{k=1}^{m_n} a_{nk} X_k.$$

Some convergence theorems for S_n have been obtained by Chow [1], Chow and Lai [2], Hanson and Koopman [4], Pruitt [5] and Stout [6].

In [2] Chow and Lai have proved strong theorems for the case $a_{nk} = f(n)c_{nk}$ where $f(n) \downarrow 0$ and (c_{nk}) satisfies some summable conditions like $\limsup_n \sum_k c_{nk}^2 < \infty$ or $\limsup_n \sum_k |c_{nk}| < \infty$. In this paper we generalize some of these results to more general (a_{nk}).

In addition, we give a characterization of general moment condition like $Ef(|X_1|) < \infty$ by almost sure convergence to zero of $X_n a_n(f)$. For example, one such known result ([2] Theorem 1) states that $E|X_1|^\alpha < \infty$ for any $\alpha \geq 1$ if and only if $n^{-1/\alpha}X_n \to 0$ a.s.

2. Results

We shall use the following definition. An array (a_{nk}) is said to converge to a sequence (a_n) *almost uniformly* as $k \to \infty$, if for every $\varepsilon > 0$ there exists $K(\varepsilon)$ such that $|a_{nk} - a_n| < \varepsilon$ for all n and all k, except at most $K(\varepsilon)$ k for each n.

It is obvious that if $(a_{nk}) \underset{k}{\to} (a_n)$ almost uniformly then $a_{nk} \underset{k}{\to} a_n$ for all n. Note that, for arrays, uniform convergence implies almost uniform convergence. But the converse is not true. The array in the proof of Corollary 2 is an example.

Theorem 1. *Let $X_1, X_2, \ldots$ be i.i.d. mean 0 random variables. Then $E\, e^{t|X_1|} < \infty$ for all $t > 0$ if and only if $S_n = \sum_{k=1}^{\infty} a_{nk} X_k \to 0$ a.s. for every array of real numbers (a_{nk}) satisfying*

(a) $A_n := \sum\limits_{k=1}^{\infty} a_{nk}^2 < \infty$ *for all n,*

(b) $\dfrac{a_{nk}^2}{A_n} \underset{k}{\longrightarrow} 0$ *almost uniformly,*

(c) $\sum\limits_{1}^{\infty} e^{-\frac{a}{\sqrt{A_n}}} < \infty$ *for some $a > 0$.*

This theorem improves Theorem 2 in [2], which deals with $a_{nk} = c_{n-k}/\log n$ where $\sum_1^{\infty} c_n^2 < \infty$. This array clearly satisfies a), b) and c) of Theorem 1.

Theorem 2. *Let (X_n) be any sequence of i.i.d. mean 0 random variables, (a_{nk}) be any array of real numbers and (m_n) be any sequence of positive integers such that $m_n \to \infty$. Then $\sum\limits_{k=1}^{m_n} a_{nk} X_k \to 0$ a.s. if there exists a sequence of positive numbers (c_n) satisfying*

(a) $\sum\limits_{k=1}^{m_n} |a_{nk}| \le c_{m_n} \quad \forall n \ge 1,$

(b) c_n *is monotone non-increasing and tending to zero,*

(c) $c_n X_n \to 0$ *a.s.*

Remark. We say that $S_n = \sum\limits_{1}^{m_n} a_{nk} X_k$ converges to zero almost surely and *absolutely* if $S_n' = \sum\limits_{1}^{m_n} |a_{nk}|\,|X_k| \to 0$ a.s. Because the proof of Theorem 2 holds when S_n is replaced by S_n', Theorem 2 states the convergence of S_n in absolute sense too. Consequently under the conditions of Theorem 2 $\sum\limits_{1}^{m_n'} a_{nk} X_k \to 0$ a.s. for all sequences (m_n') such that $m_n' \to \infty$ and $m_n' \le m_n$ for all n.

We shall give below some corollaries of this theorem. We shall use the following notations [2]. Let $e_1(x) = e^x$, $e_2(x) = e_1(e^x)$, etc., and let $\log_2 x = \log\log x$, $\log_3 x = \log(\log_2 x)$, etc. By convention we shall also write $\log_1 x = \log x$, $\log_0 x = e_0(x) = 1$ and $e_k = e_k(1)$. For definiteness let us define $\log_k x = 1$ for all $x > 0$ such that $\log_k x < 1$ or $\log_k x$ is not defined.

Corollary 1. *Let $X_1, X_2, ...$ be i.i.d. mean 0 random variables. For any $\alpha > 0$ and $k = 1, 2, ...$, the following statements are equivalent:*
(a) $E\, e_k(t\,|X_1|^\alpha) < \infty \quad \forall t > 0$;
(b) $\lim_n (log_k n)^{-1/\alpha}\, X_n = 0$ *a.s.;*
(c) $\lim_n \sum_1^{m_n} a_{ni}\, X_i = 0$ *a.s. for every sequence (m_n) and array (a_{ni}) such that*

$$m_n \to \infty \text{ and } \quad \sum_1^{m_n} |a_{ni}| = O\,((log_k m_n)^{-1/\alpha}).$$

This corollary clearly improves Theorem 5 in [2]. It applies to more general array (a_{nk}) and to every sequence $m_n \to \infty$. Of course the most important case is $m_n = n \ \ \forall n$.

Corollary 2. *Let $X_1, X_2, ...$ be i.i.d. mean 0 random variables. For any $\alpha \geq 1$ the following statements are equivalent:*
(a) $E\,|X_1|^\alpha < \infty$;
(b) $\lim_n n^{-1/\alpha}\, X_n = 0$ *a.s.;*
(c) $\lim_n \sum_1^{m_n} a_{nk}\, X_k = 0$ *a.s. for every sequence (m_n) and array (a_{nk}) such that*

$$m_n \to \infty \text{ and } \ \sum_1^{m_n} |a_{nk}| = O\,(m_n^{-1/\alpha}).$$

This corollary is essentially weaker than Theorem 1 in [2]. We write it down to show a simple consequence of Theorem 2. It would be stronger than Theorem 1 in [2] if the last condition in (c) could be replaced by $\sum_1^{m_n} a_{nk}^2 = O\,(m_n^{-1/\alpha})$. But our method of proof is not suitable to derive such a result.

By Corollaries 1 and 2, we can see that the finiteness of expectation of some function of X_1 is equivalent to a condition like (c) of Theorem 2. By the theorem below, we obtain such equivalent conditions for more general functions. Hence Theorem 2 extends its applicability.

Theorem 3. *Let $X_1, X_2, ...$ be i.i.d. random variables.*
(a) *For any $\alpha > 0$ and $\beta \geq 0$*

$$E|X_1|^\alpha (log^+|X_1|)^\beta < \infty \quad \text{if and only if} \quad \lim_n \frac{X_n\, log^{\beta/\alpha} n}{n^{1/\alpha}} = 0 \text{ a.s.}$$

(b) *For any $\alpha > 0,\ \beta > 0$ and $\gamma \geq 0$*

$$Ee^{\alpha\,|X_1|^\beta}\,|X_1|^\gamma < \infty \quad \text{if and only if} \quad \limsup_n \frac{|X_n|}{(log\, n)^{1/\beta}} \leq \frac{1}{\alpha^{1/\beta}} \text{ a.s.}$$

Theorem 2 and Theorem 3 together lead to the following corollaries.

Corollary 3. *Let $X_1, X_2, \ldots$ be i.i.d. mean 0 random variables. For any $\alpha > 0$ and $\beta \geq 0$, the following statements are equivalent:*

(a) $E\,|X_1|^{\alpha}\,(log^{+}|X_1|)^{\beta} < \infty$;

(b) $\lim_n \dfrac{X_n\, log^{\beta/\alpha} n}{n^{1/\alpha}} = 0$ *a.s.*;

(c) $\lim_n \sum_{k=1}^{m_n} a_{nk}\, X_k = 0$ *a.s. for every sequence (m_n) and array (a_{nk}) such that* $m_n \to \infty$ *and* $\sum_{k=1}^{m_n} |a_{nk}| = O\,(m_n^{-1/\alpha}\, log^{\beta/\alpha}\, m_n)$.

Corollary 4. *Let $X_1, X_2, \ldots$ be i.i.d. mean 0 random variables. Suppose $E\, e^{\alpha\,|X_1|^{\beta}}\,|X_1|^{\gamma} < \infty$ for any $\alpha > 0,\ \beta > 0$ and $\gamma \geq 0$. Then $\sum_{k=1}^{m_n} a_{nk}\, X_k \to 0$ a.s. for every sequence (m_n) and array (a_{nk}) such that $m_n \to \infty$ and $\sum_{k=1}^{m_n} |a_{nk}| = o\,(log^{-1/\beta}\, m_n)$.*

From Theorem 3 we can also obtain other consequences.

If the common distribution function of i.i.d. random variables $X_1, X_2 \ldots$ is exponential then $E\, e^{\alpha|X_1|} < \infty$ if and only if $\alpha < \lambda$. Hence by Theorem 3 $\limsup_n |X_n|/\log n \leq 1/\alpha$ a.s. if and only if $1/\alpha > 1/\lambda$. Because this equivalence holds for all $\alpha > 0$, $\limsup_n\, |X_n|\,/\log n$ must be equal $1\,/\,\lambda$ a.s.

By the same method, we can derive similar conclusions for other distribution functions. For example, we have the following statement, written for well known distribution functions.

If $X_1, X_2, \ldots$ are i.i.d. random variables, then almost surely

$$\limsup_n \frac{|X_n|}{\log n} = \begin{cases} \frac{1}{\alpha}, & \text{if } X_1 \text{ is Laplace with parameter } \alpha \\ \frac{1}{\lambda}, & \text{if } X_1 \text{ is gamma with parameter } \alpha,\ \lambda \\ 2, & \text{if } X_1 \text{ is } \chi^2 \end{cases}$$

$$\limsup_n \frac{|X_n|}{\sqrt{2\log n}} = \begin{cases} \sigma, & \text{if } X_1 \text{ is } N(0,\sigma^2) \\ \alpha, & \text{if } X_1 \text{ is Rayleigh with parameter } \alpha \end{cases}$$

$$\limsup_n \frac{|X_n|}{\log^{1/\alpha} n} = \frac{1}{\lambda}, \quad \text{if } X_1 \text{ is Weibull with parameters } \alpha,\ \lambda$$

$$\lim_n \frac{X_n}{n^{1/\alpha}} = 0 \text{ if and only if } \begin{cases} \alpha < a, & \text{if } X_1 \text{ is Pareto with parameters } a,\ b \\ \alpha < a, & \text{if } X_1 \text{ is Student's } t \text{ with parameter } a \\ 0 < \alpha < 1, & \text{if } X_1 \text{ is Cauchy.} \end{cases}$$

3. Proofs

Proof of Theorem 1. Without loss of generality suppose $E\, X_1^2 = 1$. Set $\varphi(t) := E\, e^{t\,X_1}$ for all real t. Then $\varphi(t)$ is an entire function and $\varphi(0) = 1,\ \varphi'(0) = 0$ and $\varphi''(0) = 1$. Hence there exists $t_0 > 0$ such that for all $|t| \leq t_0 \quad \varphi(t) \leq 1 + t^2$.

By (b) for any real t there exists $K(t)$ such that

$$\frac{|a_{nk}|}{\sqrt{A_n}}\,|t| < t_0$$

for all n and all k except for at most $K(t)$ k for each n. Hence we have, setting $S_{nm} = \sum_{k=1}^{m} a_{nk}\, X_k$,

$$E\, e^{t \frac{S_{nm}}{\sqrt{A_n}}} = E\, e^{t \sum_{k \notin I_m(t)} a_{nk} X_k \,/\, \sqrt{A_n}} \prod_{k \in I_m(t)} \varphi\Big(\frac{a_{nk}}{\sqrt{A_n}}\, t\Big)$$
$$\leq \big(E\, e^{|t|\,|X_1|}\big)^{K(t)} \prod_{k \in I_m(t)} \Big(1 + \frac{a_{nk}^2}{A_n}\, t^2\Big) \leq \big(E\, e^{|t|\,|X_1|}\big)^{K(t)}\, e^{t^2},$$

where $I_m(t) = \{k \leq m;\ |(a_{nk} \,/\, \sqrt{A_n})t| < t_0\}$. Also we have

$$E\, e^{t \frac{|S_{nm}|}{\sqrt{A_n}}} \leq E\big(e^{t \frac{S_{nm}}{\sqrt{A_n}}} + e^{-t \frac{S_{nm}}{\sqrt{A_n}}}\big) \leq 2\, e^{t^2} \big(E\, e^{|t|\,|X_1|}\big)^{K(t)}.$$

Hence by Fatou lemma and (a), setting the last term by $H(t)$, we have

$$E\, e^{t \frac{|S_n|}{\sqrt{A_n}}} \leq H(t).$$

For any $\varepsilon > 0$, by Markov inequality and c), the last inequality leads to

$$\sum_{n=1}^{\infty} P(|S_n| > \varepsilon) = \sum_{n=1}^{\infty} P\big(e^{t|S_n|/\sqrt{A_n}} > e^{t\varepsilon/\sqrt{A_n}}\big)$$
$$\leq \sum_{n=1}^{\infty} e^{-t\varepsilon \,/\, \sqrt{A_n}}\, E\, e^{t|S_n| \,/\, \sqrt{A_n}} \leq H(t) \sum_{n=1}^{\infty} e^{-t\varepsilon \,/\, \sqrt{A_n}} < \infty$$

if t is chosen such that $t\varepsilon > a$. So we obtain that $S_n \to 0$ completely and therefore almost surely.

In the converse, for any given sequence (c_n) such that $c^2 = \sum_{1}^{\infty} c_n^2 < \infty$ and $c_1^2 > 0$ define $a_{nk} := c_{n-k} \,/\, \log n$ for $k \leq n$ and $a_{nk} := 0$ for $k > n$. Then a) holds for (a_{nk}) with $A_n \leq c^2 \,/\, \log^2 n$. Condition b) also holds because, as $c_n \to 0$, for any $\varepsilon > 0$ there exists $K(\varepsilon)$ such that $c_n^2/c_1^2 < \varepsilon$ for $n > K(\varepsilon)$. Consequently

$$\frac{a_{nk}^2}{A_n} = \frac{c_{n-k}^2}{\sum_{i=1}^{n} c_{n-i}^2} < \frac{c_{n-k}^2}{c_1^2} < \varepsilon$$

if $n - k > K(\varepsilon)$ i.e. if $k < n - K(\varepsilon)$. Hence there are only at most $K(\varepsilon)\, k$ for which the chain of inequalities above is not true, as $a_{nk} = 0$ for $k > n$. Condition c) holds for any $a > c$ as $\sum_{1}^{\infty} e^{-a/\sqrt{A_n}} \leq \sum_{1}^{\infty} e^{(-a/c)\log n} = \sum_{1}^{\infty} n^{-a/c} < \infty$.

To this array (a_{nk}) Theorem 2 of [2] is applicable. Hence, by assuming $S_n \to 0$ a.s., we obtain that $E\, e^{t|X_1|} < \infty$ for all $t > 0$.

Proof of Theorem 2. Since $c_n \downarrow$, for any $\varepsilon > 0$ we have, setting $S_n = \sum_1^{m_n} a_{nk} X_k$,

$$|S_n| \le \Big(\sum_{k=1}^{m_n} |a_{nk}|\Big) \max_{k \le m_n} |X_k| \le c_{m_n} \max_{k \le m_n} |X_k|$$
$$\le \max\big(c_{m_n} \max_{k \le N} |X_k|, \max_{N<k\le m_n} c_k |X_k|\big)$$

for any N. Since $c_n X_n \to 0$ a.s. for any $\varepsilon > 0$ and almost all ω there exists $N = N(\omega)$ such that $c_n |X_n| < \varepsilon$ for all $n > N(\omega)$. Hence for almost all ω

$$|S_n| \le \max\big(c_{m_n} \max_{k \le N(\omega)} |X_k|, \varepsilon\big).$$

Because $c_n \to 0$ and $m_n \to \infty$ we obtain that

$$\limsup_n |S_n| \le \varepsilon \quad a.s.,$$

which leads to the conclusion since ε can be chosen arbitrarily small.

Proof of Corollary 1. By Theorem 5 in [2], (a) is equivalent to (b). Statement (c) implies (b), because both (c) and (26) of Theorem 5 in [2] are applicable to $a_{nk} := (\log_k n)^{-1/\alpha}/(n-k)^2$ for $k < n$ and $a_{nk} := 0$ otherwise, and to $m_n := n$. Hence (b) holds by Theorem 5 in [2].

Conversely, suppose there exist an array (a_{nk}) and a sequence (m_n) satisfying the conditions in (c). Then there exists a constant K such that $\sum_1^{m_n} |a_{ni}| \le K(\log_k m_n)^{-1/\alpha}\ \forall n$. Define $c_n = K(\log_k n)^{-1/\alpha}\ \forall n$. Then c_n satisfies all conditions of Theorem 2. Hence, by Theorem 2, (b) implies (c). ∎

Proof of Corollary 2. Define $a_{nk} = n^{-1/\alpha}/(n-k)^2$ for $k < n$ and $a_{nk} = 0$ otherwise, and $m_n = n$. By Theorem 1 in [2] we see that (c) implies (b) with these (a_{nk}), (m_n) and (a) is equivalent to (b). Lastly, arguing similarly as in the proof of Corollary 1, we can show that (b) implies (c). ∎

Lemma 1. *Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be a function such that $f(x)$ is monotone increasing on $[b,\infty)$ for some $b \ge 0$ and is bounded on $[0,b]$ if $b > 0$. Define f^{-1} as the inverse function of f restricted on $[b,\infty)$ and as any positive function on $[0, f(b))$ if $b > 0$. Then $E f(|X_1|) < \infty$ if and only if $\limsup_n |X_n| / f^{-1}(an) \le 1$ a.s. for some and therefore for all real $a > 0$.*

Proof. To show the "only if" part of the conclusion, suppose $E f(|X_1|) < \infty$. Let us fix any $a > 0$ and let N be any positive integer such that $aN > f(b)$. We have, since f^{-1} is monotone increasing on $[f(b),\infty)$,

$$\infty > E f(|X_1|) \ge \sum_{i=N}^{\infty} ai\, P\big(f^{-1}(ai) < |X_1| \le f^{-1}(a(i+1))\big)$$
$$= a\left\{N P\big(|X_1| > f^{-1}(aN)\big) + \sum_{i=N+1}^{\infty} P\big(|X_1| > f^{-1}(ai)\big)\right\}.$$

Consequently, since X_n are i.i.d.,

$$\sum_{n=1}^{\infty} P\Big(\frac{|X_n|}{f^{-1}(an)} > 1\Big) < \infty.$$

By Borel–Cantelli lemma [3], $\limsup_n \dfrac{|X_n|}{f^{-1}(an)} \le 1$ a.s.

Conversely, let us fix any $a > 0$ and let N be any positive integer such that $aN > f(b)$. Then we have

$$\begin{aligned} E\, f(|X_1|) &\le E\, f(|X_1|)\, 1_{\{|X_1|\le f^{-1}(aN)\}} + E\, f(|X_1|)\, 1_{\{|X_1|> f^{-1}(aN)\}} \\ &\le \max_{0\le x\le f^{-1}(aN)} f(x) + \sum_{i=N}^{\infty} a(i+1)\, P\big(f^{-1}(ai) < |X_1| \le f^{-1}\big(a(i+1)\big)\big). \end{aligned}$$

The last sum, as is shown before,

$$\le a + a\Big\{N\, P\big(|X_1| > f^{-1}(aN)\big) + \sum_{i=N+1}^{\infty} P\big(|X_i| > f^{-1}(ai)\big)\Big\},$$

where the last sum is finite by Borel–Cantelli lemma, since X_n are independent and $\limsup\limits_n |X_n| \,/\, f^{-1}(an) \le 1$ a.s. Hence we obtain the finiteness of $E\, f(|X_1|)$, since f is bounded on $[0, b]$. ■

Proof of Theorem 3. To show a), set $g(x) = x^{\alpha}(\log^+ x)^{\beta}$ for $x \ge 0$. Then $g(x)$ is monotone increasing and g^{-1} exists on the set $[1, \infty)$. For $x \in [0, 1)$ define $g^{-1}(x) = 1$. By Lemma 1 $E\,|X_1|^{\alpha}(\log^+ |X_1|)^{\beta} < \infty$ if and only if $\limsup_n \dfrac{|X_n|}{g^{-1}(an)} \le 1$ a.s. for all $a > 0$. Because the exact form of g^{-1} is unknown, we shall estimate its behavior at $+\infty$ by the following function. Put $h(x) = \Big(\dfrac{\alpha^{\beta} x}{\ln^{\beta} x}\Big)^{1/\alpha}$ for $x \ge 2$ and $h(x) = 1$ for $0 \le x < 2$. We shall show that $h(an) \,/\, g^{-1}(an) \to 1$ for all $a > 0$.

Note that, for large enough n,

$$\frac{g\big(h(an)\big)}{g\big(g^{-1}(an)\big)} = \Big(\frac{\beta \log \alpha}{\log (an)} + 1 - \frac{\log \log^{\beta}(an)}{\log (an)}\Big)^{\beta} \longrightarrow 1$$

as $n \to \infty$. We shall prove a more general statement: for every two sequences $0 < a_n \to \infty$, $0 < b_n \to \infty$ if $g(a_n) \,/\, g(b_n) \to 1$ then $a_n \,/\, b_n \to 1$. Suppose there are such (a_n) and (b_n), but in contrary $\limsup_n a_n \,/\, b_n = c > 1$. Then there exist n_i such that $a_{n_i} \,/\, b_{n_i} \to c$. Hence $a_{n_i}^{\alpha} \,/\, b_{n_i}^{\alpha} \to c^{\alpha}$ and $a_{n_i} > b_{n_i}$ for large enough n_i. So $\limsup_n \log a_{n_i} \,/\, \log b_{n_i} \ge 1$. Consequently $\limsup_n a_n^{\alpha} \log^{\beta} a_n \,/\, (b_n^{\alpha} \log^{\beta} b_n) \ge c^{\alpha} > 1$, which contradicts the assumption $g(a_n) \,/\, g(b_n) \to 1$. So $\limsup_n a_n \,/\, b_n \le 1$. Similarly we obtain that $\liminf_n a_n \,/\, b_n \ge 1$, hence $\lim_n a_n \,/\, b_n = 1$.

So we have, by Lemma 1, $E\,|X_1|^{\alpha}\,(\log^+ |X_1|)^{\beta} < \infty$ if and only if for all $a > 0$

$$\limsup_n \frac{|X_n|}{h(an)} = \limsup_n \frac{|X_n|}{g^{-1}(an)} \frac{g^{-1}(an)}{h(an)} \le 1 \quad \text{a.s.,}$$

which is equivalent to

$$\limsup_n \frac{|X_n| \log^{\beta/\alpha} n}{n^{1/\alpha}} \le \alpha^{\beta/\alpha}\, a^{1/\alpha} \quad \text{a.s.}$$

for all $a > 0$. Since $a > 0$ can be chosen arbitrarily small the last inequality is equivalent to $\limsup_n |X_n| \log^{\beta/\alpha} n \,/\, n^{1/\alpha} = 0$ a.s.

For proving b) set $g(x) = e^{\alpha x^\beta} x^\gamma$ for $x \ge 0$. Also define $h(x) = (\log \alpha^{\gamma/\beta} x - \log(\log x)^{\gamma/\beta})^{1/\beta}\, \alpha^{-1/\beta}$ for $x \ge d$, and $h(x) = 1$ for $0 \le x < d$, where d is chosen large enough such that $h(x)$ is well defined. Then it is easy to show $g(h(n)) \,/\, g(g^{-1}(n)) \to 1$, where g^{-1} is the inverse function of g. As before, in order to show $h(n) \,/\, g^{-1}(n) \to 1$, let us prove that for every sequences $a_n \to \infty$, $b_n \to \infty$ if $g(a_n) \,/\, g(b_n) \to 1$ then $a_n \,/\, b_n \to 1$. Suppose we have such (a_n) and (b_n) but in contrary $\limsup_n a_n \,/\, b_n = c > 1$. Then there exist subsequences a_{n_i}, b_{n_i} such that $a_{n_i} \,/\, b_{n_i} \to c$. Therefore $(a_{n_i}^\beta - b_{n_i}^\beta) \,/\, b_{n_i}^\beta \to c^\beta - 1 > 0$ and $(a_{n_i}^\gamma - b_{n_i}^\gamma) \,/\, b_{n_i}^\gamma \to c^\gamma - 1 > 0$ if $\gamma > 0$. Consequently, since $b_{n_i}^\beta \to \infty$,

$$\frac{g(a_{n_i})}{g(b_{n_i})} = e^{\alpha \frac{a_{n_i}^\beta - b_{n_i}^\beta}{b_{n_i}^\beta} b_{n_i}^\beta} \frac{a_{n_i}^\gamma - b_{n_i}^\gamma}{b_{n_i}^\gamma} + e^{\alpha \frac{a_{n_i}^\beta - b_{n_i}^\beta}{b_{n_i}^\beta} b_{n_i}^\beta} \longrightarrow \infty,$$

which contradicts the assumption. So $\limsup_n a_n \,/\, b_n \le 1$. Similarly we have $\liminf_n a_n \,/\, b_n \ge 1$. Hence we obtain that $\lim_n a_n \,/\, b_n = 1$.

So we have, since $h(n) \,/\, \log^{1/\beta} n \to 1 \,/\, \alpha^{1/\beta}$, $\limsup_n \dfrac{|X_n|}{g^{-1}(n)} \le 1$ a.s. if and only if $\limsup\limits_n \dfrac{|X_n|}{h(n)} \le 1$ a.s. if and only if $\limsup \dfrac{|X_n|}{\log^{1/\beta} n} \le \dfrac{1}{\alpha^{1/\beta}}$ a.s. ∎

Proof of Corollary 3. Defining $c_n = K\, n^{-1/\alpha} \log^{\beta/\alpha} n$ and acting similarly as in the proof of Corollary 1, by Theorem 2, we obtain that (b) implies (c). Conversely (c) implies (b) because by (c) $\lim_n n^{-1/\alpha} \log^{\beta/\alpha} n \sum\limits_{k=1}^{n} (n-k+1)^{-2} X_k = 0$ a.s. Then by Lemma 3 in [2] we obtain (b), arguing similarly as in the proof of Theorem 1 in [2].

Proof of Corollary 4. Set $b_{m_n} := \sum\limits_{1}^{m_n} |a_{nk}|$ and define c_k, $k = 1, 2, ...$, such that

$$c_k \log^{1/\beta} k = \max_{\{n; m_n \ge k\}} \{b_{m_n} \log^{1/\beta} m_n\}.$$

We shall show that c_n satisfies all conditions of Theorem 2.

We have $c_n \log^{1/\beta} n$ is monotone non-increasing and tending to zero, since $b_{m_n} \log^{1/\beta} m_n \to 0$ by the assumption. Hence $c_n = (c_n \log^{1/\beta} n)(\log^{-1/\beta} n)$, as the product of two monotone non-increasing and tending to zero sequences, has the same properties. By the definition of c_n we have $b_{m_n} \le c_{m_n}$ for all n. By Theorem 3 $\limsup\limits_n \dfrac{|X_n|}{\log^{1/\beta} n} \le \dfrac{1}{\alpha^{1/\beta}}$ a.s. Hence almost surely

$$\limsup_n |X_n c_n| = \limsup_n \frac{|X_n|}{\log^{1/\beta} n} \cdot c_n \log^{1/\beta} n \le \frac{1}{\alpha^{1/\beta}} \lim_n c_n \log^{1/\beta} n = 0.$$

So c_n satisfies all conditions of Theorem 2. By Theorem 2 we obtain the conclusion. ■

References

1. Y. S. Chow, Some convergence theorems for independent random variables, *Ann. Math. Statist.* **37** (1966) 1482–1493.
2. Y. S. Chow and T. L. Lai, Limiting behavior of weighted sums of independent random variables, *Ann. Probab.* **1** (1973) 810–824.
3. Y. S. Chow and H. Teicher, *Probability Theory*, Springer, New York, Heidelberg, Berlin, 1978.
4. D. L. Hanson and L. H. Koopman, On the convergence rate of the law of large numbers for linear combinations of independent random variables, *Ann. Math. Statist.* **36** (1965) 559–564.
5. W. E. Pruitt, Summability of independent random variables. *J. Math. Mech.* **15** (1966) 769–776.
6. W. F. Stout, Some results on the complete and almost sure convergence of linear combinations of independent random variables and martingale differences, *Ann. Math. Statist.* **39** (1968) 1549–1562.