On the Almost Sure Convergence of Weighted Sums of I.I.D. Random Variables

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Abstract. We generalize some theorems of Chow and Lai [2] to general weighted sums of i.i.d. random variables. A characterization of moment conditions like $E|X|^\alpha < \infty$ or $E|X|^\alpha (\log^+|X|)^\beta < \infty$ is also given.

1. Introduction

Let $X_1, X_2, \ldots$ be independent identically distributed random variables with zero means. Let $(a_{nk}), n, k = 1, 2, \ldots$ be any array of real numbers and $(m_n)$ be any sequence of positive integers such that $m_n \to \infty$. The problem is to find best conditions for almost sure convergence to zero of

$$S_n = \sum_{k=1}^{m_n} a_{nk}X_k.$$ 

Some convergence theorems for $S_n$ have been obtained by Chow [1], Chow and Lai [2], Hanson and Koopman [4], Pruitt [5] and Stout [6].

In [2] Chow and Lai have proved strong theorems for the case $a_{nk} = f(n)c_{nk}$ where $f(n) \downarrow 0$ and $(c_{nk})$ satisfies some summable conditions like $\limsup_n \sum_k c_{nk}^2 < \infty$ or $\limsup_n \sum_k |c_{nk}| < \infty$. In this paper we generalize some of these results to more general $(a_{nk})$.

In addition, we give a characterization of general moment condition like $Ef(|X_1|) < \infty$ by almost sure convergence to zero of $X_n/a_n(f)$. For example, one such known result ([2] Theorem 1) states that $E|X_1|^\alpha < \infty$ for any $\alpha \geq 1$ if and only if $n^{-1/\alpha}X_n \to 0$ a.s.
2. Results

We shall use the following definition. An array \((a_{nk})\) is said to converge to a sequence \((a_n)\) almost uniformly as \(k \to \infty\), if for every \(\varepsilon > 0\) there exists \(K(\varepsilon)\) such that \(|a_{nk} - a_n| < \varepsilon\) for all \(n\) and all \(k\), except at most \(K(\varepsilon)\) for each \(n\).

It is obvious that if \((a_{nk}) \to_k (a_n)\) almost uniformly then \(a_{nk} \to a_n\) for all \(n\).

Note that, for arrays, uniform convergence implies almost uniform convergence. But the converse is not true. The array in the proof of Corollary 2 is an example.

**Theorem 1.** Let \(X_1, X_2, \ldots\) be i.i.d. mean 0 random variables. Then \(E e^{t|X_1|} < \infty\) for all \(t > 0\) if and only if \(S_n = \sum_{k=1}^{\infty} a_{nk} X_k \to 0\) a.s. for every array of real numbers \((a_{nk})\) satisfying

\[
\begin{align*}
\text{(a)} & \quad A_n := \sum_{k=1}^{\infty} a^2_{nk} < \infty \text{ for all } n, \\
\text{(b)} & \quad \frac{a^2_{nk}}{A_n} \to 0 \text{ almost uniformly,} \\
\text{(c)} & \quad \sum_{k} e^{-a_{nk}} < \infty \text{ for some } a > 0.
\end{align*}
\]

This theorem improves Theorem 2 in [2], which deals with \(a_{nk} = c_n - k / \log n\) where \(\sum_{k=1}^{\infty} c^2_n < \infty\). This array clearly satisfies a), b) and c) of Theorem 1.

**Theorem 2.** Let \((X_n)\) be any sequence of i.i.d. mean 0 random variables, \((a_{nk})\) be any array of real numbers and \((m_n)\) be any sequence of positive integers such that \(m_n \to \infty\). Then \(\sum_{k=1}^{m_n} a_{nk} X_k \to 0\) a.s. if there exists a sequence of positive numbers \((c_n)\) satisfying

\[
\begin{align*}
\text{(a)} & \quad \sum_{k=1}^{m_n} |a_{nk}| \leq c_{m_n} \forall n \geq 1, \\
\text{(b)} & \quad c_n \text{ is monotone non-increasing and tending to zero,} \\
\text{(c)} & \quad c_n X_n \to 0 \text{ a.s.}
\end{align*}
\]

**Remark.** We say that \(S_n = \sum_{k=1}^{m_n} a_{nk} X_k\) converges to zero almost surely and absolutely if \(S'_n = \sum_{k=1}^{m_n} |a_{nk}| |X_k| \to 0\) a.s. Because the proof of Theorem 2 holds when \(S_n\) is replaced by \(S'_n\), Theorem 2 states the convergence of \(S_n\) in absolute sense too. Consequently under the conditions of Theorem 2 \(\sum_{k=1}^{m'_n} a_{nk} X_k \to 0\) a.s. for all sequences \((m'_n)\) such that \(m'_n \to \infty\) and \(m'_n \leq m_n\) for all \(n\).

We shall give below some corollaries of this theorem. We shall use the following notations [2]. Let \(e_1(x) = e^x, e_2(x) = e_1(e^x), \ldots\), and let \(\log_2 x = \log \log_2 x, \log_3 x = \log(\log_2 x), \ldots\). By convention we shall also write \(\log_1 x = \log x, \log_0 x = e_0(x) = 1\) and \(e_k = e_k(1)\). For definiteness let us define \(\log_k x = 1\) for all \(x > 0\) such that \(\log_k x < 1\) or \(\log_k x\) is not defined.
Corollary 1. Let $X_1, X_2, \ldots$ be i.i.d. mean 0 random variables. For any $\alpha > 0$ and $k = 1, 2, \ldots$, the following statements are equivalent:

(a) $E e_k(t | X_1^\alpha) < \infty \ \forall t > 0$;
(b) $\lim_n (\log n)^{-1/\alpha} X_n = 0$ a.s.;
(c) $\lim_n \sum_{i=1}^{m_n} a_{ni} X_i = 0$ a.s. for every sequence $(m_n)$ and array $(a_{ni})$ such that

$$m_n \to \infty \text{ and } \sum_{i=1}^{m_n} |a_{ni}| = O \left( (\log m_n)^{-1/\alpha} \right).$$

This corollary clearly improves Theorem 5 in [2]. It applies to more general array $(a_{nk})$ and to every sequence $m_n \to \infty$. Of course the most important case is $m_n = n \ \forall n$.

Corollary 2. Let $X_1, X_2, \ldots$ be i.i.d. mean 0 random variables. For any $\alpha \geq 1$ the following statements are equivalent:

(a) $E |X_1|^\alpha < \infty$;
(b) $\lim_n n^{-1/\alpha} X_n = 0$ a.s.;
(c) $\lim_n \sum_{k=1}^{m_n} a_{nk} X_k = 0$ a.s. for every sequence $(m_n)$ and array $(a_{nk})$ such that

$$m_n \to \infty \text{ and } \sum_{k=1}^{m_n} |a_{nk}| = O \left( m_n^{-1/\alpha} \right).$$

This corollary is essentially weaker than Theorem 1 in [2]. We write it down to show a simple consequence of Theorem 2. It would be stronger than Theorem 1 in [2] if the last condition in (c) could be replaced by $\sum_{k=1}^{m_n} a_{nk}^2 = O \left( m_n^{-1/\alpha} \right)$. But our method of proof is not suitable to derive such a result.

By Corollaries 1 and 2, we can see that the finiteness of expectation of some function of $X_1$ is equivalent to a condition like (c) of Theorem 2. By the theorem below, we obtain such equivalent conditions for more general functions. Hence Theorem 2 extends its applicability.

Theorem 3. Let $X_1, X_2, \ldots$ be i.i.d. random variables.

(a) For any $\alpha > 0$ and $\beta \geq 0$

$$E |X_1|^\alpha (\log^+ |X_1|)^\beta < \infty \quad \text{if and only if} \quad \lim \frac{X_n \log^{\beta/\alpha} n}{n^{1/\alpha}} = 0 \text{ a.s.}$$

(b) For any $\alpha > 0$, $\beta > 0$ and $\gamma \geq 0$

$$E e^{\alpha |X_1|^\beta} |X_1|^{\gamma} < \infty \quad \text{if and only if} \quad \limsup \frac{|X_n|}{(\log n)^{1/\beta}} \leq \frac{1}{\alpha^{1/\beta}} \text{ a.s.}$$

Theorem 2 and Theorem 3 together lead to the following corollaries.
Corollary 3. Let $X_1, X_2, \ldots$ be i.i.d. mean 0 random variables. For any $\alpha > 0$ and $\beta \geq 0$, the following statements are equivalent:
(a) $E |X|^\alpha (\log^+ |X|)^\beta < \infty$;
(b) $\lim_n \frac{X_n \log^{\beta/\alpha} n}{n^{1/\alpha}} = 0$ a.s.;
(c) $\lim_n \sum_{k=1}^{m_n} a_{nk} X_k = 0$ a.s. for every sequence $(m_n)$ and array $(a_{nk})$ such that $m_n \to \infty$ and $\sum_{k=1}^{m_n} |a_{nk}| = O \left( m_n^{-1/\alpha} \log^{\beta/\alpha} m_n \right)$.

Corollary 4. Let $X_1, X_2, \ldots$ be i.i.d. mean 0 random variables. Suppose $E e^{\alpha |X|^\beta} |X|^\gamma < \infty$ for any $\alpha > 0$, $\beta > 0$ and $\gamma \geq 0$. Then $\sum_{k=1}^{m_n} a_{nk} X_k \to 0$ a.s.

for every sequence $(m_n)$ and array $(a_{nk})$ such that $m_n \to \infty$ and $\sum_{k=1}^{m_n} |a_{nk}| = o \left( \log^{-1/\beta} m_n \right)$.

From Theorem 3 we can also obtain other consequences.

If the common distribution function of i.i.d. random variables $X_1, X_2, \ldots$ is exponential then $E e^{\alpha |X|} |X|^\beta < \infty$ if and only if $\alpha < \lambda$. Hence by Theorem 3 $\lim_{n \to \infty} |X_n|/\log n \leq 1/\alpha$ a.s. if and only if $1/\alpha > 1/\lambda$. Because this equivalence holds for all $\alpha > 0$, $\lim_{n \to \infty} |X_n|/\log n$ must be equal $1/\lambda$ a.s.

By the same method, we can derive similar conclusions for other distribution functions. For example, we have the following statement, written for well known distribution functions.

If $X_1, X_2, \ldots$ are i.i.d. random variables, then almost surely
\[
\limsup_n \frac{|X_n|}{\log n} = \begin{cases} 
\frac{1}{\alpha}, & \text{if } X_1 \text{ is Laplace with parameter } \alpha \\
\frac{1}{\lambda}, & \text{if } X_1 \text{ is gamma with parameter } \alpha, \lambda \\
2, & \text{if } X_1 \text{ is } \chi^2 
\end{cases}
\]
\[
\limsup_n \frac{|X_n|}{\sqrt{2 \log n}} = \begin{cases} 
\sigma, & \text{if } X_1 \text{ is } N(0, \sigma^2) \\
\alpha, & \text{if } X_1 \text{ is Rayleigh with parameter } \alpha 
\end{cases}
\]
\[
\limsup_n \frac{|X_n|}{\log^{1/\alpha} n} = \begin{cases} 
\frac{1}{\lambda}, & \text{if } X_1 \text{ is Weibull with parameters } \alpha, \lambda 
\end{cases}
\]
\[
\lim_n \frac{X_n}{n^{1/\alpha}} = 0 \text{ if and only if } \begin{cases} 
\alpha < a, & \text{if } X_1 \text{ is Pareto with parameters } a, b \\
\alpha < a, & \text{if } X_1 \text{ is Student’s } t \text{ with parameter } a \\
0 < \alpha < 1, & \text{if } X_1 \text{ is Cauchy.}
\end{cases}
\]

3. Proofs

Proof of Theorem 1. Without loss of generality suppose $E X_1^2 = 1$. Set $\varphi(t) := E e^{tX_1}$ for all real $t$. Then $\varphi(t)$ is an entire function and $\varphi(0) = 1, \varphi'(0) = 0$ and $\varphi''(0) = 1$. Hence there exists $t_0 > 0$ such that for all $|t| \leq t_0$ $\varphi(t) \leq 1 + t^2$.

By (b) for any real $t$ there exists $K(t)$ such that
\[
\frac{|a_{nk}|}{\sqrt{A_n}} |t| < t_0
\]
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for all \( n \) and all \( k \) except for at most \( K(t) \) \( k \) for each \( n \). Hence we have, setting 
\[
S_{nm} = \sum_{k=1}^{m} a_{nk} X_k,
\]
\[
E e^{\frac{t S_{nm}}{\sqrt{A_n}}} = E e^{t \sum_{k \in I_m(t)} a_{nk} X_k/\sqrt{A_n}} \prod_{k \in I_m(t)} \varphi\left(\frac{a_{nk}}{\sqrt{A_n}} t\right)
\leq (E e^{t|X_1|})^{K(t)} \prod_{k \in I_m(t)} \left(1 + \frac{a_{nk}^2}{A_n} t^2\right) \leq (E e^{\frac{t}{c} |X_1|})^{K(t)} e^{t^2},
\]
where \( I_m(t) = \{k \leq m; |(a_{nk} / \sqrt{A_n})| t < t_0\} \). Also we have
\[
E e^{\frac{t S_{nm}}{\sqrt{A_n}}} \leq E\left(e^{\frac{\frac{t}{2} S_{nm}}{\sqrt{A_n}}} + e^{-\frac{\frac{t}{2} S_{nm}}{\sqrt{A_n}}}\right) \leq 2 e^{t^2} \left(E e^{\frac{t}{c} |X_1|}\right)^{K(t)}.
\]
Hence by Fatou lemma and (a), setting the last term by \( H(t) \), we have
\[
E e^{\frac{t S_{nm}}{\sqrt{A_n}}} \leq H(t).
\]
For any \( \epsilon > 0 \), by Markov inequality and c), the last inequality leads to
\[
\sum_{n=1}^{\infty} P(|S_n| > \epsilon) = \sum_{n=1}^{\infty} P\left(e^{\frac{t}{c} |S_n|/\sqrt{A_n}} > e^{t \epsilon / \sqrt{A_n}}\right)
\leq \sum_{n=1}^{\infty} e^{-t \epsilon / \sqrt{A_n}} E e^{\frac{t |S_n|}{\sqrt{A_n}}} \leq H(t) \sum_{n=1}^{\infty} e^{-t \epsilon / \sqrt{A_n}} < \infty
\]
if \( t \) is chosen such that \( t\epsilon > 0 \). So we obtain that \( S_n \rightarrow 0 \) completely and therefore almost surely.

In the converse, for any given sequence \( (c_n) \) such that \( c^2 = \sum_{1}^{\infty} c_n^2 < \infty \) and \( c_1^2 > 0 \) define \( a_{nk} := c_{n-k} / \log n \) for \( k \leq n \) and \( a_{nk} := 0 \) for \( k > n \). Then a) holds for \( (a_{nk}) \) with \( A_n \leq c^2 / \log^2 n \). Condition b) also holds because, as \( c_n \rightarrow 0 \), for any \( \epsilon > 0 \) there exists \( K(\epsilon) \) such that \( c_n^2/c_1^2 < \epsilon \) for \( n > K(\epsilon) \). Consequently
\[
a_{nk}^2/A_n = \sum_{i=1}^{n-k} c_{n-i}^2 < \frac{c_{n-k}}{c_1^2} < \epsilon
\]
if \( n - k > K(\epsilon) \) i.e. if \( k < n - K(\epsilon) \). Hence there are only at most \( K(\epsilon) k \) for which the chain of inequalities above is not true, as \( a_{nk} = 0 \) for \( k > n \). Condition c) holds for any \( a > c \) as
\[
\sum_{1}^{\infty} e^{-a/\sqrt{A_n}} \leq \sum_{1}^{\infty} e^{-(a/c) \log n} = \sum_{1}^{\infty} n^{-a/c} < \infty.
\]
To this array \( (a_{nk}) \) Theorem 2 of [2] is applicable. Hence, by assuming \( S_n \rightarrow 0 \) a.s., we obtain that \( E e^{\frac{t}{c} |X_1|} < \infty \) for all \( t > 0 \).
Proof of Theorem 2. Since $c_n \downarrow$, for any $\varepsilon > 0$ we have, setting $S_n = \sum_{k=1}^{m_n} a_{nk} X_k$,

$$|S_n| \leq \left( \sum_{k=1}^{m_n} |a_{nk}| \right) \max_{k \leq m_n} |X_k| \leq c_{m_n} \max_{k \leq m_n} |X_k|$$

$$\leq \max \left( c_{m_n} \max_{k \leq N} |X_k|, \max_{N < k \leq m_n} c_k |X_k| \right)$$

for any $N$. Since $c_n X_n \to 0$ a.s. for any $\varepsilon > 0$ and almost all $\omega$ there exists $N = N(\omega)$ such that $c_n |X_n| < \varepsilon$ for all $n > N(\omega)$. Hence for almost all $\omega$

$$|S_n| \leq \max \left( c_{m_n} \max_{k \leq N(\omega)} |X_k|, \varepsilon \right).$$

Because $c_n \to 0$ and $m_n \to \infty$ we obtain that

$$\limsup_n |S_n| \leq \varepsilon \quad \text{a.s.,}$$

which leads to the conclusion since $\varepsilon$ can be chosen arbitrarily small.

Proof of Corollary 1. By Theorem 5 in [2], (a) is equivalent to (b). Statement (c) implies (b), because both (c) and (26) of Theorem 5 in [2] are applicable to $a_{nk} := (\log_k n)^{-1/\alpha} / (n - k)^2$ for $k < n$ and $a_{nk} := 0$ otherwise, and to $m_n := n$. Hence (b) holds by Theorem 5 in [2].

Conversely, suppose there exist an array $(a_{nk})$ and a sequence $(m_n)$ satisfying the conditions in (c). Then there exists a constant $K$ such that $\sum_{k=1}^{m_n} |a_{nk}| \leq K (\log_k m_n)^{-1/\alpha} \forall n$. Define $c_n = K (\log_k n)^{-1/\alpha} \forall n$. Then $c_n$ satisfies all conditions of Theorem 2. Hence, by Theorem 2, (b) implies (c).

Proof of Corollary 2. Define $a_{nk} = n^{-1/\alpha} / (n - k)^2$ for $k < n$ and $a_{nk} = 0$ otherwise, and $m_n = n$. By Theorem 1 in [2] we see that (c) implies (b) with these $(a_{nk})$, $(m_n)$ and (a) is equivalent to (b). Lastly, arguing similarly as in the proof of Corollary 1, we can show that (b) implies (c).

Lemma 1. Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be a function such that $f(x)$ is monotone increasing on $[b, \infty)$ for some $b \geq 0$ and is bounded on $[0, b]$ if $b > 0$. Define $f^{-1}$ as the inverse function of $f$ restricted on $[b, \infty)$ and as any positive function on $[0, f(b))$ if $b > 0$. Then $E f(|X_1|) < \infty$ if and only if $\limsup_n |X_n| / f^{-1}(an) \leq 1$ a.s. for some and therefore for all real $a > 0$.

Proof. To show the “only if” part of the conclusion, suppose $E f(|X_1|) < \infty$. Let us fix any $a > 0$ and let $N$ be any positive integer such that $aN > f(b)$. We have, since $f^{-1}$ is monotone increasing on $[f(b), \infty)$,

$$\infty > E f(|X_1|) \geq \sum_{i=N}^{\infty} ai P \left( f^{-1}(ai) < |X_1| \leq f^{-1}(a(i + 1)) \right)$$

$$= a \left\{ N P \left( |X_1| > f^{-1}(aN) \right) + \sum_{i=N+1}^{\infty} P \left( |X_1| > f^{-1}(ai) \right) \right\}.$$
Consequently, since $X_n$ are i.i.d.,

$$\sum_{n=1}^{\infty} P \left( \frac{|X_n|}{f^{-1}(an)} > 1 \right) < \infty.$$ 

By Borel–Cantelli lemma [3], \( \limsup_{n} \frac{|X_n|}{f^{-1}(an)} \leq 1 \) a.s.

Conversely, let us fix any \( a > 0 \) and let \( N \) be any positive integer such that \( aN > f(b) \). Then we have

$$Ef(|X_1|) \leq Ef(|X_1|) 1_{\{|X_1| \leq f^{-1}(aN)\}} + Ef(|X_1|) 1_{\{|X_1| > f^{-1}(aN)\}}$$

$$\leq \max_{0 \leq x \leq f^{-1}(aN)} f(x) + \sum_{i=N}^{\infty} a(i+1) P \left( f^{-1}(ai) < |X_1| \leq f^{-1}(a(i+1)) \right).$$

The last sum, as is shown before,

$$\leq a + a \left\{ NP \left( |X_1| > f^{-1}(aN) \right) + \sum_{i=N+1}^{\infty} P \left( |X_1| > f^{-1}(ai) \right) \right\},$$

where the last sum is finite by Borel–Cantelli lemma, since \( X_n \) are independent and \( \limsup_{n} |X_n| / f^{-1}(an) \leq 1 \) a.s. Hence we obtain the finiteness of \( Ef(|X_1|) \), since \( f \) is bounded on \([0,b]\).

**Proof of Theorem 3.** To show a), set \( g(x) = x^a(\log^+ x)^{\beta} \) for \( x \geq 0 \). Then \( g(x) \) is monotone increasing and \( g^{-1} \) exists on the set \([1,\infty)\). For \( x \in [0,1) \) define \( g^{-1}(x) = 1 \). By Lemma 1 \( E|X_1|^a(\log^+ |X_1|)^{\beta} < \infty \) if and only if \( \limsup_{n} g^{-1}(an) \leq 1 \) a.s. for all \( a > 0 \). Because the exact form of \( g^{-1} \) is unknown, we shall estimate its behavior at \( +\infty \) by the following function. Put \( h(x) = \left( \frac{\alpha^\beta x}{\ln^\beta x} \right)^{1/\alpha} \) for \( x \geq 2 \) and \( h(x) = 1 \) for \( 0 \leq x < 2 \). We shall show that \( h(an) / g^{-1}(an) \rightarrow 1 \) for all \( a > 0 \).

Note that, for large enough \( n \),

$$\frac{g(h(an))}{g(g^{-1}(an))} = \left( \frac{\beta \log \alpha}{\log(an)} + 1 - \frac{\log \log \beta(an)}{\log(an)} \right)^\beta \rightarrow 1$$

as \( n \rightarrow \infty \). We shall prove a more general statement: for every two sequences \( 0 < a_n \rightarrow \infty, 0 < b_n \rightarrow \infty \) if \( g(a_n) / g(b_n) \rightarrow 1 \) then \( a_n / b_n \rightarrow 1 \). Suppose there are such \( (a_n) \) and \( (b_n) \), but in contrary \( \limsup_{n} a_n / b_n = c > 1 \). Then there exist \( n_i \) such that \( a_{n_i} / b_{n_i} \rightarrow c \). Hence \( a_{n_i}^\alpha / b_{n_i}^\alpha \rightarrow c^\alpha \) and \( a_{n_i} > b_{n_i} \), for large enough \( n_i \). So \( \limsup_{n_i} \log a_{n_i} / \log b_{n_i} \geq 1 \). Consequently \( \limsup_{n_i} a_{n_i}^\alpha \log^\beta a_{n_i} / (b_{n_i}^\alpha \log^\beta b_{n_i}) \geq c^\alpha > 1 \), which contradicts the assumption \( g(a_n) / g(b_n) \rightarrow 1 \). So \( \limsup_{n_i} a_{n_i} / b_{n_i} \leq 1 \). Similarly we obtain that \( \liminf_{n_i} a_{n_i} / b_{n_i} \geq 1 \), hence \( \lim_{n} a_{n} / b_{n} = 1 \).

So we have, by Lemma 1, \( E|X_1|^a(\log^+ |X_1|)^{\beta} < \infty \) if and only if for all \( a > 0 \)
\[
\limsup_n \frac{|X_n|}{h(an)} = \limsup_n \frac{|X_n|}{g^{-1}(an)} \leq \frac{g^{-1}(an)}{h(an)} \leq 1 \text{ a.s.,}
\]
which is equivalent to
\[
\limsup_n \frac{|X_n| \log^{\beta/\alpha_n}}{n^{1/\alpha}} \leq \frac{\alpha^{\beta/\alpha} a^{1/\alpha}}{n^{1/\alpha}} \text{ a.s.}
\]
for all \( a > 0 \). Since \( a > 0 \) can be chosen arbitrarily small the last inequality is equivalent to \( \limsup_n [X_n] \log^{\beta/\alpha_n} / n^{1/\alpha} = 0 \) a.s.

For proving b) set \( g(x) = e^{\alpha x^\beta} x^\gamma \) for \( x \geq 0 \). Also define \( h(x) = (\log^{\gamma/\beta} x - \log(\log^{\gamma/\beta} x)\gamma/\beta) \) for \( x \geq d \), and \( h(x) = 1 \) for \( 0 \leq x < d \), where \( d \) is chosen large enough such that \( h(x) \) is well defined. Then it is easy to show \( g(h(n)) / g^{-1}(n) \rightarrow 1 \), where \( g^{-1} \) is the inverse function of \( g \). As before, in order to show \( h(n) / g^{-1}(n) \rightarrow 1 \), let us prove that for every sequences \( a_n \rightarrow \infty \), \( b_n \rightarrow \infty \) if \( g(a_n) / g(b_n) \rightarrow 1 \) then \( a_n / b_n \rightarrow 1 \). Suppose we have such (\( a_n \)) and (\( b_n \)) but in contrary \( \limsup_n a_n / b_n = c > 1 \). Then there exist subsequences \( a_{n_k} \), \( b_{n_k} \) such that \( a_{n_k} / b_{n_k} \rightarrow c \). Therefore \( (a_{n_k} - b_{n_k}) / b_{n_k} \rightarrow c - 1 > 0 \) and \( (a_{n_k} - b_{n_k}) / b_{n_k} \rightarrow c^{-1} \rightarrow 1 - 1 > 0 \) if \( c > 0 \). Consequently, since \( b_{n_k} \rightarrow \infty \),
\[
\frac{g(a_{n_k})}{g(b_{n_k})} = e^{\alpha a_{n_k}^{\beta/\gamma} b^{\gamma/\beta}_{n_k}} \left[ \frac{a_{n_k}^{\beta/\gamma} b^{\gamma/\beta}_{n_k} - \alpha a_{n_k}^{\beta/\gamma} b^{\gamma/\beta}_{n_k}}{\alpha a_{n_k}^{\beta/\gamma} b^{\gamma/\beta}_{n_k}} \right] = e^{\alpha a_{n_k}^{\beta/\gamma} b^{\gamma/\beta}_{n_k} / b_{n_k}^{\gamma/\beta_{n_k}}} \rightarrow \infty,
\]
which contradicts the assumption. So \( \limsup_n a_n / b_n \leq 1 \). Similarly we have \( \liminf_n a_n / b_n \geq 1 \). Hence we obtain that \( \lim_n a_n / b_n = 1 \).

So we have, since \( h(n) / \log^{1/\beta} n \rightarrow 1 / \alpha^{1/\beta} \), \( \limsup_n \frac{|X_n|}{g^{-1}(n)} \leq 1 \text{ a.s.} \text{ if and only if} \limsup_n \frac{|X_n|}{h(n)} \leq 1 \text{ a.s.} \text{ if and only if} \limsup_n \frac{|X_n|}{\log^{1/\beta} n} \leq 1 \text{ a.s.} \]

**Proof of Corollary 3.** Defining \( c_n = K n^{-1/\alpha} \log^{\beta/\alpha} n \) and acting similarly as in the proof of Corollary 1, by Theorem 2, we obtain that (b) implies (c). Conversely (c) implies (b) because by (c) \( \lim_n n^{-1/\alpha} \log^{\beta/\alpha} n \sum_{k=1}^n (n-k+1)^{-2} X_k = 0 \) a.s. Then by Lemma 3 in [2] we obtain (b), arguing similarly as in the proof of Theorem 1 in [2].

**Proof of Corollary 4.** Set \( b_m := \sum_{k=1}^m |a_{nk}| \) and define \( c_k, \ k = 1, 2, ..., \) such that
\[
c_k \log^{1/\beta} k = \max_{n,m \geq k} \{ b_m \log^{1/\beta} m_n \}.
\]
We shall show that \( c_n \) satisfies all conditions of Theorem 2.

We have \( c_n \log^{1/\beta} n \) is monotone non-increasing and tending to zero, since \( b_m \log^{1/\beta} m_n \rightarrow 0 \) by the assumption. Hence \( c_n = (c_n \log^{1/\beta} n)(\log^{-1/\beta} n) \), as the product of two monotone non-increasing and tending to zero sequences, has the same properties. By the definition of \( c_n \) we have \( b_m \leq c_m \) for all \( n \). By Theorem 3 \( \limsup_n \frac{|X_n|}{\log^{1/\beta} n} \leq \frac{1}{\alpha^{1/\beta}} \) a.s. Hence almost surely
\[ \limsup_n |X_n c_n| = \limsup_n \frac{|X_n|}{\log^{1/\beta} n} \cdot c_n \log^{1/\beta} n \leq \frac{1}{\alpha^{1/\beta}} \lim_n c_n \log^{1/\beta} n = 0. \]

So \( c_n \) satisfies all conditions of Theorem 2. By Theorem 2 we obtain the conclusion.

References