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Hartogs Spaces, Spaces Having the Forelli Property and Hartogs Holomorphic Extension Spaces

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Abstract. In this paper the notions on Hartogs spaces and Forelli spaces are given. The invariance of Hartogs and Forelli spaces through holomorphic coverings is established. Moreover, under the assumption on the holomorphically convex Kählerity we show that the three following classes of complex spaces: the Hartogs holomorphic extension spaces, the Hartogs spaces and the spaces having the Forelli property are coincident.

1. Introduction

During the past 20 years, the study of various forms of Hartogs theorem has been done by many authors. Terada [12] has shown that if $f(z, w)$ is a complex-valued function defined for $z \in U \subset \mathbb{C}^n$, $w \in V \subset \mathbb{C}^m$, U and V are open sets, and if f is holomorphic in w for all $z \in U$ and holomorphic in z for all w in some non-pluripolar set $A \subset V$ then f is holomorphic on $U \times V$. Later, Siciak [9], Zaharjuta [14], Nguyen and Zeriahi [13] investigated results on the holomorphic extendability of complex-valued separately holomorphic functions defined on sets of the form $(U \times F) \cup (E \times V)$ where $E \subset U$, $F \subset V$ are either L-regular or non-pluripolar. More recently, Shiffman [11] extended the results of Tereda for separately holomorphic maps with values in complex spaces having the Hartogs holomorphic extension property. Among the findings of Shiffmann in [11], the following result is interesting: let X be a complex space having Hartogs extension property and U, V be domains in $\mathbb{C}^N$, $\mathbb{C}^M$ respectively, and

A be a non-pluripolar subset of V. Suppose that $f : U \times V \to X$ so that
(i) $f_z \in Hol(V, X)$ for almost all $z \in U$,
(ii) $f^w \in Hol(U, X)$ for all $w \in A$.
Then f is equal almost everywhere to a holomorphic map $\widetilde{f} : U \times V \to X$.

This fact marks the start of our paper. Moreover in this paper we wish to improve a result which has been published in [2]. Namely in [2] they presented notions on a complex space having the separately holomorphic property (briefly (SHP)) and a complex space having the strong separately holomorphic property (briefly (SSHP)). However, possibly, the properties given in that paper are not suitable then they do not prove the invariance of these notions through a holomorphic covering. Hence, in this paper we give a new notion about Hartogs spaces and establish the invariance of these spaces under holomorphic coverings. Secondly the notion about spaces having the Forelli property is presented. Finally we study the relation between Hartogs holomorphic extension spaces, Hartogs spaces and spaces having the Forelli property.

2. Preliminaries

All complex spaces considered in this paper are assumed to be reduced and to have a countable topology.

Let X be a complex space and $U \subset \mathbb{C}^N$ be an open set. We let $\mathrm{Hol}\,(U, X)$ denote the set of holomorphic maps from U to X.

For a map $f : U \times V \to X$ where $U \subset \mathbb{C}^N$, $V \subset \mathbb{C}^M$ are open sets, we let $f_z : V \to X$ and $f^w : U \to X$ be given $f_z(w) = f^w(z) = f(z, w)$ for $z \in U$, $w \in V$.

We say that f is separately holomorphic if
(i) f_z Hol$\,(V, X)$ for all $z \in U$
(ii) f^w Hol$\,(U, X)$ for all $w \in V$.

Now, let U be an open subset of $\mathbb{C}^N$ and $\varphi : U \to [-\infty, +\infty)$ be an upper semicontinuous function which is not identical $-\infty$ on any connected component of U. The function φ is said to be plurisubharmonic on U if for each $a \in U$, $b \in \mathbb{C}^N$, the function $\lambda \longmapsto \varphi(a+\lambda b)$ is subharmonic on the set $\{\lambda \in \mathbb{C} : a+\lambda b \in U\}$. In this case, we write $\varphi \in PSH(U)$.

Let U be an open subset of $\mathbb{C}^N$ and $E \subset U$. The set E is said to be pluripolar if for each $a \in E$ there exists a connected neighborhood V of a, $V \subset U$, and a function $\varphi \in PSH(V)$ such that $E \cap V \subset \varphi^{-1}(-\infty)$. A result of Josefson [5] showed that $E \subset U$ is pluripolar if and only if there exists $\varphi \in PSH(U)$, $\varphi \not\equiv -\infty$ and $E \subset \varphi^{-1}(-\infty)$.

3. Hartogs Spaces, Spaces Having the Forelli Property and Hartogs Holomorphic Extension Spaces

This section is devoted to giving our notion on Hartogs spaces, and spaces having the Forelli property and the relation between these spaces and Hartogs holomorphic extension spaces.

First we give the following

Definition 3.1. *Let X be a complex space. X is called a Hartogs space if for every domain $U \subset \mathbb{C}^N$, $V \subset \mathbb{C}^M$ and every map $f : U \times V \to X$ satisfying the conditions*

(i) *There exists a pluripolar subset $E \subset U$ such that for all $z \in U \setminus E$, f_z* Hol (V, X)

(ii) *There exists a non pluripolar subset $F \subset V$ such that for all $w \in F$ the map f^w* Hol (U, X)

then there exists a holomorphic map $\widehat{f} : U \times V \to X$ and a pluripolar subset $M \subset U \times V$ such that

$$\widehat{f}\big|_{U\times V\setminus M} = f\big|_{U\times V\setminus M}.$$

Remark. In fact by the locality and the uniqueness of holomorphic extension then in the Definition 3.1 we can assume that U and V are balls in $\mathbb{C}^N$ and $\mathbb{C}^M$ respectively.

Now we recall the definition of the Hartogs holomorphic extension space (detailly see [11]).

Definition 3.2. *Let X be a complex space. X is called a Hartogs holomorphic extension space if every holomorphic map f from a Riemann domain D over a Stein manifold to X can be holomorphically extended to its envelope of holomorphy $\widehat{D}$.*

By Theorem 5 in [11] it follows that every Hartogs holomorphic extension space is a Hartogs space .

Next we give the following.

Definition 3.3. *Let X be a complex space.*

X is said to have the Forelli property (briefly, $X \in (FP)$) if every map $f : \mathbb{B}^N(0,1) \to X$ such that f is of $\mathcal{C}^\infty$ - class in a neighborhood of $0 \in \mathbb{B}^N(0,1)$ and $f\big|_{\mathbb{B}^N(0,1)\cap \ell}$ is holomorphic for every complex line ℓ through $0 \in \mathbb{B}^N(0,1)$ then f is holomorphic on $\mathbb{B}^N(0,1)$, where $\mathbb{B}^N(0,1) = \Big\{z \in \mathbb{C}^N : \|z\| < 1\Big\}$.

From the Forelli theorem in [8], it follows that $\mathbb{C}^M$ and, hence, every Stein space has the Forelli property.

The following result is one of the main results of this paper.

Theorem 3.4. *Let X, Y be complex spaces and $\theta : X \to Y$ be a holomorphic covering. Then X is a Hartogs space if and only if so is Y.*

Proof. Sufficiency

Without loss of generality we may assume that $U = \mathbb{B}^N(0,1) \subset \mathbb{C}^N$ and $V = \mathbb{B}^M(0,1) \subset \mathbb{C}^M$ are the unit balls in $\mathbb{C}^N$ and $\mathbb{C}^M$ respectively. Let $E \subset U$ be a pluripolar subset and $F \subset V$ a non-pluripolar subset and $f : U \times V \to X$ a map satisfying all conditions in the Definition 3.1. Put $h = \theta \cdot f$. Notice that h

satisfies also the conditions of the Definition 3.1. By the hypothesis there exists a holomorphic map $\widehat{h} : U \times V \to Y$ and a pluripolar subset $M \subset U \times V$ such that

$$\widehat{h}\big|_{U\times V\setminus M} = h\big|_{U\times V\setminus M}. \tag{1}$$

Let $\widehat{M} = (E \times V) \cup M$. Then $\widehat{M} \subset U \times V$ is a pluripolar subset. Now for each $0 < r < 1$ it suffices to show that there exists a holomorphic map $g_r : \mathbb{B}^N(0,r) \times V \to X$ such that $g_r = f$ on $\mathbb{B}^N(0,r) \times V \setminus \widehat{M}$. Fix $0 < r < 1$. As proven in Theorem 5 of [11], we can find a ball $\mathbb{B}_0 \subset\subset V$ and a holomorphic map $f_r : \mathbb{B}^N(0,r) \times \mathbb{B}_0 \to X$ such that

$$f_r\big|_{(\mathbb{B}^N(0,r)\setminus E)\times\mathbb{B}_0} = f\big|_{(\mathbb{B}^N(0,r)\setminus E)\times\mathbb{B}_0}. \tag{2}$$

Hence

$$f_r\big|_{(\mathbb{B}^N(0,r)\times\mathbb{B}_0)\setminus\widehat{M}} = f\big|_{(\mathbb{B}^N(0,r)\times\mathbb{B}_0)\setminus\widehat{M}}. \tag{3}$$

Choose $(x_0, y_0) \in \Big(\mathbb{B}^N(0,r) \times \mathbb{B}_0\Big) \setminus \widehat{M}$. Let $g_r : \mathbb{B}^N(0,r) \times V \to X$ be a holomorphic lift of $\widehat{h}\big|_{\mathbb{B}^N(0,r)\times V}$ satisfying the condition

$$g_r(x_0, y_0) = f(x_0, y_0).$$

Then $\theta \cdot g_r\big|_{\mathbb{B}^N(0,r)\times V} = \widehat{h}\big|_{\mathbb{B}^N(0,r)\times V}$. On the other hand, since

$$\begin{aligned} &\theta f_r\big|_{(\mathbb{B}^N(0,r)\times\mathbb{B}_0)\setminus\widehat{M}} = \theta f\big|_{(\mathbb{B}^N(0,r)\times\mathbb{B}_0)\setminus\widehat{M}} \\ &= h\big|_{(\mathbb{B}^N(0,r)\times\mathbb{B}_0)\setminus\widehat{M}} = \widehat{h}\big|_{(\mathbb{B}^N(0,r)\times\mathbb{B}_0)\setminus\widehat{M}} = \theta g_r\big|_{(\mathbb{B}^N(0,r)\times\mathbb{B}_0)\setminus\widehat{M}} \end{aligned}$$

and

$$g_r(x_0, y_0) = f(x_0, y_0) = f_r(x_0, y_0),$$

then

$$g_r\big|_{\mathbb{B}^N(0,r)\times\mathbb{B}_0} = f_r.$$

Now by the holomorphicity of f_z on V for $z \in \mathbb{B}^N(0,1) \setminus E$ and from the equality

$$f_z\big|_{\mathbb{B}_0} = g_{r,z}\big|_{\mathbb{B}_0}$$

it follows that $f = g_r$ on $\Big(\mathbb{B}^N(0,r) \times V\Big) \setminus \widehat{M}$ and the conclusion follows.

Necessity. As in the proof of the sufficient condition we may assume that

$$U = B^N(0,1) = \big\{z \in \mathbb{C}^N : \|z\| < 1\big\}$$

and

$$V = B^M(0,1) = \big\{w \in \mathbb{C}^M : \|w\| < 1\big\}.$$

Let $f : U \times V \to Y$ be a map satisfying all conditions of Definition 3.1. Given an arbitrary $0 < r < 1$. The proof of Theorem 5 in [11] implies the existence of a ball $B_0 \subset V$ such that $B_0 \cap F$ is not pluripolar and a holomorphic map $f_r : B^N(0,r) \times B_0 \to Y$ such that

$$f_r\big|_{(B^N(0,r)\setminus E)\times B_0} = f\big|_{(B^N(0,r)\setminus E)\times B_0}.$$

Let $h_r : B^N(0,r)\times B_0 \to X$ be a holomorphic lift of f_r. Fix $y_0 \in B_0$. For $x \in B^N(0,r)\setminus E$, by the hypothesis, $f_x \in Hol(V,Y)$. Let $\widehat{f}_{r,x}$ be a holomorphic lift of f_x satisfying the condition

$$\widehat{f}_{r,x}(y_0) = h_r(x,y_0).$$

Notice that $\widehat{f}_{r,x} \in Hol(V,X)$. Since

$$\theta\widehat{f}_{r,x}\big|_{B_0} = f_x\big|_{B_0} = \theta h_{r,x}\big|_{B_0}$$

and

$$\widehat{f}_{r,x}(y_0) = h_r(x,y_0)$$

then we deduce that

$$\widehat{f}_{r,x}(y) = h_r(x,y)$$

for all $y \in B_0$ and $x \in (B^N(0,r)\setminus E)$. Now we consider the map $g_r : B^N(0,r)\times V \to X$ given by

$$g_r(x,y) = \begin{cases} h_r(x,y) & \text{for} \quad (x,y) \in B^N(0,r)\times B_0 \\ \widehat{f}_{r,x}(y) & \text{for} \quad x \in B^N(0,r)\setminus E,\ y \in V \\ a & \text{for} \quad x \in E,\ y \in V\setminus B_0 \end{cases}$$

where $a \in X$ is some point.

Now for $x \in B^N(0,r)\setminus E$, $g_{r,x}(y) = \widehat{f}_{r,x}(y)$ for all $y \in V$ and, hence, $g_{r,x} \in Hol(V,X)$. On the other hand, for $y \in B_0$, $g_r^y(x) = h_r^y(x)$, $x \in B^N(0,r)$. Hence $g_r^y \in Hol(B^N(0,r),X)$. By the hypothesis, there exists a holomorphic map $\widetilde{g}_r : B^N(0,r)\times V \to X$ and a pluripolar subset $M(r) \subset B^N(0,r)\times V$ such that

$$\widetilde{g}_r\big|_{(B^N(0,r)\times V)\setminus M(r)} = g_r\big|_{(B^N(0,r)\times V)\setminus M(r)}.$$

Put $\widetilde{M}(r) = M(r)\cup(E\times V)$. Then $\widetilde{M}(r) \subset U\times V$ is pluripolar and

$$\widetilde{g}_r\big|_{(\mathbb{B}^N(0,r)\times V)\setminus \widetilde{M}(r)} = g_r\big|_{(\mathbb{B}^N(0,r)\times V)\setminus \widetilde{M}(r)}.$$

Set $\widetilde{h}_r = \theta\widetilde{g}_r$. Then $\widetilde{h}_r : B^N(0,r)\times V \to Y$ is holomorphic and

$$\widetilde{h}_r\big|_{(B^N(0,r)\times V)\setminus \widetilde{M}(r)} = f\big|_{(B^N(0,r)\times V)\setminus \widetilde{M}(r)}.$$

Now let $r < r'$. Then $\widetilde{h}_r$ and $\widetilde{h}'_r$ are holomorphic on $\mathbb{B}^N(0,r)\times V$ and

$$\begin{aligned} \widetilde{h}_r\big|_{B^N(0,r)\times V\setminus \widetilde{M}(r)\cup\widetilde{M}(r')} &= f\big|_{B^N(0,r)\times V\setminus \widetilde{M}(r)\cup\widetilde{M}(r')} \\ &= \widetilde{h}_{r'}\big|_{B^N(0,r)\times V\setminus \widetilde{M}(r)\cup\widetilde{M}(r')}. \end{aligned} \tag{4}$$

From the pluripolarity of $\widetilde{M}(r)\cup\widetilde{M}(r')$ and (4) we derive that

$$\widetilde{h}_r\big|_{B^N(0,r)\times V} = \widetilde{h}_{r'}\big|_{B^N(0,r)\times V}.$$

Now choose an increasing sequence of positive numbers $\{r_n\} \uparrow 1$ and by using the above argument we claim that the family $\{\widetilde{h}_{r_n} : 0 < r_n < 1\}$ defines a holomorphic map $\widetilde{h} : U \times V \to Y$ such that

$$\widetilde{h}|_{U\times V\setminus\widetilde{M}} = f|_{U\times V\setminus\widetilde{M}}.$$

where $\widetilde{M} = \bigcup_{n=1}^{\infty} \widetilde{M}(r_n)$ is a pluripolar subset of $U \times V$.
Theorem 3.4 is completely proved. ■

Next, as above, we deal with the invariance of the Forelli property through holomorphic coverings. Namely, we prove the following.

Theorem 3.5. *Let $\theta : X \to Y$ be a holomorphic bundle with Stein fibers. Then*
(i) *If $Y \in (FP)$ then so is X.*
(ii) *If θ is a holomorphic covering and $X \in (FP)$ then so is Y.*

Proof. (i) Let Y have the (FP) and $f : \mathbb{B}^N(0,1) \to X$ be a map satisfying all conditions of the Definition 3.3. Then $g = \theta.f : \mathbb{B}^N(0,1) \to Y$ is also a map satisfying all conditions as f. By the hypothesis $g : \mathbb{B}^N(0,1) \to Y$ is holomorphic. On the other hand, the Forelli theorem for scalar functions in [8] implies that there exists $0 < \alpha < 1$ such that $f : \mathbb{B}^N(0,\alpha) \to X$ is holomorphic, where $\mathbb{B}^N(0,\alpha) = \{z \in \mathbb{C}^N : \|z\| < \alpha\}$. As in [8] consider the map $\varphi = \varphi_N : \mathbb{B}^N_{*,N} \to \mathbb{C}^N$ given by

$$\varphi = \varphi_N(z_1, \cdots, z_N) = \Big(\frac{z_1}{z_N}, \cdots, \frac{z_{N-1}}{z_N}, z_N\Big),$$

where

$$\mathbb{B}^N_{*,N} = \{(z_1, \cdots, z_N) \in \mathbb{B}^N(0,1) : z_N \neq 0\}.$$

It is clear that $\varphi = \varphi_N$ is biholomorphic onto its image. Set

$$T = \varphi(\mathbb{B}^N_{*,N}) = \bigcup_{R>0} \mathbb{B}^{N-1}_R \times \triangle^*\Big(0, \sqrt{\frac{1}{1+R^2}}\Big)$$

and $h = f \circ \varphi^{-1} : T \to X$. Fix $R > 0$. Then h is holomorphic on $\mathbb{B}^{N-1}_R \times \triangle^*\Big(0, \sqrt{\frac{\alpha^2}{1+R^2}}\Big)$ and by the hypothesis $h(z', .)$ is holomorphic on $\triangle^*\Big(0, \sqrt{\frac{1}{1+R^2}}\Big)$ for all $z' \in \mathbb{B}^{N-1}_R$. Now we show that there exists a pluripolar set $S(R) \subset B^{N-1}_R$ such that h is holomorphic on $\Big(\mathbb{B}^{N-1}_R \setminus S(R)\Big) \times \triangle^*\Big(0, \sqrt{\frac{1}{1+R^2}}\Big)$. Take a strictly decreasing sequence $\{r_n\}$ of positive numbers satisfying $0 < r_n < \sqrt{\frac{\alpha^2}{1+R^2}}$ and $\{r_n\} \downarrow 0$ as $n \to \infty$. For each $n \geq 1$ consider h on $\mathbb{B}^{N-1}_R \times \Big\{r_n \leq |z_N| \leq \sqrt{\frac{1}{1+R^2}} - r_n\Big\}$. Obviously, h is holomorphic on $\mathbb{B}^{N-1}_R \times \Big\{r_n < |z_N| < \sqrt{\frac{\alpha^2}{1+R^2}}\Big\}$ and for each $z' \in \mathbb{B}^{N-1}_R$ $h(z', .)$ is holomorphic on $\Big\{r_n \leq |z_N| \leq \sqrt{\frac{1}{1+R^2}} - r_n\Big\}$. Theorem 1 in [11] implies that there exists a closed pluripolar set $S_n(R) \subset B^{N-1}_R$ such that h is holomorphic on $\Big(\mathbb{B}^{N-1}_R \setminus S_n(R)\Big) \times \Big\{r_n \leq |z_N| \leq \sqrt{\frac{1}{1+R^2}} - r_n\Big\}$. Moreover

we can assume that $S_n(R) \subset S_{n+1}(R)$ for $n \geq 1$. Put $S(R) = \bigcup\limits_{n=1}^{\infty} S_n(R)$. Then $S(R) \subset \mathbb{B}_R^{N-1}$ is a pluripolar set and h is holomorphic on $\Big(\mathbb{B}_R^{N-1} \setminus S(R)\Big) \times \triangle^*\Big(0, \sqrt{\frac{1}{1+R^2}}\Big)$. Now we prove that h is holomorphic on $\mathbb{B}_R^{N-1} \times \triangle^*\Big(0, \sqrt{\frac{1}{1+R^2}}\Big)$. Let $z'_0 \in S$ be an arbitrary point.
Set

$$G = \{z_N \in \triangle^*\Big(0, \sqrt{\frac{1}{1+R^2}}\Big) : h \text{ is holomorphic at} (z'_0, z_N)\}.$$

Obviously, G is open in $\triangle^*\Big(0, \sqrt{\frac{1}{1+R^2}}\Big)$. Now we prove that G is closed in $\triangle^*\Big(0, \sqrt{\frac{1}{1+R^2}}\Big)$. Let $z_N^o \in \partial G \bigcap \triangle^*\Big(0, \sqrt{\frac{1}{1+R^2}}\Big)$. Then $z_0 = (z'_0, z_N^o) \in \mathbb{B}_R^{N-1} \times \triangle^*\Big(0, \sqrt{\frac{1}{1+R^2}}\Big)$. Let $\widehat{g} = g \circ \varphi^{-1}$. Then $\widehat{g}$ is holomorphic on $\mathbb{B}_R^{N-1} \times \triangle^*\Big(0, \sqrt{\frac{1}{1+R^2}}\Big)$. Choose a Stein neighborhood V of $\widehat{g}(z_0)$ in Y. Then $\theta^{-1}(V)$ is also Stein. Next we take a neighborhood $U' \times U_N$ of $z_0 = (z'_0, z_N^o)$ in $\mathbb{B}_R^{N-1} \times \triangle^*\Big(0, \sqrt{\frac{1}{1+R^2}}\Big)$ such that $\widehat{g}(U' \times U_N) \subset V$. Hence $h : (U' \setminus S) \times U_N \to \theta^{-1}(V)$ is holomorphic and if we put $U_0 = U_N \bigcap G$ then $h : U' \times U_0 \to \theta^{-1}(V)$ is holomorphic. Theorem 3 in [11] implies that h is holomorphic on $U' \times U_N$. Hence $z_N^o \in G$. Thus G is an open - closed subset in $\triangle^*\Big(0, \sqrt{\frac{1}{1+R^2}}\Big)$. On the other hand, since $h : \mathbb{B}_R^{N-1} \times \triangle^*\Big(0, \sqrt{\frac{\alpha^2}{1+R^2}}\Big) \to X$ is holomorphic then $G \neq \emptyset$. Hence $G = \triangle^*\Big(0, \sqrt{\frac{1}{1+R^2}}\Big)$ and h is holomorphic on $\mathbb{B}_R^{N-1} \times \triangle^*\Big(0, \sqrt{\frac{1}{1+R^2}}\Big)$. Notice that $R > 0$ is arbitrary then h is holomorphic on T and, hence, f is holomorphic on $\mathbb{B}_\alpha^N \bigcup \bigcup\limits_{j=1}^{N} \mathbb{B}_{*,j}^N = \mathbb{B}^N(0,1)$ and the desired conclusion follows.

(ii) Assume that θ is a holomorphic covering and $X \in (FP)$. Given $f : \mathbb{B}^N(0,1) \to Y$ a map satisfying all conditions of the Definition 3.3. Fix $x_0 \in X$. For each complex line ℓ through $0 \in \mathbb{B}^N(0,1)$ there exists a unique holomorphic lift $g_\ell : \mathbb{B}^N(0,1) \bigcap \ell \to X$ such that

$$g_\ell(0) = x_0 \ \text{ and } \ \theta g_\ell = f\big|_{\mathbb{B}^N(0,1) \bigcap \ell}.$$

Define $g : \mathbb{B}^N(0,1) \to X$ given by

$$g\big|_{\mathbb{B}^N(0,1) \bigcap \ell} = g_\ell.$$

It remains to check that g is of $\mathcal{C}^\infty$-class in a neighborhood of $0 \in \mathbb{B}^N(0,1)$. Take a neighborhood V of $f(0)$ in Y such that $\theta^{-1}(V) = \sqcup_{j=1}^\infty V_j$ where $\theta\big|_{V_j} : V_j \cong V$. Choose $\delta > 0$ such that $f(\mathbb{B}^N(0,\delta)) \subset V$ and f is of $\mathcal{C}^\infty$-class on it. Then $g(\mathbb{B}_\delta^N) \subset \theta^{-1}(V)$. Since $\theta(x_0) = f(0)$ then there exists $j \geq 1$ such that $x_0 \in V_j$. Hence $g(\mathbb{B}^N(0,\delta)) \subset V_j$. This yields that g is of $\mathcal{C}^\infty$-class on $\mathbb{B}^N(0,\delta)$. By the hypothesis g and, hence, f is holomorphic on $\mathbb{B}^N(0,1)$. Theorem 3.5 is completely proved. ∎

Now we present the following result about the relation between the three classes of complex spaces: the Hartogs holomorphic extension spaces, the Hartogs and the Forelli ones.

Theorem 3.6. *Let X be a holomorphically convex Kähler complex space. Then the three following assertions are equivalent:*
(i) *X is a Hartogs holomorphic extension space;*
(ii) *X is a Hartogs space;*
(iii) *X is a space having the Forelli property.*

Proof. (i) $\Rightarrow$ (ii). Let X be a Hartogs holomorphic extension space and $U \subset \mathbb{C}^N$, $V \subset \mathbb{C}^M$ domains, $E \subset U$ a pluripolar subset, $F \subset V$ a non-pluripolar subset and $f : U \times V \to X$ a map satisfying the condition: $\forall z \in U \setminus E, f_z \in$ Hol (V, X) and $\forall w \in F, f^w \in$ Hol (U, X). Let $\{U_n\}_{n=1}^{\infty}$ be an increasing sequence of relatively compact subdomains of U with $U_n \subset\subset U_{n+1} \subset\subset U$ and $\bigcup_{n=1}^{\infty} U_n = U$. By virtue of the proof of Theorem 5 in [11], for each $n \geq 1$ there exists a holomorphic extension $g_n : U_n \times V \to X$ with

$$g_n\big|_{(U_n \setminus E) \times V} = f\big|_{(U_n \setminus E) \times V}. \tag{5}$$

By setting $M = E \times V \subset U \times V$ we claim that M is a pluripolar subset of $U \times V$ and for every $n < m$ we have

$$g_n\big|_{(U_n \times V) \setminus M} = f\big|_{(U_n \times V) \setminus M} = g_m\big|_{(U_n \times V) \setminus M}. \tag{6}$$

Hence,

$$g_n\big|_{(U_n \times V)} = g_m\big|_{(U_n \times V)}. \tag{7}$$

The equality (7) says that the family of holomorphic maps $\{g_n : n \geq 1\}$ defines a holomorphic map $g : U \times V \to X$ with

$$g\big|_{U \times V \setminus M} = f\big|_{U \times V \setminus M}.$$

and the conclusion follows.
(ii) $\Rightarrow$ (iii). Given $f : \mathbb{B}^N(0,1) \to X$ a map as in the statement of Definition 3.4. As in the proof of (i) of Theorem 3.5 there exists $0 < \alpha < 1$ such that $f : \mathbb{B}^N(0,\alpha) \to X$ is holomorphic where $\mathbb{B}^N(0,\alpha) = \{z \in \mathbb{C}^N : \|z\| < \alpha\}$. Next consider the map $\varphi = \varphi_N : \mathbb{B}^N_{*,N} \to \mathbb{C}^N$ given by

$$\varphi = \varphi_N(z_1, \cdots, z_N) = \Big(\frac{z_1}{z_N}, \cdots, \frac{z_{N-1}}{z_N}, z_N\Big)$$

where

$$\mathbb{B}^N_{*,N} = \big\{(z_1, \cdots, z_N) \in \mathbb{B}^N(0,1) : z_N \neq 0\big\}.$$

Set

$$T = \varphi(\mathbb{B}^N_{*,N}) = \bigcup_{R>0} \mathbb{B}^{N-1}_R \times \triangle^*\Big(0, \sqrt{\frac{1}{1+R^2}}\Big)$$

and $h = f\circ\varphi^{-1} : T \to X$. Fix $R > 0$. Then as in the proof of Theorem 3.5 there exists a pluripolar set $S(R) \subset \mathbb{B}_R^{N-1}$ such that h is holomorphic on $\big(\mathbb{B}_R^{N-1} \setminus S(R)\big) \times \triangle^*\big(0, \sqrt{\frac{1}{1+R^2}}\big)$. Now by applying the definition of a Hartogs space to $E = S(R), F = \triangle^*\Big(0, \sqrt{\frac{\alpha^2}{1+R^2}}\Big)$, we deduce that there exists a holomorphic map $\widehat{h} : \mathbb{B}_R^{N-1} \times \triangle^*\big(0, \sqrt{\frac{1}{1+R^2}}\big) \to X$ and a pluripolar subset $M(R) \subset \mathbb{B}_R^{N-1} \times \triangle^*\big(0, \sqrt{\frac{1}{1+R^2}}\big)$ with

$$\widehat{h}\Big|_{\mathbb{B}_R^{N-1}\times\triangle^*\big(0,\sqrt{\frac{1}{1+R^2}}\big)\setminus M(R)} = h\Big|_{\mathbb{B}_R^{N-1}\times\triangle^*\big(0,\sqrt{\frac{1}{1+R^2}}\big)\setminus M(R)}. \tag{8}$$

Notice that $\widetilde{S}(R) = S(R) \times \triangle^*\Big(0, \sqrt{\frac{1}{1+R^2}}\Big)$ is a pluripolar subset in $\mathbb{B}_R^{N-1} \times \triangle^*\big(0, \sqrt{\frac{1}{1+R^2}}\big)$ and as in the proof of Theorem 3.6 $h\Big|_{\mathbb{B}_R^{N-1}\times\triangle^*\big(0,\sqrt{\frac{1}{1+R^2}}\big)\setminus\widetilde{S}(R)}$ is holomorphic. Now we need to prove that h is holomorphic on $\mathbb{B}_R^{N-1} \times \triangle^*\big(0, \sqrt{\frac{1}{1+R^2}}\big)$. From the above argument we can assume that $M(R) \subset \widetilde{S}(R)$. Let $z_0' \in S(R)$ be an arbitrary point. Set

$$G = \Big\{z_N \in \triangle^*\Big(0, \sqrt{\frac{1}{1+R^2}}\Big) : h \text{ is holomorphic at } (z_0', z_N)\Big\}.$$

As in Theorem 3.5 we need to prove that G is closed in $\triangle^*\big(0, \sqrt{\frac{1}{1+R^2}}\big)$. Let $z_N^0 \in \partial G \cap \triangle^*\big(0, \sqrt{\frac{1}{1+R^2}}\big)$. Then $z_0 = \big(z_0', z_N^0\big) \in \mathbb{B}_R^{N-1} \times \triangle^*\big(0, \sqrt{\frac{1}{1+R^2}}\big)$. Choose a Stein neighborhood V of $\widehat{h}(z_0)$ in X. Next we take a neighborhood $U' \times U_N$ of $z_0 = \big(z_0', z_N^0\big)$ in $\mathbb{B}_R^{N-1} \times \triangle^*\big(0, \sqrt{\frac{1}{1+R^2}}\big)$ such that $\widehat{h}\big(U' \times U_N\big) \subset V$. Then from (8) and $M(R) \subset \widetilde{S}(R) = S(R) \times \triangle^*\big(0, \sqrt{\frac{1}{1+R^2}}\big)$ we infer that $h : \big(U' \setminus S(R)\big) \times U_N \to V$ is holomorphic. On the other hand, if we put $V_0 = U_N \cap G$ then $h : U' \times V_0 \to V$ is holomorphic. Now Theorem 3 in [11] implies that h is holomorphic on $U' \times U_N$. Hence $z_n^0 \in G$. Thus G is an open-closed subset of $\triangle^*\big(0, \sqrt{\frac{1}{1+R^2}}\big)$ and because $h : \mathbb{B}_R^{N-1} \times \triangle^*\big(0, \sqrt{\frac{\alpha^2}{1+R^2}}\big) \to X$ is holomorphic then $G \neq \emptyset$. This leads to the equality $G = \triangle^*\big(0, \sqrt{\frac{1}{1+R^2}}\big)$. Now we conclude that h is holomorphic on T and f is holomorphic on $\mathbb{B}_\alpha^N \cup \cup_{j=1}^N \mathbb{B}_{*,j}^N = \mathbb{B}^N(0, 1)$.

(iii) $\Rightarrow$ (i). Let X satisfy (iii) but X is not a Hartogs holomorphic extension space. Then by [3] and [4] we can find a non-constant holomorphic map $\varphi : \mathbb{C}P^1 \longrightarrow X$. Since φ is not constant then $\varphi : \mathbb{C}P^1 \longrightarrow \varphi(\mathbb{C}P^1)$ is a branched covering. Consider the map $f : \mathbb{C}^2 \longrightarrow \mathbb{C}P^1$ given by

$$f(z, w) = \begin{cases} [(z-1) : (w-1)] & \text{if } (z, w) \neq (1, 1) \\ [1 : 1] & \text{if } (z, w) = (1, 1) \end{cases}$$

Then f is holomorphic in $\mathbb{B}^2$ and, consequently, so is φf. Given $a = (z^0, w^0) \in \mathbb{C}^2 \setminus \{(0, 0)\}$. If $z^0 \neq w^0$ then $\{\lambda \in \mathbb{C} : \lambda z^0 - 1 = 0 = \lambda w^0 - 1\} = \emptyset$.

Hence the restriction of f to the complex line $\ell = \{(\lambda z^0, \lambda w^0) : \lambda \in \mathbb{C}\}$ is the function $\widetilde{f}(\lambda) = [\lambda z^0 - 1 : \lambda w^0 - 1]$ which is holomorphic on ℓ. For the case $z^0 = w^0$ we notice that the restriction of f to the complex line $d = \{(\lambda z^0, \lambda w^0) : \lambda \in \mathbb{C}\}$ is equal to $[1 : 1]$. Hence $\varphi \circ f$ is holomorphic on every complex line through $0 \in \mathbb{C}^2$ and by the hypothesis it is holomorphic on $\mathbb{C}^2$. In particular it is continuous at $(1, 1)$.

Since φ is a branched covering we deduce that the set $B = \{\lim f(z, w) : (z, w) \to (1, 1)\}$ is finite. This is impossible, because $\{\lim_{(z,w)\to(0,0)} \frac{z}{w}\} \subset B$.

Remark. Theorem 3.6 is not true if the assumption on the Kählerity of X is removed. Indeed, suppose in order to get a contradiction that the above theorem is still true without the assumption on the Kählerity of X. Consider the Hopf surface $H = \mathbb{C} \setminus \{0\}/z \sim 2z$. Then H is not a Kähler manifold. Let $\theta : \mathbb{C} \setminus \{0\} \to H$ be the canonical map. Then θ is a holomorphic covering. Since $\mathbb{C} \setminus \{0\}$ is a Hartogs holomorphic extension space then it is a Hartogs one. Theorem 3.3 implies that so is H. Now by using our above hypothesis we derive that H is a Hartogs holomorphic extension space which is absurd.

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