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On the Laws of Large Numbers for Blockwise Martingale Differences and Blockwise Adapted Sequences

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Abstract. In this paper we establish the laws of large numbers for blockwise martingale differences and for blockwise adapted sequences which are stochastically dominated by a random variable. Some well-known results from the literature are extended.

1. Introduction and Notations

Let $\{\mathcal{F}_n, n \geq 1\}$ be an increasing σ-fields and let $\{X_n, n \geq 1\}$ be a sequence of random variables. We recall that the sequence $\{X_n, n \geq 1\}$ is said to be adapted to $\{\mathcal{F}_n, n \geq 1\}$ if each X_n is measurable with respect to $\mathcal{F}_n$. The sequence $\{X_n, n \geq 1\}$ is said to be stochastically dominated by a random variable X if there exists a constant $C > 0$ such that $P\{|X_n| \geq t\} \leq CP\{|X| \geq t\}$ for all nonnegative real numbers t and for all $n \geq 1$.

Related to the adapted sequences, Hall and Heyde [3] proved the following theorem.

Theorem 1.1. (see [3], Theorem 2.19) *Let $\{\mathcal{F}_n, n \geq 1\}$ be an increasing σ-fields and $\{X_n, n \geq 1\}$ is adapted to $\{\mathcal{F}_n, n \geq 1\}$. If $\{X_n, n \geq 1\}$ is stochastically dominated by a random variable X with $E|X| < \infty$, then*

$$\frac{1}{n}\sum_{i=1}^{n}(X_i - E(X_i|\mathcal{F}_{i-1})) \xrightarrow{P} 0 \text{ as } n \to \infty. \tag{1.1}$$

In the case, when $E(|X| \log^+ |X|) < \infty$ or X_n are independent, the convergence *in* (1.1) *can be strengthened to a.s. convergence.*

Moricz [4] introduced the concept of blockwise m-dependence for a sequence of random variables and extended the classical Kolmogorov strong law of large numbers to the blockwise m-dependence case. Later, the strong law of large numbers for arbitrary blockwise independent random variables was also studied by Gaposhkin [1]. He then showed in [2] that some properties of independent sequences of random variables remain satisfied for the sequences consisting of independent blocks. However, the same problem for sequences of blockwise independent and identically distributed random variables and for blockwise martingale differences is not yet studied.

The main results of this paper are Theorems 3.1, 3.3. Theorem 3.1 establishes the strong law of large numbers for arbitrary blockwise martingale differences. In Theorem 3.3, we set up the law of large numbers for the so called *blockwise adapted sequences* which are stochastically dominated by a random variable X. Some well-known results from the literature are extended.

Let $\{\omega(n), n \geq 1\}$ be a strictly increasing sequence of positive integers with $\omega(1) = 1$. For each $k \geq 1$, we set $\Delta_k = \big[\omega(k), \omega(k+1)\big)$. We recall that a sequence $\{X_i, i \geq 1\}$ of random variables is blockwise independent with respect to blocks $[\Delta_k]$, if for any fixed k, the sequence $\{X_i\}_{i\in\Delta_k}$ is independent.

Now let $\{\mathcal{F}_i, i \geq 1\}$ be a sequence of σ-fields such that for any fixed k, the sequence $\{\mathcal{F}_i, i \in \Delta_k\}$ is increasing. The sequence $\{X_i, i \geq 1\}$ of random variables is said to be *blockwise adapted* to $\{\mathcal{F}_i, i \geq 1\}$, if each X_i is measurable with respect to $\mathcal{F}_i$. The sequence $\{X_i, \mathcal{F}_i, i \geq 1\}$ called *a blockwise martingale difference* with respect to blocks $[\Delta_k]$, if for any fixed k, the sequence $\{X_i, \mathcal{F}_i\}_{i\in\Delta_k}$ is a martingale difference. Let

$$\begin{aligned}
N_m &= \min\{n | \omega(n) \geq 2^m\},\\
s_m &= N_{m+1} - N_m + 1,\\
\varphi(i) &= \max_{k\leq m} s_k \text{ if } i \in [2^m, 2^{m+1}),\\
\Delta^{(m)} &= [2^m, 2^{m+1}), m \geq 0,\\
\Delta_k^{(m)} &= \Delta_k \cap \Delta^{(m)}, m \geq 0, k \geq 1,\\
p_m &= \min\{k : \Delta_k^{(m)} \neq \emptyset\},\\
q_m &= \max\{k : \Delta_k^{(m)} \neq \emptyset\}.
\end{aligned}$$

Since $\omega(N_m - 1) < 2^m, \omega(N_m) \geq 2^m, \omega(N_{m+1}) \geq 2^{m+1}$ for each $m \geq 1$, the number of nonempty blocks $[\Delta_k^{(m)}]$ is not larger than $s_m = N_{m+1} - N_m + 1$. Assume $\Delta_k^{(m)} \neq \emptyset$, let $r_k^{(m)} = \min\{r : r \in \Delta_k^{(m)}\}$.

Throughout this paper, C denotes a unimportant positive constant which is allowed to be changed.

2. Lemmas

In the sequel we will need the following lemmas.

Lemma 2.1. (Doob's Inequality) *If* $\{X_i, \mathcal{F}_i\}_{i=1}^N$ *is a martingale difference,* $E|X|^p < \infty$ $(1 < p < \infty)$*, then*

$$E|\max_{k\le N}\sum_{i=1}^{k} X_i|^p \le (\frac{p}{p-1})^p E|\sum_{i=1}^{N} X_i|^p.$$

Lemma 2.2. *If* $\{x_n, n \ge 0\}$ *is a sequence of numbers such that* $\lim_{n\to\infty} x_n = 0$ *and* $q > 1$*, then*

$$\lim_{n\to\infty} q^{-n}\sum_{k=0}^{n} q^{k+1}x_k = 0.$$

Proof. Let $s = q + \sum_{i=0}^{\infty} q^{-i}$. For any $\epsilon > 0$, there exists k_0 such that $|x_k| < \dfrac{\epsilon}{2s}$ for all $k \ge k_0$. Since $\lim_{n\to\infty} q^{-n} = 0$, so, there exists $n_0 \ge k_0$ such that for all $n \ge n_0$, we have

$$\big|q^{-n}\sum_{k=0}^{k_0} q^{k+1}x_k\big| < \frac{\epsilon}{2}.$$

It follows that

$$\begin{aligned}\big|q^{-n}\sum_{k=0}^{n} q^{k+1}x_k\big| &\le \big|q^{-n}\sum_{k=0}^{k_0} q^{k+1}x_k\big| + \big|q^{-n}\sum_{k_0+1}^{n} q^{k+1}x_k\big| \\ &\le \frac{\epsilon}{2} + \frac{\epsilon}{2s}(q+1+\frac{1}{q}+\cdots) \\ &= \frac{\epsilon}{2}+\frac{\epsilon}{2} = \epsilon \text{ for all } n \ge n_0.\end{aligned}$$

■

3. Main Results

With the notations and lemmas as above, the main results may now be established. The following theorem is analogous to Theorem 1 in [1].

Theorem 3.1. *Let* $\{X_i, \mathcal{F}_i, i \ge 1\}$ *be a blockwise martingale difference. If*

$$\sum_{i=1}^{\infty}\frac{E|X_i|^2}{i^2} < \infty, \tag{3.1}$$

then

$$\frac{\sum_{i=1}^{n} X_i}{n\varphi^{\frac{1}{2}}(n)} \to 0 \quad a.s.\ as\ n\to\infty. \tag{3.2}$$

Proof. Let

$$\gamma_k^{(m)} = \max_{n\in\Delta_k^{(m)}} \sum_{i=r_k^{(m)}}^{n} X_i, \quad m \geq 0, k \geq 1,$$

$$\gamma_m = 2^{-m-1}\varphi^{-\frac{1}{2}}(2^m) \sum_{k=p_m}^{q_m} \gamma_k^{(m)}, \quad m \geq 0.$$

By using Lemma 2.1 for the martingale differences $\{X_i\}_{i\in\Delta_k^{(m)}}$, we get

$$\begin{aligned} E|\gamma_k^{(m)}|^2 &\leq 4E\Big|\sum_{i\in\Delta_k^{(m)}} X_i\Big|^2 \\ &\leq 4\sum_{i\in\Delta_k^{(m)}} E|X_i|^2, \quad m\geq 0,\ k\geq 1. \end{aligned}$$

This implies

$$\begin{aligned} E|\gamma_m|^2 &\leq 2^{-2m-2}\sum_{k=p_m}^{q_m} E|\gamma_k^{(m)}|^2 \text{ (by the Cauchy-Schwarz inequality)} \\ &\leq 4\sum_{i=2^m}^{2^{m+1}-1} \frac{E|X_i|^2}{i^2},\ m\geq 0. \end{aligned}$$

Thus $\sum_{m=0}^{\infty} E|\gamma_m|^2 < \infty$. By the Chebyshev inequality and the Borel-Canteli Lemma, we have

$$\lim_{m\to\infty} \gamma_m = 0 \text{ a.s.} \tag{3.3}$$

On the other hand, for each $k \geq 1$, we have

$$0 \leq 2^{-m}\varphi^{-\frac{1}{2}}(2^m)\sum_{k=0}^{m}\sum_{i=p_k}^{q_k}\gamma_i^{(k)} \leq 2^{-m}\sum_{k=0}^{m} 2^{k+1}\gamma_k. \tag{3.4}$$

Then by (3.3), (3.4) and Lemma 2.2, we get $\lim_{m\to\infty} 2^{-m}\varphi^{-\frac{1}{2}}(2^m)\sum_{k=0}^{m}\sum_{i=p_k}^{q_k}\gamma_i^{(k)} =$ 0 a.s. Assume $k \geq 1, n \in \Delta_k^{(m)}$, we have

$$\begin{aligned} 0 &\leq \Big|n^{-1}\varphi^{-\frac{1}{2}}(n)\sum_{i=1}^{n} X_i\Big| \\ &\leq 2^{-m}\varphi^{-\frac{1}{2}}(2^m)\sum_{j=0}^{m}\sum_{i=p_j}^{q_j}\gamma_i^{(j)} \to 0 \text{ a.s. } (m\to\infty). \end{aligned}$$

The proof is complete. ∎

Corollary 3.2. *If* $\omega(k) = 2^{k-1}$ *(or* $\omega(k) = [q^{k-1}], q > 1$*),* $k \geq 1$ *and* $\{X_i, \mathcal{F}_i, i \geq 1\}$ *is a blockwise martingale difference with respect to blocks* $[\Delta_k]$*, then*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} X_i = 0 \ a.s.$$

Proof. In that case $\varphi(i) = O(1)$, The Corollary follows immediately from Theorem 3.1. ∎

Theorem 3.3. *Let $\{\mathcal{F}_i, i \geq 1\}$ be a sequence of σ-fields such that for any fixed k, the sequence $\{\mathcal{F}_i, i \in \Delta_k\}$ is increasing and $\{X_i, i \geq 1\}$ is blockwise adapted to $\{\mathcal{F}_i, i \geq 1\}$. If $\{X_i, i \geq 1\}$ is stochastically dominated by a random variable X with $E|X| < \infty$, then*

$$\frac{1}{n\varphi^{\frac{1}{2}}(n)} \sum_{i=1}^{n} (X_i - a_i) \xrightarrow{P} 0 \ as \ m \to \infty, \tag{3.5}$$

where $a_i = EX_i$ if $i = r_k^{(m)}$ and $a_i = E(X_i|\mathcal{F}_{i-1})$ if $i \neq r_k^{(m)}$.

In the case, when $E(|X| \log^+ |X|) < \infty$ or the $\{X_n, n \geq 1\}$ is blockwise independent, then the convergence in (3.5) *can be strengthened to a.s. convergence.*

Proof. Let $X_i^{'} = X_i I\{|X_i| \leq i\}$, $b_i = EX_i^{'}$ if $i = r_k^{(m)}$ and $b_i = E(X_i^{'}|\mathcal{F}_{i-1})$ if $i \neq r_k^{(m)}$ for $k \geq 1$ and $m \geq 0$. We have

$$\begin{aligned}
\sum_{i=1}^{\infty} i^{-2} E(X_i^{'} - b_i)^2 &\leq \sum_{i=1}^{\infty} i^{-2} E|X_i^{'}|^2 \\
&\leq 2 \sum_{i=1}^{\infty} i^{-2} \int_0^i xP(|X_i| > x)dx \\
&\leq C \sum_{i=1}^{\infty} i^{-2} \int_0^i xP(|X_i| > x)dx \\
&= C \sum_{i=1}^{\infty} i^{-2} \sum_{k=1}^{i} \int_{k-1}^{k} xP(|X| > x)dx \\
&\leq C \sum_{i=1}^{\infty} i^{-2} \sum_{k=1}^{i} kP(|X| > k-1) \\
&= C \sum_{k=1}^{\infty} kP(|X| > k-1) \sum_{i=k}^{\infty} i^{-2} \\
&\leq C \sum_{k=1}^{\infty} P(|X| > k-1) < \infty,
\end{aligned}$$

since $E|X| < \infty$. Note that $\{X_i^{'} - b_i, \ \mathcal{F}_i, i \geq 1\}$ is a blockwise martingale difference, by using the proof of Theorem 3.1, we get

$$\lim_{n\to\infty} \frac{1}{n\varphi^{\frac{1}{2}}(n)} \sum_{i=1}^{n} (X_i^{'} - b_i) = 0 \text{ a.s.} \tag{3.6}$$

Next,

$$\sum_{i=1}^{\infty} P(X_i \neq X_i^{'}) = \sum_{i=1}^{\infty} P(|X_i| > i)$$
$$\leq C \sum_{i=1}^{\infty} P(|X| > i) < \infty,$$

so that the sequences $\{X_n\}$ and $\{X_n^{'}\}$ are tail equivalent, and hence from (3.6),

$$\lim_{n\to\infty} \frac{1}{n\varphi^{\frac{1}{2}}(n)} \sum_{i=1}^{n} (X_i - b_i) = 0 \text{ a.s. } . \tag{3.7}$$

Now, since

$$E\big[|X_n| I(|X_n| > n)\big] = \int_n^{\infty} P(|X_n| > x)dx$$
$$\leq C \int_n^{\infty} P(|X| > x)dx \to 0,$$

it follows that

$$n^{-1} E\Big|\sum_{i=1}^{n} (a_i - b_i)\Big| \leq n^{-1} \sum_{i=1}^{n} E\big[|X_i| I(|X_i| > i)\big] \to 0.$$

Therefore

$$n^{-1} \sum_{i=1}^{n} (a_i - b_i) \to 0 \text{ in probability,}$$

implying (3.5). If $\{X_n, n \geq 1\}$ is bockwise independent, we let $\mathcal{F}_i = \sigma\{X_j, r_k^{(m)} \leq j \leq i\}$ if $i \in \Delta_k^{(m)}$ for $m \geq 0$ and $k \geq 1$. Then each $a_n - b_n$ is a constant, and so the a.s. convergence version of (3.5) holds. To complete the proof we now suppose that $E(|X| \log^+ |X|) < \infty$. It suffices to prove that

$$n^{-1} \big| \sum_{i=1}^{n} (a_i - b_i) \big| \leq n^{-1} \sum_{i=1}^{n} E\big[|X_i| I(|X_i| > i) \big| \mathcal{F}_{i-1}\big] \to 0 \text{ a.s., as } n \to \infty. \tag{3.8}$$

Since

$$\sum_{n=1}^{\infty} n^{-1} E\big[|X_n| I(|X_n| > n)\big] = \sum_{n=1}^{\infty} n^{-1} \int_n^{\infty} P\big(|X_n| > n\big)dx$$
$$\leq C \sum_{n=1}^{\infty} n^{-1} \int_n^{\infty} P(|X| > x)dx$$
$$= C \sum_{n=1}^{\infty} n^{-1} \sum_{i=n}^{\infty} \int_{i<x\leq i+1} P(|X| > x)dx$$
$$\leq C \sum_{i=1}^{\infty} P(|X| > i) \sum_{n=1}^{i} n^{-1}$$
$$\leq C \sum_{i=1}^{\infty} (1 + \log i) P(|X| > i) < \infty,$$

it follows that

$$\sum_{n=1}^{\infty} n^{-1} E\big[|X_n| I(|X_n| > n) \big| \mathcal{F}_{n-1}\big] < \infty \text{ a.s.}$$

Thus using Kronecker's Lemma, we get (3.8). The proof of theorem is completed. ■

The following corollary is a strong law of large numbers for sequences of blockwise independent and identically distributed random variables.

Corollary 3.4. *Let $\{X_i, i \geq 1\}$ be a sequence of blockwise independent (with respect to blocks $[\Delta_k]$) and identically distributed random variables. If $E|X_1| < \infty$, then*

$$\lim_{n\to\infty} \frac{1}{n\varphi^{\frac{1}{2}}(n)} \sum_{i=1}^{n} X_i = 0 \quad a.s.$$

Similar to Corollary 3.2, we have the following.

Corollary 3.5. *Let $\omega(k) = 2^{k-1}$ (or $\omega(k) = [q^{k-1}], q > 1$), $k \geq 1$, and let $\{X_i, i \geq 1\}$ be a sequence of random variable, $\{\mathcal{F}_i, i \geq 1\}$ a sequence of σ-fields such that for any fixed k, the sequences $\{\mathcal{F}_i, i \in \Delta_k\}$ is increasing and each X_i is measurable with respect to $\mathcal{F}_i$. If $\{X_n, n \geq 1\}$ is stochastically dominated by a random variable X with $E|X| < \infty$, then (1.1) holds.*

In the case, when $E(|X| \log^+ |X|) < \infty$ or $\{X_n\}$ is blockwise independent with respect to blocks $[\Delta_k]$, the convergence in (1.1) can be strengthened to a.s. convergence.

Note here that Corollary 3.5 extends Theorem 1.1. The next corollary extends a classical result of Kolmogorov.

Corollary 3.6. *Let $\omega(k) = 2^{k-1}$ (or $\omega(k) = [q^{k-1}], q > 1$), $k \geq 1$ and $\{X_i, i \geq 1\}$ be a sequence of blockwise independent (with respect to blocks $[\Delta_k]$) and identically distributed random variables. If $E|X_1| < \infty$, then*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} X_i = 0 \quad a.s.$$

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