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Almost Periodic Solutions of Evolution Equations Associated with *C*-Semigroups: An Approach Via Implicit Difference Equations*

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Abstract. The paper is concerned with the existence of almost periodic mild solutions to evolution equations of the form $\dot{u}(t) = Au(t) + f(t)$ $(*)$, where A generates a *C-semigroup* and f is almost periodic. We investigate the existence of almost periodic solutions of $(*)$ by means of associated implicit difference equations which are well-studied in recent works on the subject. As results we obtain various sufficient conditions for the existence of almost periodic solutions to $(*)$ which extend previous ones to a more general class of ill-posed equations involving *C*-semigroups. The paper is supported by a research grant of the Vietnam National University, Hanoi.

In this paper we are concerned with the existence of almost periodic solutions to equations of the form

$$\frac{du}{dt} = Au + f(t), \tag{1}$$

where A is a (unbounded) linear operator which generates a *C-semigroup* of linear operators on Banach space $\mathbb{X}$ and f is an almost periodic function in the sense of Bohr (for the definition and properties see [1, 6, 12]). We refer the reader to [5, 10, 27, 28, 29] for more information on the definitions and properties of C-semigroups and related ill-posed equations and to [3, 7, 34] for more information

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on the asymptotic behavior of solutions of ill-posed equations associated with C-semigroups.

The existence of almost periodic solutions to ill-posed evolution equations of the form (1) has not been treated in mathematical literature yet except for a recent paper by Chen, Minh and Shaw (see [3]) although many nice results on the subject are available for well-posed equations (see e.g. [8, 17, 19, 32] and the refenreces therein). In this paper we study the existence of almost periodic mild solutions of Eq. (1) by means of the associated implicit difference equation. This approach goes back to the period map method which is very well known in the theory of ordinary differential equations. Subsequently, this method has been extended to study the existence of almost periodic solutions in [20]. By this approach we obtain a necessary and sufficient condition (see Theorem 2.3) for the existence of periodic solutions which extends a result in [18, 24, 32] to the case of C-semigroups. As far as almost periodic mild solutions are concerned, we obtain a sufficient condition (see Theorem 2.5) which extends a result in [20] to ill-posed equations associated with C-semigroups.

1. Preliminaries

1.1. Notation

Throughout the paper, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{X}$ stand for the sets of real, complex numbers and a complex Banach space, respectively; $L(\mathbb{X})$, $C(J,\mathbb{X})$, $BUC(\mathbb{R},\mathbb{X})$, $AP(\mathbb{X})$ denote the spaces of linear bounded operators on $\mathbb{X}$, all $\mathbb{X}$-valued continuous functions on a given interval J, all $\mathbb{X}$-valued bounded uniformly continuous and almost periodic functions in Bohr's sense (see definition below) with sup-norm, respectively. For a linear operator A, we denote by $D(A)$, $\sigma(A)$ the domain of A and the spectrum of A.

1.2. Spectral theory of functions

In the present paper $sp(u)$ stands for the Beurling spectrum of a given bounded uniformly continuous function u, which is defined by

$$sp(u) := \{\xi \in \mathbb{R} : \forall \varepsilon > 0, \exists \varphi \in L^1(\mathbb{R}) : \operatorname{supp} \tilde{\varphi} \subset (\xi - \varepsilon, \xi + \varepsilon), \varphi * u \neq 0\},$$

where

$$\tilde{\varphi}(s) := \int_{-\infty}^{\infty} e^{-ist} f(t) dt; \quad \varphi * u(s) := \int_{-\infty}^{\infty} \varphi(s-t) u(t) dt.$$

The notion of Beurling spectrum of a function $u \in BUC(\mathbb{R},\mathbb{X})$ coincides with the one of Carleman spectrum, which consists of all $\xi \in \mathbb{R}$ such that the Carleman–Fourier transform of u, defined by

$$\hat{u}(\lambda) := \begin{cases} \int_0^\infty e^{-\lambda t} u(t) dt & (\mathrm{Re}\lambda > 0) \\ -\int_0^\infty e^{\lambda t} u(-t) dt & (\mathrm{Re}\lambda < 0), \end{cases}$$

has no holomorphic extension to any neighborhood of $i\xi$ (see [24, Prop. 0.5, p. 22]). ■

Proposition 1.1. *Let $u \in BUC(\mathbb{R}, \mathbb{X})$. Then*
(i) *$sp(u)$ is closed,*
(ii) *$sp(u(\cdot)) = sp(u(\cdot + h))$, for all $h \in \mathbb{R}$,*

Proof. We refer the reader to [24, Prop. 0.4, Prop. 0.6, Theorem 0.8, p. 20 - 25]. ■

1.3. Almost Periodic Functions

We recall that a subset $E \subset \mathbb{R}$ is said to be *relatively dense* if there exists a number $l > 0$ (*inclusion length*) such that every interval $[a, a + l]$ contains at least one point of E. Let f be a continuous function on $\mathbb{R}$ taking values in a Banach space $\mathbb{X}$. f is said to be *almost periodic in the sense of Bohr* if to every $\epsilon > 0$ there corresponds a relatively dense set $T(\epsilon, f)$ (*of ϵ-periods*) such that

$$\sup_{t\in\mathbb{R}} \|f(t+\tau) - f(t)\| \le \epsilon, \ \forall \tau \in T(\epsilon, f).$$

If for all reals λ such that the following integrals

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(t) e^{-i\lambda t} dt$$

exist, then

$$a(\lambda, f) := \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(t) e^{-i\lambda t} dt$$

is called *Fourier coefficients* of f. As is well known (see e.g. [12]), if f is an almost periodic function taking values in $\mathbb{X}$, then there are at most countably reals λ (*Fourier exponents*) such that $a(\lambda, f) \neq 0$, the set of which will be denoted by $\sigma_b(f)$ and called *Bohr spectrum* of f. Throughout the paper we will use the relation $sp(f) = \overline{\sigma_b(f)}$ and denote by $AP(\mathbb{X})$ the space of all almost periodic functions taking values in $\mathbb{X}$ with sup norm. We summarize several main properties of almost periodic functions, whose proofs can be found in [12], in the following theorem:

Theorem 1.2. *The following assertions hold:*
(i) *$f \in BC(\mathbb{R}, \mathbb{X})$ is almost periodic if and only if for every sequence $\{\tau_n\}_{n=1}^{\infty} \subset \mathbb{R}$ the sequence of functions $\{f_{\tau_n} = f(\tau_n + \cdot)\}_{n=1}^{\infty}$ contains at least a convergent subsequence.*
(ii) *If f is $\mathbb{X}$-valued almost periodic, then $f \in BUC(\mathbb{R}, \mathbb{X})$.*
(iii) *$AP(\mathbb{X})$ is a closed subspace of $BUC(\mathbb{R}, \mathbb{X})$.*
(iv) *If $f \in AP(\mathbb{X})$ and f' exists as an element of $BUC(\mathbb{R}, \mathbb{X})$, then $f' \in AP(\mathbb{X})$.*

1.4. Spectral Theory of Bounded Sequences

First we define the spectrum of a bounded sequence $g := \{g(n)\}_{n\in\mathbb{Z}}$ in $\mathbb{X}$ used in this paper. Recall that the set of all bounded sequences in $\mathbb{X}$ forms a

Banach space $l_\infty(\mathbb{X})$ with norm $\|g\| := \sup_n \|g(n)\|_{\mathbb{X}}$. We will denote by $S(k)$ the k-translation in $l_\infty(\mathbb{X})$, i.e., $(S(k)g)(n) = g(n+k)$, $\forall g, n$.

Definition 1.3. *The subset of all λ of the unit circle $\Gamma := \{z \in \mathbb{C} : |z| = 1\}$ at which*

$$\hat{g}(\lambda) := \begin{cases} \sum_{n=0}^{\infty} \lambda^{-n-1} S(n) g, \ \forall |\lambda| > 1, \\ -\sum_{n=1}^{\infty} \lambda^{n-1} S(-n) g, \ \forall |\lambda| < 1, \end{cases}$$

has no holomorphic extension to any neighborhood in $\mathbb{C}$ of λ, is called the spectrum of the sequence $g := \{g(n)\}_{n\in\mathbb{Z}}$ and will be denoted by $\sigma(g)$.

We recall that a bounded sequence x is said to be almost periodic if it belongs to the following subspace of $l_\infty(\mathbb{X})$

$$APZ(\mathbb{X}) := \overline{\text{span}\,\{\lambda^{\cdot} z, \lambda \in \Gamma, z \in \mathbb{X}\}}, \tag{2}$$

where $(\lambda^{\cdot} z)(n) := \lambda^n z, \ \forall n \in \mathbb{Z}$.

We list below some properties of the spectrum of $g = \{g(n)\}$.

Proposition 1.4. *Let $g := \{g(n)\}$ be a two-sided bounded sequence in $\mathbb{X}$. Then the following assertions hold:*

(i) *$\sigma(g)$ is closed.*

(ii) *If g^n is a sequence in $l_\infty(\mathbb{X})$ converging to g such that $\sigma(g^n) \subset \Lambda$ for all $n \in \mathbb{N}$, where Λ is a closed subset of the unit circle, then $\sigma(g) \subset \Lambda$.*

(iii) *If $g \in l_\infty(\mathbb{X})$ and A is a bounded linear operator on the Banach space $\mathbb{X}$, then $\sigma(Ag) \subset \sigma(g)$, where $Ag \in l_\infty(\mathbb{X})$ is given by $(Ag)(n) := Ag(n), \forall n \in \mathbb{Z}$.*

(iv) *Let the space $\mathbb{X}$ do not contain any subspace which is isomorphic to c_0 (the Banach space of numerical sequences which converge to 0) and $x \in l_\infty(\mathbb{X})$ be a sequence such that $\sigma(x)$ is countable. Then x is almost periodic.*

1.5. C-semigroups: Definition and Basic Properties

Definition 1.5. *Let $\mathbb{X}$ be a Banach space and let C be an injective operator in $L(\mathbb{X})$. A family $\{S(t);\ t \geq 0\}$ in $L(\mathbb{X})$ is called a C-semigroup if the following conditions are satisfied:*

(i) *$S(0) = C$,*

(ii) *$S(t+s)C = S(t)S(s)C$ for $t,\ s \geq 0$,*

(iii) *$S(\cdot)x$: $[0,\infty) \to \mathbb{X}$ is continuous for any $x \in \mathbb{X}$,*

(iv) *There are $M \geq 0$ and $a \in \mathbb{R}$ such that $\|S(t)\| \leq Me^{at}$ for $t \geq 0$.*

We define an operator G as follows:

$$D(G) = \{x \in \mathbb{X} : \lim_{h\to 0^+} (S(h)x - Cx)/h \in R(C)\}$$
$$Gx = C^{-1} \lim_{h\to 0^+} (S(h)x - Cx)/h, \quad \forall x \in D(G).$$

This operator is called the *generator* of $(S(t))_{t\geq 0}$. It is known that G is closed but the domain of G is not necessarily dense in $\mathbb{X}$.

Lemma 1.6. *Let C be an injective linear operator and let $(S(t))_{t\geq 0}$ be a C-semigroup with generator A. Then, the following assertions hold true:*

(i) $S(t)S(s) = S(s)S(t), \quad \forall t, s \geq 0,$

(ii) *If $x \in D(A)$, then $S(t)x \in D(A)$, $AS(t)x = S(t)Ax$ and*

$$\int_0^t S(\xi)Axd\xi = S(t)x - Cx, \quad \forall t \geq 0,$$

(iii) $\int_0^t S(\xi)xd\xi \in D(A)$ *and* $A\int_0^t S(\xi)xd\xi = S(t)x - Cx$ *for every* $x \in \mathbb{X}$ *and* $t \geq 0,$

(iv) *A is closed and satisfies $C^{-1}AC = A$,*

(v) $R(C) \subset \overline{D(A)}$.

For more information about C-semigroups we refer the reader to [5, 10, 13, 14].

2. Main Results

We always assume that C is an injection and $\{T(t); t \geq 0\}$ is a C-semigroup with generator A. Below we introduce some notions of solutions. We denote by J an interval of the form (α, β), $[\alpha, \beta)$, $(\alpha, \beta]$ or $[\alpha, \beta]$.

Definition 2.1.

(i) *An $\mathbb{X}$-valued function $u \in C^1(J, \mathbb{X})$ is called a (classical) solution on J to Eq. (1) for a given $f \in C(J, \mathbb{X})$ if $u(t) \in D(A)$, $\forall t \in J$ and u, f satisfy Eq. (1) for all $t \in J$.*

(ii) *An $\mathbb{X}$-valued function u on J is called a mild solution on J to Eq. (1) for a given $f \in C(\mathbb{R}, \mathbb{X})$ if $u(t)$ is continuous in t and satisfies*

$$Cu(t) = T(t-s)u(s) + \int_s^t T(t-r)f(r)dr, \quad \forall t \geq s; t, s \in J. \tag{3}$$

As shown in [3] every classical solution is a generalized one. However, given an initial value $u(t_0) = x \in \mathbb{X}$ we do not know if there exists a mild solution Eq. (1) starting at this point.

Lemma 2.2. *Let $R(C)$ be closed, $x \in R(C)$ and let $f(t) \in R(C)$ be continuous for all $t \in [t_0, \infty)$. Then there exists a unique mild solution u to Eq. (1) on $[t_0, \infty)$ such that $u(t_0) = x$ and $u(t) \in R(C)$, $\forall t \in [t_0, \infty)$.*

Proof. By the Open Mapping Theorem and the assumptions the operator C^{-1} is continuous from $R(C)$ to $\mathbb{X}$. Using the indentity $T(t)C = CT(t), \forall t \in [0, \infty)$ we define a function u as

$$u(t) = T(t-t_0)C^{-1}x + \int_{t_0}^t T(t-\xi)C^{-1}f(\xi)d\xi, \quad \forall t \geq t_0.$$

Since C^{-1} is continuous and $x, f(t) \in R(C)$ we have that u is a continuous function defined on $[t_0, \infty)$ and $u(t) \in R(C)$, $\forall t \in [t_0, \infty)$. We now show that u is a mild solution of Eq. (1) on $[t_0, \infty)$. In fact, we have

$$\begin{aligned}
C^2u(t) &= CT(t-t_0)x + C\int_{t_0}^t T(t-\xi)f(\xi)d\xi \\
&= CT(t-s)T(s-t_0)x + C\int_{t_0}^t T(t-\xi)f(\xi)d\xi \\
&= CT(t-s)T(s-t_0)x + C\int_{t_0}^s T(t-s)T(s-\xi)f(\xi)d\xi + C\int_s^t T(t-\xi)f(\xi)d\xi \\
&= T(t-s)Cu(s) + C\int_s^t T(t-\xi)f(\xi)d\xi, \quad \forall t \geq s \geq t_0.
\end{aligned}$$

Since C is an injection and by the identity $CT(t) = T(t)C$ the above yields that

$$Cu(t) = T(t-s)u(s) + \int_s^t T(t-\xi)f(\xi)d\xi, \quad \forall t \geq s \geq t_0,$$

and so, by definition, u is a mild solution of Eq. (1) on $[t_0, \infty)$. The uniqueness of such a solution is obvious. ■

2.1. Periodic Solutions

We will use the following notation in the remaining part of this paper

$$\rho_C(T(1)) := \{\lambda \in \mathbb{C} : (\lambda C - T(1)) : R(C) \to R(C) \text{ is bijective}\}.$$

Note that this definition is possible because of $CT(1) = T(1)C$.

Theorem 2.3. *Let $R(C)$ be closed. Then Eq.* (1) *has a unique* 1*-periodic mild solution u with $u(t) \in R(C)$ for every* 1*-periodic $f \in C(\mathbb{R}, R(C))$ provided that* $1 \in \rho_C(T(1))$.

Proof. Suppose that $1 \in \rho_C(T(1))$. We now prove that for every 1-periodic continuous f with $f(t) \in R(C)$, $\forall t \in \mathbb{R}$ there exists a unique 1-periodic mild solution u to Eq. (1) such that $u(t) \in R(C)$, $\forall t \in \mathbb{R}$. In fact, let $x := \int_0^1 T(1-\xi)f(\xi)d\xi$. Then, by assumption, we have $x \in R(C)$. Next, consider the element $y \in R(C)$ such that $(C - T(1))y = x$ whose existence is guaranteed by the assumption. By Lemma 2.2 there exists a unique mild solution u to Eq. (1) on $[0, \infty)$ such that $u(t) \in R(C)$, $\forall t \in [0, \infty)$ and $u(0) = y$. Next, we have

$$Cu(1) = T(1)y + x.$$

Since C is injective, $u(1) = y$. By the 1-periodicity of f, this shows that u can be extended to a 1-periodic mild solution of Eq. (1). The uniqueness of such 1-periodic solutions follows from the uniqueness of the element y from the equation $(C - T(1))y = x$. ■

2.2. Almost Periodic Solutions

We will give a sufficient conditions for the existence of almost periodic mild solutions to Eq. (1).

Proposition 2.4. *Let C be an injection with closed range and let f be an almost periodic function such that $f(t) \in R(C)$, $\forall t \in \mathbb{R}$. Then, any mild solution u on $\mathbb{R}$ of Eq.* (1) *is almost periodic provided that the sequence $\{u(n)\}_{n\in\mathbb{Z}}$ is almost periodic.*

Proof. The proof of this proposition is suggested by that of [20]. We first prove the sufficiency: Suppose that the sequence x is almost periodic. Note that the function $w(t) := su(n) + (1-s)u(n+1)$, if $t = sn + (1-s)(n+1)$, $s \in [0,1]$, as a function defined on the real line, is almost periodic. Also, the function taking t into $g(t) := (w(t), f(t))$ is almost periodic (see [12, p.6]). As is seen, the sequence $\{g(n)\} = \{(w(n), f(n))\}$ is almost periodic. Hence, for every positive ϵ the following set is relatively dense (see [6, pp. 163-164])

$$T := \mathbb{Z} \cap T(g, \epsilon), \tag{4}$$

where $T(g,\epsilon) := \{\tau \in \mathbb{R} : \sup_{t\in\mathbb{R}} \|g(t+\tau) - g(t)\| < \epsilon\}$, i.e., the set of ϵ periods of g. Hence, for every $m \in T$ we have

$$\|f(t+m) - f(t)\| < \epsilon, \ \forall t \in \mathbb{R}, \tag{5}$$

$$\|u(n+m) - u(n)\| < \epsilon, \ \forall n \in \mathbb{Z}. \tag{6}$$

Moreover, by the Open Mapping Theorem, $C^{-1} : R(C) \to \mathbb{X}$ is bounded. Thus, using the identity $CT(t) = T(t)C$ we have

$$\begin{aligned}
\|u(n+m+s) - u(n+s)\| &\leq \|C^{-1}T(s)(u(n+m) - u(n))\| \\
&\quad + \|\int_0^s T(s-\xi)C^{-1}(f(n+\xi+m) - f(n+\xi))d\xi\| \\
&\leq \|C^{-1}\|_{R(C)\to\mathbb{X}}[Ne^{\omega}\|u(n+m) - u(n)\| \\
&\quad + N\frac{e^{\omega}}{\omega}\sup_{t\in\mathbb{R}}\|f(m+t) - f(t)\|].
\end{aligned}$$

In view of (5) and (6) m is a $\epsilon \max\left\{1, \frac{1}{\omega}\right\}Ne^{\omega}$-period of the function u. Finally, since T is relatively dense for every ϵ, we see that u is an almost periodic mild solution of Eq. (1). ■

Below we always assume that the conditions of Proposition 2.4 are satisfied. We consider the difference equation

$$Cx(n+1) = T(1)x(n) + g(n), \quad n \in \mathbb{Z}, \tag{7}$$

where

$$g(n) := \int_n^{n+1} T(n+1-\xi)f(\xi)d\xi, \quad n \in \mathbb{Z}.$$

Obviously, $g(n) \in R(C)$ for all $n \in \mathbb{Z}$. Consider the space $\mathbb{Y} := APZ_{\Lambda}(R(C)$ with translation $S : x(\cdot) \mapsto x(\cdot + 1)$, where $\Lambda := \sigma(g)$. We rewrite Eq. (8) in the abstract form

$$(\tilde{C}S - \tilde{T}(1))x = g, \tag{8}$$

where $\tilde{C}$ and $\tilde{T}(1)$ are the operators of multiplication by C and $T(1)$ in $\mathbb{Y}$, respectively. Since C is invertible in $R(C)$ and $g \in R(C)$, we will solve x as a solution of the equation $(S - \tilde{C}^{-1}\tilde{T}(1))x = \tilde{C}^{-1}g$. Now we have

Theorem 2.5. *Let all assumptions of Proposition* 2.4 *be satisfied. Moreover, assume that*

$$e^{isp(f)} \cap \sigma(C^{-1}T(1)|_{R(C)}) = \emptyset.$$

Then, there exists an almost periodic solution to Eq. (1)*.*

Proof. As in the above argument, it is sufficient to prove that Eq. (8) has a solution in $\mathbb{Y}$. Since the operators $S, \tilde{C}^{-1}T(1)$ are commutative and $\sigma(S) = \Lambda \subset e^{isp(f)}$ we have that

$$\sigma(S - \tilde{C}^{-1}\tilde{T}(1)) \subset \sigma(S) - \sigma(\tilde{C}^{-1}\tilde{T}(1)) = \Lambda - \sigma(\tilde{C}^{-1}\tilde{T}(1)).$$

By the assumption $e^{isp(f)} \cap \sigma(C^{-1}T(1)|_{R(C)}) = \oslash$ we have $\Lambda \cap \sigma(\tilde{C}^{-1}\tilde{T}(1)) = \oslash$. Therefore, $1 \notin \Lambda - \sigma(C^{-1}T(1))$. This shows that the operator $(S - \tilde{C}^{-1}\tilde{T}(1))$ is invertible, so Eq. (8) has at least a solution in $\mathbb{Y}$. That is the difference equations (7) has at least one almost periodic solution. Using the injectiveness of C and Lemma 2.2 we can construct a mild solution u to Eq. (1) defined on the whole line such that $u(n) = x(n)$. By Proposition 2.4, we claim that the mild solution u should be almost periodic. ∎

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