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Renewal Process for a Sequence of Dependent Random Variables

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Abstract. We investigate a renewal process $N(t) = \max\{n \geq 1 : S_n = \sum_{i=1}^{n} X_i \leq t\}$ for $t \geq 0$ where $X_1, X_2, ...$ with $P(X_i \geq 0) = 1$ $(i = 1, 2, ...)$ is a sequence of m-dependent or mixing random variables. We give such a condition under which $N(t)$ has finite moment. Strong law of large numbers and central limit theorems for the function $N(t)$ are given.

1. Preliminaries and Notations

Let $(\Omega, \mathcal{A}, P)$ be a probability space and let $X_0, X_1, X_2, ...$ be non negative random variables with $P(X_0 = 0) = 1$, $S_n = \sum_{i=1}^{n} X_i$. It is well known that if the sequence $X_1, X_2, ...$ is independent and identically distributed, then the counting process $N(t) = \max\{n \geq 1 : S_n = \sum_{i=1}^{n} X_i \leq t\}, t \geq 0$ is called a renewal process. In this article, we investigate generalized renewal process, i.e. we suppose that our basic sequence $X_0, X_1, X_2, ...$ is a sequence of m - independent or mixing radom variables. We denote $\mathcal{F}_n = \sigma(X_0, X_1, ..., X_n)$, $\mathcal{F}^k = \sigma(X_k, X_{k+1}, ...)$. Now we begin this section with some definitions.

Definition 1.1. *A sequence of random variables $(X_n)_{n\geq 0}$ is called m-dependent if the sigma-fields $\mathcal{F}_n$ and $\mathcal{F}^{n+k}$ are independent for all $k > m$.*

Definition 1.2. *We consider the following quantities*

$\alpha(n) = \sup\{|P(A.B) - P(A).P(B)| : A \in \mathcal{F}_k, B \in \mathcal{F}^{k+n}\};$
$\rho(n) = \sup\{|Cov(X.Y)|/(V(X)^{1/2}.V(Y)^{1/2}) : X \in \mathcal{F}_k, Y \in \mathcal{F}^{k+n}\};$
$\phi(n) = \sup\{|P(B|A) - P(B)| : A \in \mathcal{F}_k, P(A) > 0; B \in \mathcal{F}^{k+n}\}.$

A sequence of random variables $(X_n)_{n\geq 0}$ is said to be α-mixing (resp. ρ-mixing, ϕ-mixing) if $\lim_{n\to+\infty} \alpha(n) = 0$ (resp. $\lim_{n\to+\infty} \rho(n) = 0$, $\lim_{n\to+\infty} \phi(n) = 0$).

2. Results

Theorem 2.1. *Let $(X_n)_{n\geq 0}$ be a sequence of nonnegative random variables. Denote $p_i = P(X_i \geq a)$ where a is a positive constant and $N(t) = \max\{n \geq 1 : S_n = \sum_{1=1}^{n} X_i \leq t\}$. Suppose that either*

(i) *$(X_n)_{n\geq 0}$ is a $(m-1)$- dependent random variables, $(m \leq 1)$ such that $\sum_{i=1}^{n} p_{r+im} \geq A_r.n^{\alpha_r}$, $0 < A_r < +\infty$, $\alpha_r > 0$ for all $n \geq 1$, $m - 1 \geq r \geq 0$,*

or

(ii) *$(X_n)_{n\geq 0}$ is a ϕ-mixing sequence of random variables such that $\liminf p_n = p > 0$.*

Then

$$E[N(t)]^l < +\infty, \ \forall l.$$

We need the following lemma to prove the theorem.

Lemma 2.1. *Let $(X_n)_{n\geq 0}$ be a sequence of non negative, independent random variables such that*

$$\sum_{i=1}^{n} p_i \geq A.n^{\alpha}, \quad 0 < A < +\infty, \ \alpha > 0 \text{ for all } n \geq 1.$$

Then

$$E[N(t)]^l < +\infty, \ \forall l.$$

Proof. From the definition of $N(t)$, it is easy to see that $N(t)$ is a non decreasing function in t. We define new random variables $\bar{X}_n$ as follows: for a given positive number a, we put

$$\bar{X}_n = 1_{(a,\infty)}(X_n), \quad n \geq 1,$$
$$\bar{S}_n = \sum_{i=1}^{n} \bar{X}_i,$$

and

$$\bar{N}(t) = \max\{n \geq 1 : \bar{S}_n \leq t\}.$$

It is easy to see that

$$0 \leq N(t) \leq \bar{N}(t/a) \quad \text{for all} \quad t > 0.$$

This guarantees that , we can investigate the function $\bar{N}(t)$ instead of the function $N(t)$.

$$P(\bar{N}(j) = n) = P(\bar{X}_1 + \bar{X}_2 + ... + \bar{X}_n = j).$$

Denote by $I_n^J (1 \leq j \leq n)$ the set of all combinations of j numbers from the set $\{1, 2, ..., n\}$. For $i_1, i_2, ..., i_j \in I_n^j$, we consider the following events:

$$A\{i_1, i_2, ..., i_n\} = \{\bar{X_{i_1}} = \cdots = \bar{X_{i_j}} = 1\},$$
$$\bar{A}\{i_1, i_2, ..., i_n\} = \{\bar{X}_{i(1)} = \cdots = \bar{X}_{i(n-j)} = 0\},$$

where $\{i(1), ..., i(n-j)\}$ is the complement of $\{i_1, ..., i_j\}$, i.e.

$$\{i(1), ..., i(n-j)\} = \{1, 2, ..., n\} \setminus \{i_1, ..., i_j\}.$$

We obtain the following relations

$$\{\bar{X}_1 + \bar{X}_2 + \cdots + \bar{X}_n = j\} = \bigcup_{\{i_1, ..., i_j \in I_n^j\}} A\{i_1, ..., i_j\} \cap \bar{A}\{i_1, ..., i_j\}$$
$$\subset \bigcup_{\{i_1, ..., i_j \in I_n^j\}} \bar{A}\{i_1, ..., i_j\}.$$

We have the following probability

$$P(\bar{N}(j) = n) \leq C_n^J \prod_{s=1}^{n-j} .\{1 - p_{i_s}\}.$$

Using the inequality

$$(1 - x) < e^{-x} \quad \text{for} \quad 0 < x < 1,$$

we get

$$P(\bar{N}(j) = n) \leq C_n^J . \exp\{-\sum_{s=1}^{n-j} p_{i_s}\} \leq C_n^J . \exp\{j\} . \exp\{-\sum_{i=1}^{n} p_i\}.$$

Combining the above inequalities, we get

$$E(\bar{N}(j))^l = \sum_{n=1}^{\infty} n^l . P(\bar{N}(j) = n) \leq e^j . \sum_{n=j}^{\infty} C_n^j . n^l \exp\{-\sum_{i=1}^{n} p_i\}.$$

Since $C_n^j \leq \frac{n^j}{j!}$ and $\sum_{i=1}^n p_i \leq A.n^\alpha$, we have

$$E(\bar{N}(j))^l \leq \frac{e^j}{j!} . \sum_{n \geq j} n^{j+l} e^{-n^\alpha}.$$

But $e^{-n^\alpha} = o(\frac{1}{n^\beta})$ for all $\beta \geq 1$. So we deduce $e^{-n^\alpha} \leq \frac{1}{n^{j+l+2}}$ if n is sufficiently large. Then, we have

$$E(\bar{N}(j))^l \leq \frac{e^j}{j!}\Big(\sum_{j\leq n\leq n_0} n^{j+l}e^{-n\alpha} + \sum_{n\geq n_0} n^{j+l}\Big) \leq \sum_{n\geq n_0}\frac{1}{n^2} < \infty.$$

The lemma is proved. ∎

Proof of Theorem 2.1.

(i) Let $\bar{X}_1, \bar{X}_2, ..., \bar{X}_n$ be such random variables as in the Lemma. We estimate the probability

$$P(\bar{N}(j) = n) = P(\bar{X}_1 = \lambda_1, \bar{X}_2 = \lambda_2, ..., \bar{X}_n = \lambda_n)$$

where λ_i $(1 \leq i \leq n)$ takes only the value 0 or 1 and among them, there are j numbers being 1 while $n - j$ numbers being 0.

Suppose that $n = km + r$ for $0 \leq r < m$. We rewrite

$$A = \{\bar{X}_1 = \lambda_1, \bar{X}_2 = \lambda_2, ..., \bar{X}_n = \lambda_n\} = \bigcap_{s=1}^{m-1} B_s,$$

here

$$B_s = \{\bar{X}_s = \lambda_s, \bar{X}_{m+s} = \lambda_{m+s}, \bar{X}_{2m+s} = \lambda_{2m+s}, ..., \bar{X}_{km+s} = \lambda_{km+s}\}$$

for $0 \leq s \leq r$, and

$$B_s = \{\bar{X}_s = \lambda_s, \bar{X}_{m+s} = \lambda_{m+s}, \bar{X}_{2m+s} = \lambda_{2m+s}, ..., \bar{X}_{(k-1)m+s} = \lambda_{(k-1)m+s}\}$$

for $m - 1 \geq s \geq r$.

Note that the random variables $\bar{X}_s, \bar{X}_{m+s}, ...$ are independent, we get

$$P(A) \leq \max_{0\leq s\leq m-1} P(B_s) \leq \max_{0\leq s\leq m-1} C_k^j e^j \exp\{-\sum_{i=1}^{k} p_{im+s}\} \leq C_k^j e^j \bar{\lambda} e^{-n^{\lambda_0}}.$$

By the same argument as in the Lemma, we deduce that $E(N(j))^l < \infty$ for all $l > 0$.

(ii) Without loss of generality, we can suppose that the random variables X_n take only the values 0 or 1 and $p = \inf_n p_n > 0$. Since the mixing coefficient $\phi(n)$ tends to zero when n tends to $+\infty$, we have $0 < \phi(n_0) < 1 - q$ (here $q = 1 - p$) for a sufficiently large number n_0. Suppose that $n = kn_0 + r$ for $0 \leq r < n_0$, then we can rewrite the event A as follows:

$$\begin{aligned} A = \{\bar{N}(j) = n\} &= \bigcup_{(\lambda_1,\lambda_2,...,\lambda_n)} \{\bar{X}_1 = \lambda_1, \bar{X}_2 = \lambda_2, ..., \bar{X}_n = \lambda_n\} \\ &= \bigcup_{(\lambda_1,\lambda_2,...,\lambda_n)} \{\bar{X}_1 = \lambda_1, \bar{X}_{n_0+1} = \lambda_{n_0+1}, ..., \bar{X}_{kn_0+1} = \lambda_{kn_0+1}\} \dots \\ &\quad \{\bar{X}_r = \lambda_r, \bar{X}_{n_0+r} = \lambda_{n_0+r}, ..., \bar{X}_{kn_0+r} = \lambda_{kn_0+r}\} \dots \\ &\quad \{\bar{X}_{n_0} = \lambda_{n_0}, \bar{X}_{2n_0} = \lambda_{2n_0}, ..., \bar{X}_{kn_0} = \lambda_{kn_0}\}. \end{aligned}$$

Now we have

$$P(A) \le C_n^j P\big(\{\bar{X_{n_0}} = \lambda_{n_0}, \bar{X}_{2n_0} = \lambda_{2n_0}, ..., \bar{X}_{kn_0} = \lambda_{kn_0}\}\big).$$

We estimate the probability:

$P(\{\bar{X_{n_0}} = \lambda_{n_0}, \bar{X}_{2n_0} = \lambda_{2n_0}, ..., \bar{X}_{kn_0} = \lambda_{kn_0}\}) = P(\bar{X}_{n_0} = \lambda_{n_0})P(\bar{X}_{2n_0} = \lambda_{2n_0} \mid \bar{X}_{n_0} = \lambda_{n_0})...P(\bar{X}_{kn_0} = \lambda_{kn_0} \mid \bar{X}_{n_0} = \lambda_{n_0}, \bar{X}_{2n_0} = \lambda_{2n_0}, ..., \bar{X}_{(k-1)n_0} = \lambda_{(k-1)n_0})$.

This implies that

$$P(A) \le C_n^j (1 + \phi(n_0))^j (q + \phi(n_0))^{k-j}.$$

We get finally

$$E(N(j))^l = \sum_{n=1}^{\infty} n^l P(N(j) = n) \le (1 + \phi(n_0))^j \sum_{n \ge j} n^l C_n^j (q + \phi(n_0))^{k-j}$$
$$\le C \sum_{k=1}^{\infty} k^{l+j} (q + \phi(n_0))^k \le \infty.$$

The proof of the theorem is complete. ■

In a classical renewal theory, it is well-known that if $\{X_n, n \ge 1\}$ are i.i.d. non-negative radom variables with $\mu = EX_1 > 0$. Then we have the following Renewal Theorem :

$$\lim_{t \to \infty} \frac{EN(t)}{t} = \frac{1}{\mu}.$$

The theorem below is a generalization of Renewal Theorem to the case when $\{X_n, n \ge 1\}$ are $(m-1)$-dependent radom variables.

Theorem 2.2. *Let $\{X_n, n \ge 1\}$ be a sequence of $(m-1)$- independent, identically distributed, non-negative radom variables such that $P(X_1 = 0) < 1$ and $0 < \mu = EX_1 < \infty$.*

Denote $S_n = \sum_{i=1}^n X_i$. Define

$$N(t) = \max\{n : S_n \le t\},$$
$$T(t) = \inf\{n : S_n > t\}.$$

Then

$$ET(t) < \infty, \lim_{t \to \infty} \frac{EN(t)}{t} = \lim_{t \to \infty} \frac{ET(t)}{t} = \frac{1}{\mu}. \qquad (*)$$

Now we state the lemma, which plays an important role in proving the above theorem

Lemma 2.2. *Let $\{X_n, n \ge 1\}$ be a sequence of non negative, $(m-1)$-dependent, identically distributed random variables ($m \ge 1$). Denote $S_n = \sum\limits_{i=1}^{n} X_i$. We define a stopping time $T(t)$ as follows:*

$$T(t) = \inf\{n : S_n > t\}.$$

Then the following inequality holds

$$E[T(t)].EX_1 - (m-1)EX_1 \le ES_{T(t)} \le E[T(t)].EX_1 + (m-1)EX_1.$$

Proof. We evaluate ES_T for $T = T(t)$:

$$\begin{aligned} ES_T &= \int_\Omega S_T dP = \sum_{k=1}^\infty \int_{(T=k)} S_k dP = \sum_{k=1}^\infty \sum_{j=1}^k \int_{(T=k)} X_j dP \\ &= \sum_{j=1}^\infty \sum_{k=j}^\infty \int_{(T=k)} X_j dP = \sum_{j=1}^\infty \int_{(T\ge j)} X_j dP = \sum_{j=1}^\infty [EX_j - \int_{(T<j)} X_j dP] \\ &= \sum_{j=1}^{m-1} [EX_j - \int_{(T<j)} X_j dP] \\ &\quad + \sum_{j=m}^\infty \Big[EX_j - \Big(\int_{(T=1)} X_j dP + \cdots + \int_{(T=j-m)} X_j dP \Big) \\ &\qquad - \Big(\int_{(T=j-m+1)} X_j dP + \cdots + \int_{(T=j-1)} X_j dP \Big) \Big] \end{aligned}$$

From the above equality we have

$$ES_T \le (m-1)EX_1 + \sum_{j=m}^\infty [EX_j - \int_{(T\le j-m)} X_j dP].$$

Note that X_j is independent with respect to $\mathcal{F}_{j-m}$, so we have

$$\int_{(T\le j-m)} X_j dP = E[X_j.1_{T\le j-m}] = EX_j.P\{T \le j-m\}.$$

Combining the above inequalities, we get

$$ES_T \le (m-1)EX_1 + EX_1 \sum_{j=m}^\infty P(T > j-m) = (m-1)EX_1 + EX_1 ET.$$

On the other hand we can get the lower bound for ES_T as follows:

$$\begin{aligned} ES_T &\ge \sum_{j=m}^\infty \Big[EX_j - \int_{T\le j-m} X_j dP\Big] - (m-1)EX_j = EX_1 \sum_{j=m}^\infty P(T > j-m) \\ &\quad - (m-1)EX_1 = EX_1 ET - (m-1)EX_1 \end{aligned}$$

So the lemma is proved.

Now we are ready to prove the Theorem. ■

Proof of Theorem 2.2. Since $P(X_1 = 0) < 1$ we can choose a number $a > 0$ such that $P(X_1 > a) = p > 0$. By virtue of Theorem 2.1. we have $P(T(t) < \infty) = 1$ and $ET(t) < \infty$. (Note that $T(t) = N(t) + 1$). For a given number λ with $0 < \lambda < \mu$ we choose a number K to ensure $EX_1 I_{(X_n \leq K)} > \lambda$.

We define $\tilde{X}_n = X_n I_{(X_n \leq K)}$, $\tilde{S}_n = \sum_{i=1}^{n} \tilde{X}_i$ and a stopping time $\nu(t) = \inf\{n : \tilde{S}_n > t\}$. Then $\{\tilde{X}_n; n \geq 1\}$ are $(m-1)$-independent, identically distributed random variables and $E\nu(t) < \infty$. We get the upper bound on the term $E\tilde{S}_\nu$ as follows:

$$E\tilde{S}_{\nu(t)} = E\tilde{S}_{\nu(t)-1} + E\tilde{X}_{\nu(t)} \leq t + K.$$

On the other hand, using Lemma 2.2 we write

$$E\tilde{S}_{\nu(t)} \geq E\nu(t).E\tilde{X}_1 - (m-1)E\tilde{X}_1.$$

Combining the two above inequalities, we obtain (note that $T(t) \leq \nu(t)$)

$$\frac{ET(t)}{K+t} \leq \frac{E\nu(t)}{K+t} \leq \frac{1}{K+t}.\frac{E\tilde{S}_\nu + (m-1)E\tilde{X}_1}{E\tilde{X}_1} \leq \frac{1}{E\tilde{X}_1} + \frac{m-1}{K+t} \leq \frac{1}{\lambda} + \frac{m-1}{K+t}.$$

This implies that

$$\overline{\lim_{t\to\infty}} \frac{ET(t)}{t} \leq \frac{1}{\lambda} \leq \frac{1}{\mu}.$$

Conversely, we have

$$t < ES_{T(t)} \leq ET(t).EX_1 + (m-1)EX_1.$$

This implies that

$$\frac{ET(t)}{t} \geq \frac{1}{t}.\frac{t-(m-1)EX_1}{EX_1} = \frac{1}{EX_1} - \frac{m-1}{t}.$$

From this inequality, we obtain

$$\underline{\lim_{t\to\infty}} \frac{ET(t)}{t} \geq \frac{1}{\mu}.$$

Finally we have proved that

$$\lim_{t\to\infty} \frac{ET(t)}{t} = \frac{1}{EX_1}.$$

This ends our proof. ■

We treat a behavior of the renewal function $N(t)$ and show that the sequence of random variables $\{X_n, n \geq 1\}$ obeys Strong law of large numbers if and only if its renewal function satisfies the condition

$$P\Big(\lim_{t\to\infty}\frac{N(t)}{t}=1/a\Big)=1 \quad \forall t>0.$$

Theorem 2.3. *Let $(X_n)_{n\geq 0}$ be a sequence of nonnegative random variables. Then the following statements are equivalent*

(i) $P\Big(\lim_{n\to\infty}\dfrac{S_n}{n}=a\Big)=1;$

(ii) $P\Big(\lim_{t\to\infty}\frac{N(t)}{t}=1/a\Big)=1.$

Proof.

(i) Let Ω_0 be a subset of Ω such that $P(\Omega_0)=1$ and $\lim\limits_{n\to\infty}\frac{S_n(\omega)}{n}=a.$

Fix $\omega\in\Omega_0$. For a given positive number $\epsilon \quad (\epsilon<a)$, there exists a number $n_\epsilon=n(\epsilon,a)$ such that

$$a-\epsilon<\lim_{n\to\infty}\frac{S_n(\omega)}{n}<a+\epsilon \text{ for all } n\geq n_\epsilon. \tag{1}$$

From (1) we have

$$n(a-\epsilon)<S_n(\omega)<n(a+\epsilon). \tag{2}$$

From (2) and the definitions of the functions $N(t),T(t)$ we get

$$N(n(a-\epsilon))<T(n(a-\epsilon))<n, \tag{3}$$
$$n<N(n(a+\epsilon))<T(n(a+\epsilon)). \tag{4}$$

For $\quad t\geq t_\epsilon=n_\epsilon(a+\epsilon) \quad$ we have $N(t)\geq N(t_\epsilon)\geq n_\epsilon$. So (1) implies that

$$a-\epsilon<\frac{S_{N(t)}}{N(t)}<a+\epsilon. \tag{5}$$

Since $\dfrac{S_{N(t)}}{t}\leq 1$ we obtain

$$\frac{S_{N(t)}}{N(t)}.\frac{N(t)}{t}=\frac{S_{N(t)}}{t}\leq 1.$$

So

$$\frac{N(t)}{t}\leq\frac{1}{\frac{S_{N(t)}}{N(t)}}\leq\frac{1}{a-\epsilon}.$$

This implies that

$$\overline{\lim_{t\to\infty}}\frac{N(t)}{t}\leq\frac{1}{a-\epsilon}. \tag{6}$$

Since (6) holds for any $\epsilon>0$ we deduce

$$\overline{\lim_{t\to\infty}}\frac{N(t)}{t}\leq\lim_{\epsilon\to 0}\frac{1}{a-\epsilon}=\frac{1}{a}. \tag{7}$$

Similarly, for $t\geq t_\epsilon$ we have

$$T(t)\geq T(t_\epsilon)\geq N(t_\epsilon)\geq t_\epsilon.$$

Therefore for every $\epsilon > 0$ we get

$$a - \epsilon \leq \frac{S_{T(t)}}{T(t)} \leq a + \epsilon.$$

From the last inequality and $\frac{S_{T(t)}}{t} \geq 1$, it follows that

$$1 \leq \frac{S_{T(t)}}{t} = \frac{S_{T(t)}}{T(t)}.\frac{T(t)}{t}.$$

Hence the following inequality holds

$$\frac{T(t)}{t} \geq \frac{1}{\frac{S_{T(t)}}{T(t)}} \geq \frac{1}{a + \epsilon}.$$

Letting $t \to \infty$ we get

$$\underline{\lim}_{t \to \infty} \frac{T(t)}{t} \geq \frac{1}{a + \epsilon}.$$

Note that $T(t) = N(t) + 1$, the last inequality implies that

$$\underline{\lim}_{t \to \infty} \frac{N(t)}{t} \geq \frac{1}{a + \epsilon}. \tag{8}$$

(8) guaranteies that (when letting $\epsilon \to \infty$)

$$\underline{\lim}_{t \to \infty} \frac{N(t)}{t} \geq \frac{1}{a}. \tag{9}$$

Combining (7) and (9) we get finally

$$P(\lim_{t \to \infty} \frac{N(t)}{t} = \frac{1}{a}) = 1.$$

Conversely, suppose that for all $\omega \in \Omega_0$ we have

$$\lim_{t \to \infty} \frac{N(t)}{t} = \frac{1}{a}.$$

For a given ϵ, we choose $\delta > 0$ such that

$$\frac{1}{a} + \delta < \frac{1}{a - \epsilon},$$
$$\frac{1}{a} - \delta < \frac{1}{a + \epsilon}.$$

Then there exists t_δ such that for all $t \geq t_\delta$ we have

$$\frac{1}{a} - \delta < \frac{N(t)}{t} < \frac{1}{a} + \delta.$$

Now we choose n_ϵ satisfying the conditions

$$n_\epsilon(a - \epsilon) > t_\delta, n_\epsilon(a + \epsilon) > t_\delta.$$

Then

$$\frac{N(n(a - \epsilon))}{n(a - \epsilon)} < \frac{1}{a} + \delta < \frac{1}{a - \epsilon} \quad \text{for} \quad n \geq n_\epsilon.$$

It follows that $N(n(a-\epsilon)) < n$. Hence

$$\frac{S_n}{n} > a - \epsilon.$$

Similarly, from $n(a+\epsilon) > t_\delta$ it follows that

$$\frac{N(n(a+\epsilon))}{n(a+\epsilon)} < \frac{1}{a} - \delta < \frac{1}{a+\epsilon} \text{ for } n \geq n_\epsilon.$$

This ensures that

$$\frac{S_n}{n} < a + \epsilon.$$

Therefore, we get finally

$$\lim_{n\to\infty} \frac{S_n}{n} = a \quad \text{for all} \quad \omega \in \Omega_0.$$

The proof of the theorem is complete. ■

Theorem 2.3 has the following remarkable corollary

Corollary 2.1.

(i) *Let $(X_n)_{n\geq 0}$ be a $(m-1)$- dependent sequence of random variables $(m \geq 1)$ such that $\forall i : 0 \leq i \leq m-1$ $(X_{i+km})_{k\geq 0}$ are identical distributed radom variables and $EX_0 = EX_1 = ... = EX_{m-1} = a$. Then*

$$P(\lim_{t\to\infty} \frac{N(t)}{t} = 1/a) = 1.$$

(ii) *Let $(X_n)_{n\geq 0}$ be a stationary, α-mixing sequence of random variables with $EX_1 = a > 0$ and $E \mid X_1 \mid^\beta < \infty$ with $\beta > 2$ and mixing coefficients $\alpha(n) = O(n^{-\theta})$ where $\theta > 2\beta/(\beta-2)$. Then*

$$P(\lim_{t\to\infty} \frac{N(t)}{t} = 1/a) = 1 > 0.$$

Proof.
(i) From assumption, it is easy to see that for each given $i \leq m-1$, the sequence $(X_{i+km})_{k\geq 0}$ obeys the Strong law of large numbers. So the whole sequence does too. This means that

$$P(\lim_{n\to\infty} \frac{S_n}{n} = a) = 1.$$

By virtue of Theorem 2.3 we deduce

$$P(\lim_{t\to\infty} \frac{N(t)}{t} = 1/a) = 1 > 0.$$

(ii) is a direct corollary of Theorem 2.3 and [4, Theorem 2.1].

We end this work by presenting the Central limit theorem for the function $N(t)$.

Theorem 2.4. *Let $(X_n)_{n\geq 0}$ be a stationary, α-mixing sequence of positive radom variables such that $\liminf_{n\to\infty} \frac{E(S_n^2)}{n} > 0$, X_1 has finite $(2+\delta)^{th}$-order moment with $\delta > 0$ and $\lim_{n\to\infty} n.\alpha_n^{\delta/(2+\delta)} = 0$. Then*

$$P\Big(\frac{N(n) - n/\mu}{(\sigma\sqrt{n})/\mu^{3/2}} < x\Big) \to \Phi(x), \quad \forall x \in R,$$

where $\Phi(x)$ is the standard normal distribution.

Proof. The theorem is a direct consequence of the theorem in [5]. ∎

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