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Some Results on Mid-Point Sets of Sets of Natural Numbers

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Abstract. In this paper the authors study some properties of the mid-point sets of sets of natural numbers using upper (lower) asymptotic density of sets of natural numbers. In this connection a set has been introduced here and studied.

1. Introduction

Let P and Q be two linear sets of points. The mid-point set $M(P,Q)$ has been defined as the set $M(P,Q) = \left\{\dfrac{x+y}{2} : x \in P,\ y \in Q\right\}$. In particular, for $P = Q$, we write $M(P,P) = M(P)$. Again whenever A and B are two linear sets of points with positive abscissae then their ratio set $R(A,B)$ is defined as $R(A,B) = \{(a/b) : a \in A,\ b \in B\}$. In particular, when $A = B$, we write $R(A,A) = R(A)$. With the usual notations $\mathbb{N}$ is the set of natural numbers and $\mathbb{R}^+$ is the set of positive rational numbers.

One may recall here the notion of asymptotic density of a set of positive integers which is extensively used by Šala′t [5] in studying some properties of ratio sets of sets of natural numbers. Later, other authors viz Bukor, Kmetova and Toth [2] worked on ratio sets of sets of natural numbers.

Let $A \subset \mathbb{N}, A \neq \emptyset$ then $A(n)$ denotes the counting function of A given by $A(n) = \sum_{a\in A,\ a\le n} 1$. The lower asymptotic density of A is given by $\liminf_{n\to\infty} \dfrac{A(n)}{n} = \underline{d}(A)$ and the upper asymptotic density is given by $\limsup_{n\to\infty} \dfrac{A(n)}{n} = \overline{d}(A)$. If

$\underline{d}(A) = \overline{d}(A)$ we call the common value $d(A)$ as the asymptotic density of A.

On the other hand, mid-point sets, primarily of Cantor type sets were studied by Randolph [4] and subsequently by Bose Majumdar [1]. Then Ganguly and Majumdar [3] proved some results on mid-point sets of two linear sets in the light of the Lebesgue density. In the present paper the authors restrict their investigations into mid-point sets of sets of natural numbers with the help of the notion of asymptotic density.

2. Main Results

We shall study some properties of $A \subset \mathbb{N}$ which guarantee the denseness of $M(A)$ in $[1, \infty)$.

Theorem 2.1. *Let $\overline{d}(A) = 1$ where $A \subset \mathbb{N}$. Then each positive rational number can be represented as the mid-point for infinite number of pairs (g, h), $g \in A$, $h \in A$.*

Proof. Assuming the theorem not to be true there must exist an $r(\in \mathbb{R}^+) = (p/q) \neq 1$, $(p, q) = 1$ such that $r = \dfrac{g+h}{2}$ only for a finite number of pairs $(g, h), g \in A, h \in A$. Let (g_i, h_i), $i = 1, 2, ..., m$, be all the pairs of numbers of A satisfying the relation $r = \dfrac{g_i + h_i}{2}$, $i = 1, 2, ..., m$. Let us denote max $(g_1, g_2, ..., g_m, h_1, h_2, ..., h_m)$ by a and form the sequence

$$a, a+1, ..., n \quad (n > a). \tag{1}$$

The numbers in the sequence (1) are characterized by the fact that the mid-point of any two of them is different from r. Now, to sequence (1) belong all the numbers $p + u$ where

$$a < p + u \leq n \quad \text{i.e.} \quad a - p < u \leq n - p. \qquad (\alpha)$$

and also the numbers $q - v$ where

$$a < q - v \leq n \;\; \text{i.e.} \;\; a - q < -v \leq n - q. \qquad (\beta)$$

Next we put $s = \max(p, q)$ and $s' = \min(p, q)$. Then relation (α) leads to $a - s < u \leq n - s'$ and (β) yields $a - s < -v \leq n - s'$. Combining these two inequalities we can state that the numbers $p + i, q - i$ belong to sequence (1) if

$$a - s < |i| \leq n - s'. \tag{2}$$

Again from the fact that the mid-point of any two numbers of A belonging to sequence (1) is different from r, we can assert that at least one of $p + i$ and $q - i$ does not belong to A if $|i|$ satisfies condition (2). Now, we denote by $T_1(T_2)$ the set of $|i|$ which satisfies (2) but for which $p + i \notin A(q - i \notin A)$ is true. Also, let $P(T_j), j = 1, 2$ denote the number of elements of the set T_j. Then $P(T_1) + P(T_2) \geq (n - s') - (a - s)$ and consequently at least one of the numbers $P(T_1)$ and $P(T_2)$ is not smaller than $(1/2)[(n - s') - (a - s)]$.

Therefore by the definition of T_1 and T_2 and also recalling $A(n) = \sum_{a\in A, a\le n} 1$ we arrive at the inequality $A(n) \le n - (1/2)((n-s') - (a-s))$. Therefore $\overline{d}(A) = \limsup_{n\to\infty} \frac{A(n)}{n} \le 1 - (1/2) < 1$ which contradicts the assumption and hence the result follows. ■

Corollary. *If* $\overline{d}(A) = 1$ *then* $M(A) = \mathbb{R}^+$.

Note. The converse of this theorem is not necessarily true. For example, the set $A = \{2, 3, 4, 6, 7, 8, 10, 11, 12, ...\}$ has upper density $1/2$ but $M(A)$ is dense in $\mathbb{R}^+$.

We now propose to study some sufficient conditions for the set $M(A)$ not to be nowhere dense in the interval $[1, \infty)$. For this end we first prove the following theorem.

Theorem 2.2. *Let the set* $A \subset \mathbb{N}$ *be such that for each* a, b *on the real line with* $1 < a < b$ *we have* $\liminf_{n\to\infty} \frac{A((2b-1)n)}{A((2a-1)n)} > 1$. *Then there exists an interval* $I \subset (1, \infty)$, *such that* $I \cap M(A) \ne \emptyset$.

Proof. Since $A \subset \mathbb{N}$, we can certainly take A to be an infinite set. It serves our purpose to prove that the intersection of the set $M(A)$ with an interval is non-empty.

From the given condition of the theorem it can be stated that there exists a natural number n_0 such that

$$\frac{A((2b-1)n)}{A((2a-1)n)} > 1 \quad \text{for} \quad n > n_0.$$

A being an infinite set we can find a $q \in A$ such that $q > n_0$ and for this q the inequality $A((2b-1)q) - A((2a-1)q) > 0$ holds true. Then there exists a number $p \in A$ such that

$$(2a-1)q < p \le (2b-1)q \Rightarrow a < \frac{p+q}{2} \le bq$$

i.e. the intersection of the set $M(A)$ with the interval (a, bq) where $bq > b$ is non-empty. In other words the set $M(A)$ is not nowhere dense in $[1, \infty)$. ■

Theorem 2.3. *If the set* $A \subset \mathbb{N}$ *has a positive asymptotic density then the mid-point set* $M(A)$ *is not nowhere dense in* $[1, \infty)$.

Proof. By definition the asymptotic density of A is given by $d(A) = \lim_{n\to\infty} \frac{A(n)}{n}$ and we have $d(A) > 0$ by assumption. For simplicity we write d for $d(A)$. Applying the result of the foregoing theorem it needs only to show that for each a, b on the positive half of the real axis with $1 < a < b$ the inequality $\liminf_{n\to\infty} \frac{A((2b-1)n)}{A((2a-1)n)} > 1$ is true.

Let us choose an $\varepsilon\,(> 0)$ so that

(1) $\varepsilon < \dfrac{d(b-a)}{a+b-1}$. Since $d = \lim_{n\to\infty} \dfrac{A(n)}{n}$ there exists an $x_0 > 0$ such that
(2) $(d-\varepsilon)x < A(x) < (d+\varepsilon)x$ for $x > x_0$. Next we choose a natural number n_0 such that $(2a-1)n > x_0$ for $n > n_0$ which obviously leads to $(2b-1)n > x_0$ for $n > n_0$. Then using (2) we get
(3) $\dfrac{A((2b-1)n)}{A((2a-1)n)} > \dfrac{(d-\varepsilon)(2b-1)n}{(d+\varepsilon)(2a-1)n} = \dfrac{(d-\varepsilon)(2b-1)}{(d+\varepsilon)(2a-1)}$ for $n > n_0$ and for pre-assigned $\varepsilon > 0$.

Now from (1) $\varepsilon(a+b-1) < d(b-a) \Rightarrow \varepsilon(2a+2b-2) < d(2b-2a)$ i.e. $(d+\varepsilon)(2a-1) < (d-\varepsilon)(2b-1) \Rightarrow \dfrac{(d-\varepsilon)(2b-1)}{(d+\varepsilon)(2a-1)} > 1$. Thus by (3) we must have $\dfrac{A((2b-1)n)}{A((2a-1)n)} > 1$ for $n > n_0$. It follows that $\liminf_{n\to\infty} \dfrac{A((2b-1)n)}{A((2a-1)n)} > 1$, $1 < a < b$ and hence the result by Theorem 2.2. ∎

Theorem 2.4. *Let A be a subset of natural numbers with positive upper asymptotic density.*
Then the set $M(A)$ given by $M(A) = \left\{c \in N : c = \dfrac{a+b}{2},\ a \in A, b \in A\right\}$ has also positive upper asymptotic density.

Proof. By the given condition $\overline{d}(A) > 0$ i.e. $\limsup_{n\to\infty} \dfrac{A(n)}{n} > 0$ where $A(n) = \sum_{a\in A, a<n} 1$. Then a positive integer n_0 can be so chosen that $(A(n))/n > 0$ for $n > n_0 \Rightarrow A(n) > 0$ for $n > n_0$. In other words for $a \in A$, $b \in A$ where $a \le n$, $b \le n$ so that $c = \dfrac{a+b}{2} \le n$ we have
(1) $A(n) = \Sigma 1 > 0$ for $n > n_0$. Hence writing M in place of $M(A)$ for convenience we get $M(n) = \sum_{c\in M, c\le n} 1$ for $n > n_0$ by virtue of (1). Hence $\dfrac{M(n)}{n} > 0$ for $n > n_0$ leading to $\limsup_{n\to\infty} \dfrac{M(n)}{n} > 0$ i.e. $\overline{d}(M(A)) > 0$ is proved. ∎

Theorem 2.5. *Let $A \subset \mathbb{N}$ satisfy the condition $\liminf_{n\to\infty} \dfrac{A((2b-1)n)}{A((2a-1)n)} > 1$ for any pair of real numbers a, b where $1 < a < b$. Then the set $M_1(A)$ defined as*

$$M_1(A) = \left\{x \in [0,\infty) : \exists\, \{p_n\} \in A,\ \{q_n\} \in A \text{ such that } x = \lim_{n\to\infty} \frac{p_n + q_n}{2n}\right\}$$

is dense in $[0,\infty)$ provided $\lim_{n\to\infty} \dfrac{q_n}{n}$ $\left(or \lim_{n\to\infty} \dfrac{p_n}{n}\right) = l$ (l a finite quantity different from x).

Proof. It serves our purpose to show that the set $M_1(A)$ has non-empty intersection with the interval (al, bl).

We can take A to be an infinite set. Then a natural number n_0 can certainly be found so that $\dfrac{A((2b-1)n)}{A((2a-1)n)} > 1$ for $n > n_0$ and also we can find sufficiently large $q_n (> n_0) \in A$ such that the inequality $A((2b-1)q_n) > A((2a-1)q_n)$ holds true for $n > n_0$. Then there exists $p_n \in A$ such that

$$(2a-1)q_n < p_n < (2b-1)q_n \quad \text{for} \quad n > n_0 \quad \text{or} \quad a\frac{q_n}{n} < \frac{p_n + q_n}{2n} < b\frac{q_n}{n}.$$

Taking limit as $n \to \infty$ we get $al \leq \lim_{n\to\infty} \dfrac{p_n + q_n}{2n} \leq bl$ i.e. $al \leq x \leq bl$ which indicates that the intersection of the set $M_1(A)$ with the interval (al, bl) is non-empty. In other words the set $M_1(A)$ is dense in the interval $[0, \infty)$. ■

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